Thursday, 5 October 2023 08:44

Logistic eqn

$$\frac{P'(t)}{P(t)} = -\rho\boxed{(P(t)-M)}$$

$\rho, M > 0$ constants

$$\ulcorner\ \boxed{\frac{P'}{P} = r} \iff (\log P)' = r$$

$$\iff \log P = \int r \, dt + c$$

$$(*)\ \boxed{\underset{F}{\log} P = rt + c} \leftarrow \ \lrcorner$$

$$\frac{P'(t)}{P(t)} = (\log P(t))' \qquad (P>0)$$

$$(\log P(t))' = -\rho(P(t)-M)$$

$$\boxed{\log P(t) = \int -\rho(P(t)-M)\,dt + c}\ ?$$

If $P(t) \neq M$

$$\frac{P'(t)}{P(t)} = -\rho\,\underbrace{(P(t)-M)} \Leftrightarrow$$

$$\boxed{\begin{array}{c} \dfrac{P'(t)}{P(t)(P(t)-M)} \\ \| \\ \Big(F(P(t)) \Big)' \end{array}} \begin{array}{l} = -\rho \\ \\ \\ = -\rho \end{array}$$

$$\Updownarrow$$

$$F(P(t)) = \int -\rho\, dt \; \boxed{+c}$$

$$\boxed{F(P(t)) = -\rho t + c} \leftarrow (\mathrm{IF})$$

implicit form

$$\underbrace{F^{-1}}(F(P(t))) = F^{-1}(-\rho t + c)$$

$$P(t) = F^{-1}(-\rho t + c)$$

But, how can we det such an F?

We want

$$\left[\underset{\uparrow}{F}(P(t))\right]' = \frac{P'(t)}{P(t)\,(P(t)-M)} \leftarrow$$

chain rule

$$F'(\underbrace{P(t)}_{p})\cdot \cancel{P'(t)} = \frac{\cancel{P'(t)}\ 1}{\underbrace{P(t)}_{p}\,(\underbrace{P(t)}_{p}-M)}$$

$$F'(p) = \frac{1}{p(p-M)}$$

$$\Longleftrightarrow \quad F(p) = \int \frac{1}{p(p-M)}\,dp$$

$$-\frac{1}{M}\underbrace{\left(\frac{1}{p} + \frac{-1}{p-M}\right)}$$

$$-\frac{1}{M}\left(\frac{\cancel{p}-M-\cancel{p}}{p(p-M}\right)$$

$$= \int -\frac{1}{M}\left(\frac{1}{p} - \frac{1}{p-M}\right)dp$$

$$= -\frac{1}{M}\left(\log|p| - \log|p-M|\right)$$

$$= \frac{1}{M}\log\frac{|p-M|}{|p|} = \frac{1}{M}\log\left|\frac{p-M}{p}\right|$$

$$F(p) = \frac{1}{M}\log\left|\frac{P-M}{P}\right|$$

$$(IF) \qquad \frac{1}{M}\log\left|\frac{P(t)-M}{P(t)}\right| = -\rho t + c$$

$$c \in \mathbb{R}$$

$$\Leftrightarrow \quad \boxed{\log}\left|\frac{P(t)-M}{P(t)}\right| = -\rho M t + \underset{c}{\cancel{cM}}$$

$$\Leftrightarrow \quad \left|\frac{P(t)-M}{P(t)}\right| = e^{-\rho M t + c}$$

$$= e^{c}\, e^{-\rho M t}$$

$$\left|\frac{P(t)-M}{P(t)}\right| = \underbrace{k\, e^{-\rho M t}} \qquad k > 0$$

$+$

$$\Leftrightarrow \quad \frac{P(t)-M}{P(t)} = (\pm k)e^{-\rho M t}$$

$$\frac{P(t)-M}{P(t)} = k\, e^{-\rho M t} \qquad k \neq 0$$

$$\text{“}\Leftrightarrow\text{”} \quad P(t) - M = k\, e^{-\rho M t}\, P(t)$$

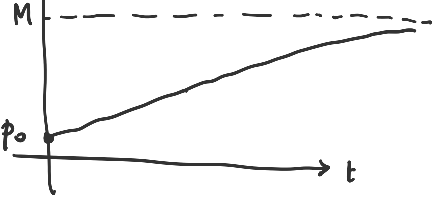
$$\Leftrightarrow \quad P(t)\left(1 - k\, e^{-\rho M t}\right) = M$$

$$\Leftrightarrow \quad \boxed{P(t) = \frac{M}{1 - k\, e^{-\rho M t}} \qquad k \neq 0.}$$

Cauchy Pb

$$\begin{cases} \dfrac{P'(t)}{P(t)} = -\rho\,(P(t) - M) & \leftarrow \\[2ex] P(0) = p_0 < M, \quad 0 < p_0 & \leftarrow \end{cases}$$

$P(t)$ ↑



Since P solves the eqn ⇒

$$P(t) = \frac{M}{1 - k e^{-\rho M t}} \qquad k \neq 0$$

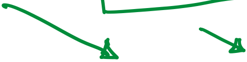
Imposing $P(0) = p_0$ we get

$$\frac{M}{1-k} = p_0 \iff \iff \frac{M}{p_0} = 1 - k$$

$$\iff k = 1 - \frac{M}{p_0}$$

$$\Rightarrow \boxed{P(t) = \frac{M}{1 - \underbrace{\left(1 - M/p_0\right)}_{-} e^{-\rho M t}}} \searrow$$

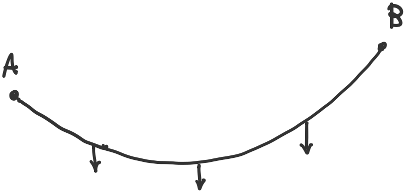
$$= \frac{M}{\underbrace{1 + \left(\frac{M}{p_0} - 1\right) e^{-\rho M t}}_{+}} \nearrow \searrow$$



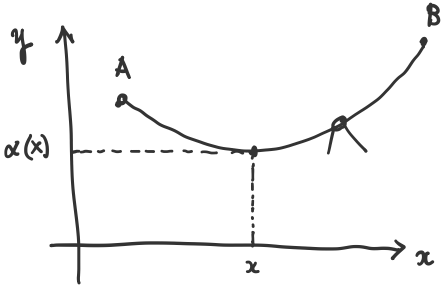
$$\lim_{t \to +\infty} P(t) = \frac{M}{1} = M$$

■

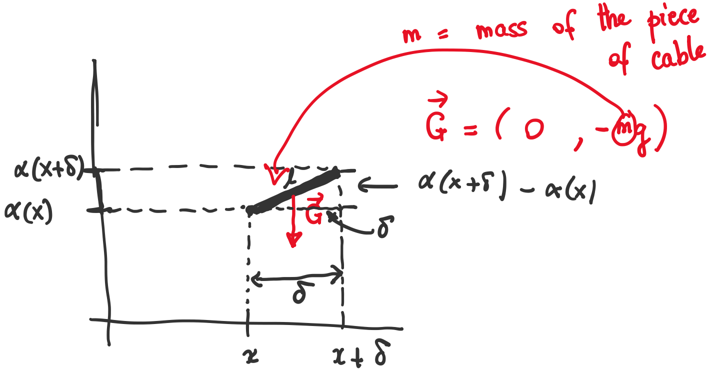
Catenary Problem



Pb: Det the shape of equil. cve of an heavy cable suspended on two fixed pts A, B.



Math. we have to det the funct. $\alpha = \alpha(x)$.



$$m = \overset{\text{length of the piece of cable}}{\ell} \cdot \rho$$

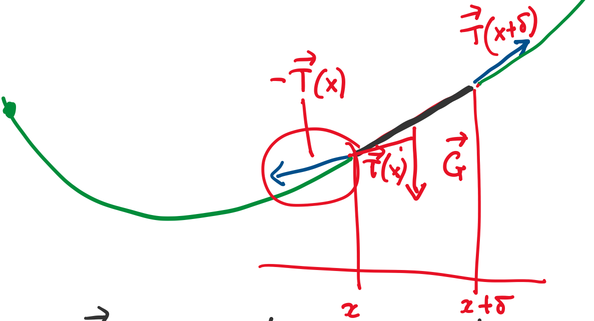
ρ = mass density (kg/m)

$$\ell = \sqrt{\delta^2 + (\alpha(x+\delta) - \alpha(x))^2} \leftarrow$$

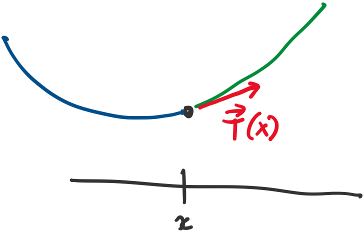
$$\overset{\delta>0}{=} \delta \sqrt{1 + \Big(\underset{\substack{\downarrow\ \delta \to 0 \\ \alpha'(x)}}{\frac{\alpha(x+\delta) - \alpha(x)}{\delta}}\Big)^2}$$

$$\ell \approx \delta \sqrt{1 + \alpha'(x)^2}$$

$$\Rightarrow \quad \vec{G} = \left(0, -g\rho\delta\sqrt{1+x'(x)^2}\right)$$



$\vec{T}(x)$ = tension a pt x



Equilibrium cond :

$$\boxed{\underset{(\quad,\quad)}{\vec{T}(x+\delta)} - \underset{(\quad,\quad)}{\vec{T}(x)} + \underset{(\quad,\quad)}{\vec{G}(x)} = \vec{0}} = (0,0)$$

$$\vec{T}(x) = \left(\underbrace{\tau(x)}, \sigma(x)\right)$$

$$\vec{T}(x+\delta) = \left(\underbrace{\tau(x+\delta)}_{}, \sigma(x+\delta)\right)$$

$$\begin{cases} \tau(x+\delta) - \tau(x) + 0 = 0 \\ \sigma(x+\delta) - \sigma(x) - g\rho\boxed{\delta}\sqrt{1+\alpha'(x)^2} = 0 \end{cases}$$

$$\frac{\tau(x+\delta) - \tau(x)}{\delta} = 0 \qquad \delta \to 0 \qquad \Rightarrow \tau'(x) \equiv 0$$

$$\Rightarrow \boxed{\tau = \text{const}}$$

$$\frac{\sigma(x+\delta) - \sigma(x)}{\delta} - g\rho\sqrt{1+\alpha'(x)^2} = 0$$

$$\Rightarrow \underbrace{\sigma'(x)}_{} - g\rho\sqrt{1+\alpha'(x)^2} = 0$$

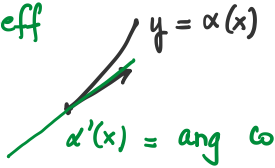
Since $\vec{T}(x)$ is tg to the cve

$\vec{T}(x) = (\tau(x), \sigma(x))$ tg to cve

$\downarrow$

ang coeff

$$\frac{\sigma(x)}{\tau(x)}$$



$\alpha'(x)$ = ang coeff of tg to α

$$\boxed{\frac{\sigma(x)}{\tau(x)} = \alpha'(x)}$$

(τ const)

$$\frac{\sigma(x)}{\tau_0} = \alpha'(x)$$

$$\sigma(x) = \tau_0 \alpha'(x)$$

$$\sigma'(x) = \tau_0 \alpha''(x)$$

$$\Rightarrow \quad \boxed{\tau_0 \alpha''(x) - g\rho\sqrt{1 + \alpha'(x)^2} = 0}$$

$$y'(x) + y(x-1) = 0$$

$$F\left(x, y(x), y'(x), \dots, y^{(n)}(x)\right) = 0$$

$$y'(x) + \int_0^1 y(t)\, dt = 0$$