Wednesday, 4 October 2023 08:47

Malthus model

$\underset{\overset{\vee}{0}}{P(t)}$ = size of the pop at time t

fulfilling

$$\frac{P'(t)}{P(t)} = r$$

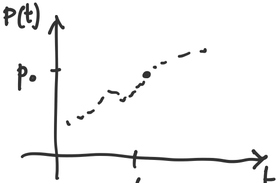
(r constant)
instantaneous
growth rate

Typically one aims to solve the following prediction pb: given the "initial" pop, determine future values of P.

Formally: det $P = P(t)$ s.t

Cauchy problem

$$\begin{cases} \boxed{\dfrac{P'(t)}{P(t)} = r} \\ P(t_0) = p_0 \end{cases}$$



$\nearrow$ $\qquad\qquad\qquad\qquad t_0 \qquad\qquad t$

(initial value cond
passage cond)

Step 1 : let's solve the diff eqn.

$$\frac{P'(t)}{P(t)} = r \quad (*)$$

$$(\log P(t))' \qquad \frac{d}{dt}$$

$$\boxed{[f(g(t))]' = f'(g(t))\underset{\uparrow}{g'(t)}}$$

$$= \frac{1}{P(t)} \cdot P'(t)$$

$$(*) \iff (\log P(t))' = r$$

$$\square' = r$$

$$f'(t) = 1 \quad \Rightarrow \quad f(t) = t + c \qquad c \in \mathbb{R}$$

$$f'(t) = r \quad \Rightarrow \quad f(t) = rt + c \qquad c \in \mathbb{R}$$

$$\Rightarrow \quad \log \underbrace{P(t)} = rt + c, \qquad \underline{c \in \mathbb{R}}$$

$$\Rightarrow \qquad (\log = \log_e)$$

$$P(t) = \underbrace{e^{rt+c}}_{}$$

$$= e^{c}\, e^{rt}$$

$$= k e^{rt} \qquad k > 0$$

Conclusion: if $P > 0$, P sol of

$$\frac{P'(t)}{P(t)} = r \iff \boxed{\begin{array}{c} P(t) = k e^{rt} \\ k > 0 \end{array}}$$

Rmk 1: a diff eqn has (in gen.) infinitely many sols. (this is shown by param k which is $k > 0$, arb.)

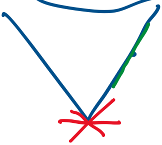
Rmk 2: What if I look for $P < 0$?

$$\frac{P'(t)}{P(t)} = r$$

$$\frac{P'(t)}{P(t)} = \left(\log|P(t)|\right)'$$

$$\left[= \frac{1}{|P(t)|} \cdot \left(|P(t)|\right)' = \frac{1}{|P(t)|}\,\mathrm{sgn}(P(t)) \cdot P'(t)\right.$$

$$|x|' \;\; \frac{d}{dx} = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$



$$= \mathrm{sgn}(x) \qquad x \neq 0$$

$$\left(\log|P(t)|\right)' = \frac{1}{|P(t)|}\,\mathrm{sgn}(P(t)) \cdot P'(t)$$

$$\frac{\mathrm{sgn}\, x}{|x|} = \begin{cases} \frac{+1}{x} = \frac{1}{x} & x > 0 \\ \frac{-1}{-x} = \frac{1}{x} & x < 0 \end{cases} = 1/x \qquad x \neq 0$$

$$|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

$$\left(\log|P(t)|\right)' = \frac{1}{P(t)} \cdot P'(t) \quad \lrcorner$$

So

$$\left(\log|P(t)|\right)' = \boxed{\frac{P'(t)}{P(t)} = r}$$

$$\Rightarrow \quad \log|P(t)| = rt + c$$

$$\Rightarrow \quad |P(t)| = e^{rt+c} = e^{c} e^{rt}$$

$$|P(t)| = \underbrace{k e^{rt}}_{+} \qquad k > 0$$

$$\Rightarrow \quad P(t) = \underbrace{\pm k}\, e^{rt} \qquad k > 0$$

$$\Rightarrow \quad P(t) = k e^{rt} \qquad k \neq 0.$$

Rmk 3: What if $k = 0$? $\Rightarrow P(t) \equiv 0$

Certainly $P \equiv 0$ is not a sol of

$$\frac{P'(t)}{P(t)} = r \quad \not\Leftrightarrow \quad \left(P'(t) = r P(t) \right)$$

$\downarrow$ $0/0$ here $P(t) \equiv 0$ is a sol)

Rmk 4. Let's apply sol formula to the Cauchy Pb

$$\left\{ \frac{P'(t)}{P(t)} = r \right.$$

$$(*) \quad P(t_0) = p_0$$

(r, t_0, p_0 are known)

We know that

$P > 0$ solves $\dfrac{P'}{P} = r$

$$\Leftrightarrow \quad P(t) = \boxed{k} e^{rt} \qquad k > 0$$

$$(*) \quad \Leftrightarrow \quad \underbrace{k \underbrace{e^{rt_0}}_{+}}_{P(t_0)} = p_0$$

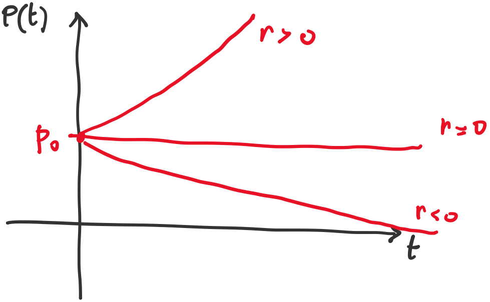
$$\Leftrightarrow \quad \boxed{k = p_0 e^{-rt_0}}$$

⇒ there is precisely a unique sol to the CP

$$P(t) = \underbrace{p_0 e^{-rt_0}}_{k} \cdot e^{rt}$$

$$\boxed{P(t) = p_0 e^{r(t-t_0)}}$$

$t_0 = 0 \qquad P(t) = p_0 e^{rt}$

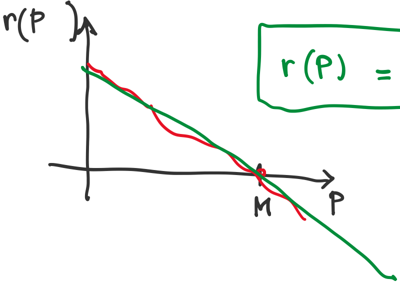


A more realistic model could be

$$\frac{P'(t)}{P(t)} = r(t, P(t))$$

Here we focus on

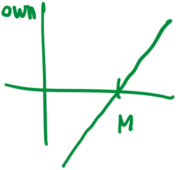
$$r = r(P(t))$$



$$\boxed{r(P) = -\rho(P-M)}$$

ρ, M known

$\begin{matrix}\vee & \vee \\ 0 & 0\end{matrix}$



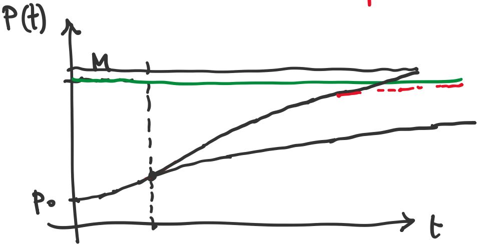
With this $r(P)$ we get

$$\boxed{\frac{P'(t)}{P(t)} = -\rho\,(P(t) - M)}$$

(logistic model)

Again we consider the typical pred pb

Cauchy pb
$$\begin{cases} \dfrac{P'(t)}{P(t)} = -\rho\,(P(t) - M) \\ P(t_0) = p_0 . \end{cases}$$



$$r(P) = mP + q = mP - mM$$

$$r(M) = 0$$

$$mM + q = 0$$

$$q = -mM$$

$$= m(P - M)$$