Numerical Methods for Astrophysics:

ORDINARY DIFFERENTIAL EQUATIONS (ODEs)
Part 1

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Ordinary Differential Equations (ODEs). Concept

ODEs ARE UBIQUITOUS IN ASTROPHYSICS

Examples:

– **equation of motion of a particle subject to Newton's gravitational force**

$$\frac{\mathrm{d}^2\mathbf{x}_i}{\mathrm{d}t^2} = -G \sum_{j=1,\, j\neq i}^{N} m_j \frac{\mathbf{x}_i - \mathbf{x}_j}{\left|\mathbf{x}_i - \mathbf{x}_j\right|^3}$$

– **equation of hydrostatic equilibrium of a star interior**

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -G\,\frac{M\,\rho}{r^2}$$

ODEs. Euler Method

General form of an ODE in 1 variable: $$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,\, t)$$

For simplicity, let's assume *t* is time

Simplest way to proceed: TAYLOR EXPANSION

$$x(t+h) = x(t) + \frac{\mathrm{d}x}{\mathrm{d}t}\,h + \frac{1}{2}\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,h^2 + \ldots$$

or with different notation

$$x(t+h) = x(t) + h\,f(x,t) + \mathcal{O}(h^2)$$

If we neglect higher order terms we can calculate *x(t+h)* as

$$\boxed{x(t+h) = x(t) + h\,f(x,t)}$$

equation of the Euler scheme

if we know the value of *x* at time *t*, then we can derive the value of *x* at time *(t+h)*

This approximation is "good" if *h* is small

ODEs. Euler Method

Equation of the Euler's method: $\boxed{x(t+h) = x(t) + h\, f(x,t)}$

To integrate the ODE between $t = t_0$ and $t = t_f$

I need to choose a $h \ll (t_f - t_0)$

and then to repeat Euler's equation for N steps with

$$N = (t_f - t_0)\, /\, h$$

Euler equation is 1st order method → errors scale as h^2

→ we can reduce the error by reducing h

but if we reduce h the computing time increases

We will see other methods that have a smaller error for the same (or similar) computing time

ODEs. Runge-Kutta family

Family of algorithms that solve ODEs via Taylor's expansion

First order: Euler method or first-order Runge-Kutta method

Second order: midpoint or second-order Runge-Kutta method

Fourth order: fourth-order Runge-Kutta method

etcetc

ODEs. Midpoint or second-order Runge-Kutta method

Evaluate the slope d*x*/d*t* of *x(t)*
not at the end of the interval *h*, but at the midpoint of the interval *h/2*

Mathematically, corresponds to a Taylor expansion around *t+h/2* instead that around *t*

$$x(t+h) = x\left(t+\frac{1}{2}h\right) + \frac{1}{2}h\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{t+\frac{1}{2}h} + \frac{1}{8}h^2\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)_{t+\frac{1}{2}h} + \mathcal{O}(h^3)$$

$$x(t) = x\left(t+\frac{1}{2}h\right) - \frac{1}{2}h\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{t+\frac{1}{2}h} + \frac{1}{8}h^2\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)_{t+\frac{1}{2}h} + \mathcal{O}(h^3)$$

Subtracting the second expression from the first, we get

$$\begin{aligned} x(t+h) &= x(t) + h\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{t+\frac{1}{2}h} + \mathcal{O}(h^3) \\ &= x(t) + h\, f\left(x\left(t+\frac{1}{2}h\right), t+\frac{1}{2}h\right) + \mathcal{O}(h^3) \end{aligned}$$

The error scales as h^3 → is a second-order scheme

BUT THERE IS A PROBLEM HERE

ODEs. Midpoint or second-order Runge-Kutta method

PROBLEM:

$$x(t+h) = x(t) + h\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{t+\frac{1}{2}h} + \mathcal{O}(h^3)$$
$$= x(t) + h\,f\left(x\left(t+\frac{1}{2}h\right), t+\frac{1}{2}h\right) + \mathcal{O}(h^3)$$

This eq. requires that we know *x(t+h/2)* which we still do not know

IMPLICIT SCHEME: method that depends on quantities that we still need to calculate (because they refer to the next step)

vs EXPLICIT SCHEME: method depending only on quantities that we already know (because they refer to current step)

ODEs. Midpoint or second-order Runge-Kutta method

PROBLEM:

$$x(t+h) = x(t) + h\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{t+\frac{1}{2}h} + \mathcal{O}(h^3)$$

$$= x(t) + h\, f\left(x\left(t+\frac{1}{2}h\right), t+\frac{1}{2}h\right) + \mathcal{O}(h^3)$$

This eq. requires that we know *x(t+h/2)* which we still do not know

We get around this problem by approximating *x (t + h/2)*
with the Euler method $x\left(t+\frac{h}{2}\right) = x(t) + \frac{h}{2} f(x(t),t)$ **where** $f(x(t),t) = \frac{\mathrm{d}x}{\mathrm{d}t}$

and then substituting into the above equation:

$$\boxed{\begin{aligned} k_1 &= \frac{h}{2} f(x(t),t) \\ k_2 &= h\, f\left(x(t)+k_1,\, t+\frac{h}{2}\right) \\ x(t+h) &= x(t) + k_2 \end{aligned}}$$

equation of the midpoint scheme

which is the practical implementation of the midpoint scheme

ODEs. Fourth-order Runge-Kutta method

We can use the same approach to go to higher order i.e.

*** by using Taylor expansion**

*** by evaluating the ODE in several intermediate time steps**

With calculations, we derive the fourth-order Runge-Kutta as

$$k_1 = \frac{1}{2}\, h\, f(x, t)$$

$$k_2 = \frac{1}{2}\, h\, f\left(x + k_1,\, t + \frac{1}{2}h\right)$$

$$k_3 = h\, f\left(x + k_2,\, t + \frac{1}{2}h\right)$$

$$k_4 = h\, f(x + k_3,\, t + h)$$

$$x(t+h) = x(t) + \frac{1}{6}\left(2\, k_1 + 4\, k_2 + 2\, k_3 + k_4\right)$$

*** Errors scale as h^5**

*** Fourth-order Runge-Kutta (RK4) is considered the best match between accuracy and not-too-complicated programming**

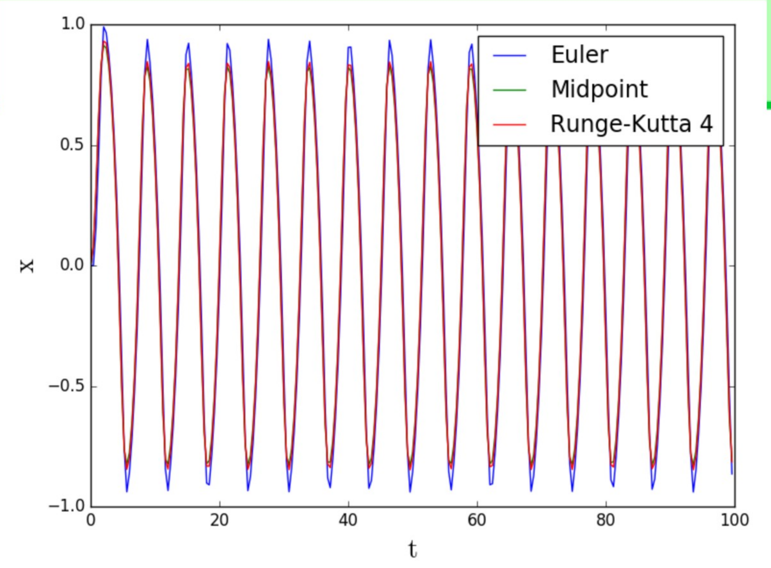
ODEs. EXERCISE on Euler, Midpoint, Runge-Kutta 4

EXERCISE:

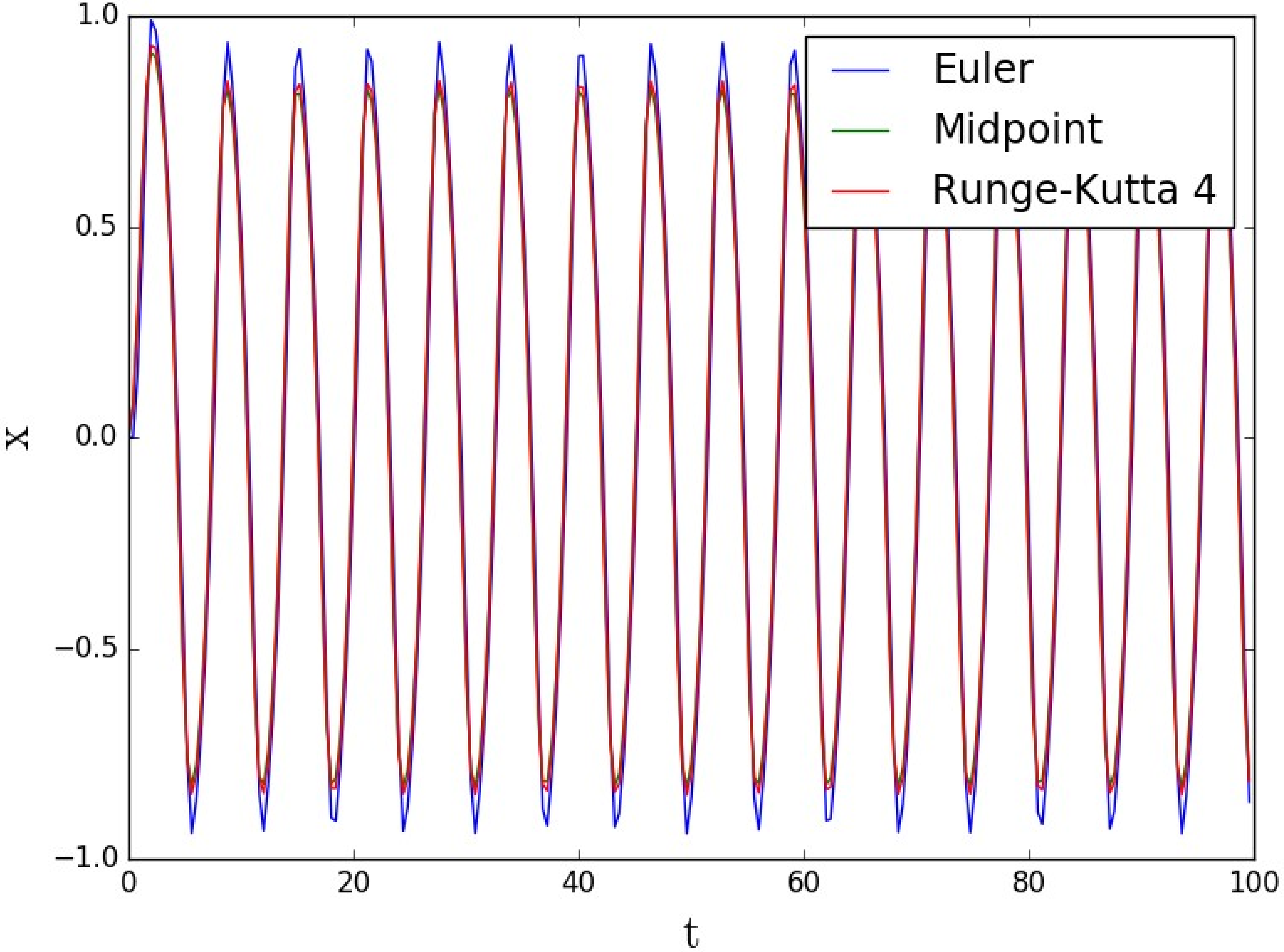
Write a python script to implement the Euler's method, the midpoint method and the fourth-order Runge-Kutta method. Use this script to integrate the following differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x^3 + \sin t \tag{146}$$

Compare the results. For a choice of initial time $t_0 = 0.0$, final time $t_{\mathrm{fin}} = 100$, initial position $x(t_0) = 0.0$ and step-size $h = 0.4$, you should obtain something similar to Figure 41.



ODEs. EXERCISE on Euler, Midpoint, Runge-Kutta 4



ODEs. Systems of ordinary differential equations

Same approach as we have seen in the previous sections, provided that the derivatives are with respect to a single variable

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x,\ y,\ t)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f_2(x,\ y,\ t)$$

They must be integrated in the same timestep, simultaneously, to avoid mismatch between x and y

In contrast, partial differential equations require a different treatment

$$\frac{\partial x}{\partial t} + \frac{\partial x}{\partial s} = f_1(x, y, t, s)$$

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial s} = f_2(x, y, t, s),$$

ODEs. Second-order and higher-order ODEs

Solving second-order (or higher-order) ODEs with one variable is trivial once we know how to solve first-order ODEs.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = f\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, t\right)$$

Can be rewritten as a SYSTEM of TWO FIRST-ORDER ODES

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = f\left(x, y, t\right) \end{cases}$$

Solve this system using the algorithms we learnt for first-order ODEs.

To solve higher ODEs, we repeat this trick till we have a system of first-order ODEs only.

ODEs. Second-order and higher-order ODEs

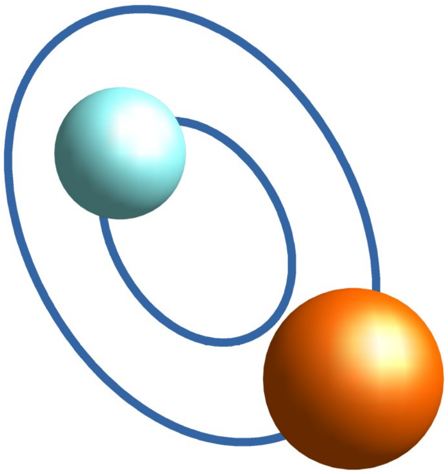
CLASSICAL EXAMPLE of 2^{nd} order ODE for astrophysicists:

equation of motion of a star in a binary system

$$\frac{\mathrm{d}^2\vec{x}_i}{\mathrm{d}t^2} = -G\,m_j\,\frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|^3}$$

can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} = \vec{v}_i \\[2ex] \dfrac{\mathrm{d}\vec{v}_i}{\mathrm{d}t} = -G\,m_j\,\dfrac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|^3} \end{cases}$$



ODEs. Astrophysical N-body problem

Integration of the equations of motion for N –bodies subject to Newton's gravity force (1687)

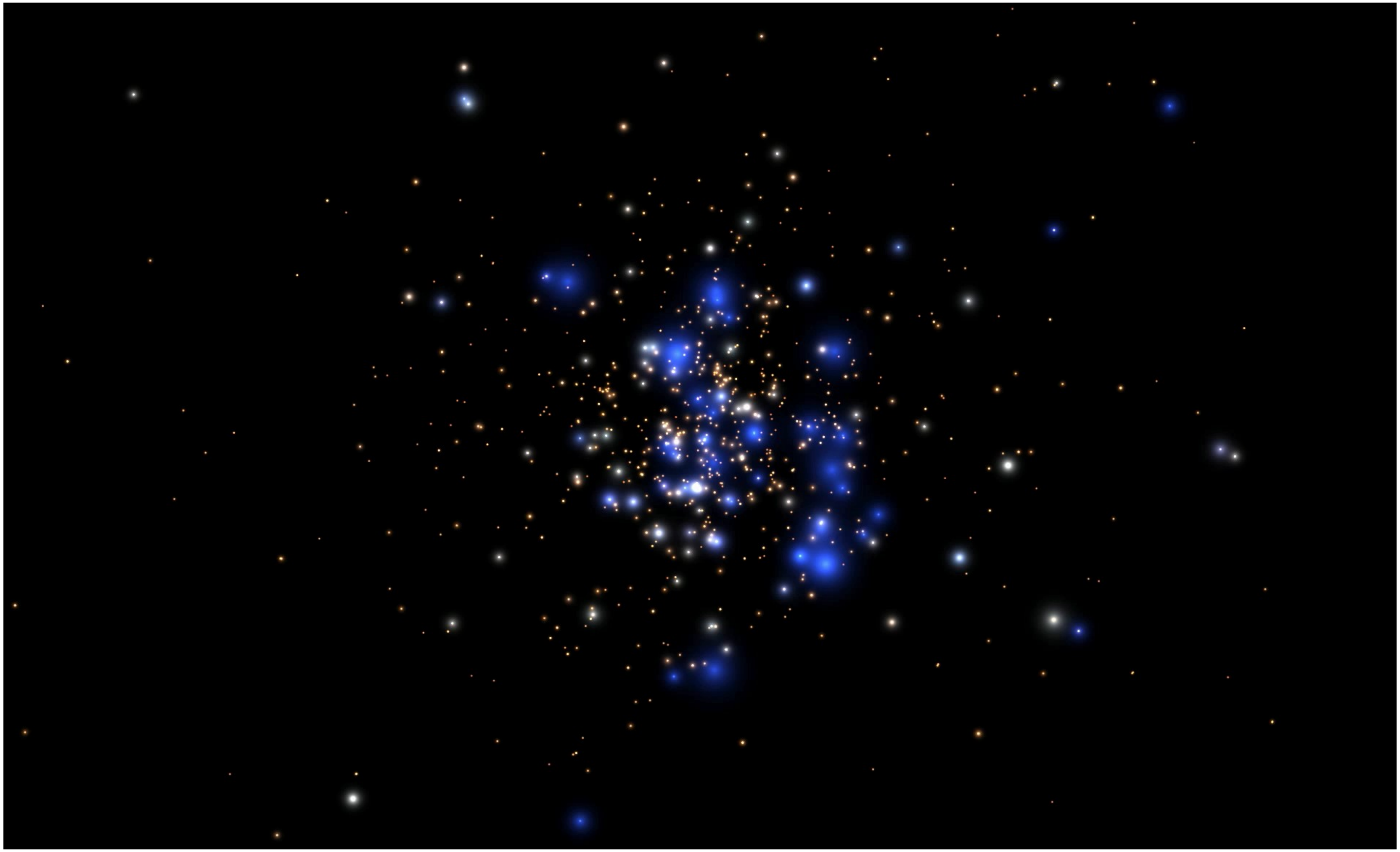
$$\frac{\mathrm{d}^2\mathbf{x}_i}{\mathrm{d}t^2} = -G \sum_{j=1,\, j\neq i}^{N} m_j \frac{\mathbf{x}_i - \mathbf{x}_j}{\left|\mathbf{x}_i - \mathbf{x}_j\right|^3}$$

can be split into a system of 2 first-order ODEs

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = -G \displaystyle\sum_{j=1,\, j\neq i}^{N} m_j \frac{\mathbf{x}_i - \mathbf{x}_j}{\left|\mathbf{x}_i - \mathbf{x}_j\right|^3}, \\ \dfrac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = \mathbf{v}_i \end{cases}$$

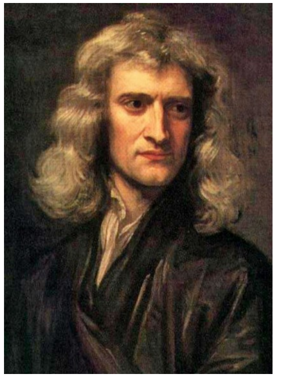
ODEs. Astrophysical N-body problem

It is the first thing you need to solve to simulate a star cluster



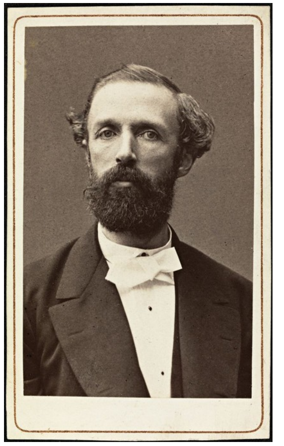
The second thing you need is stellar evolution

ODEs. Astrophysical N-body problem



Newton

Bernoulli



Oscar II of Sweden

- **This eq. can be solved analytically for N = 2 (Bernoulli solution, 1710).**
- **In 1885, a challenge was proposed, to be answered before January 21st 1889, in honour of the 60th birthday of King Oscar II of Sweden and Norway:**

  "Given a system of arbitrarily many mass points that attract each according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly."

  Nobody found the solution, although many participated (including Henry Poincaré).

- **1991: the mathematician Qiudong Wang found the first convergent power series solution for a generic number of bodies.**
  However, the solution by Q. Wang is too difficult to implement and slow to convergence.
  Thus, everybody solves Newton's equation numerically for N>=3.

Q. Wang

ODEs. Astrophysical N-body problem

Newton's equation can be solved with Runge-Kutta methods.
For example, the Euler scheme:

$$\boxed{x(t+h) = x(t) + h\, f(x,t)}$$

$$\vec{a}_i(t) = -G \sum_{j=1,\, j\neq i}^{N} m_j \frac{\vec{x}_i(t) - \vec{x}_j(t)}{|\vec{x}_i(t) - \vec{x}_j(t)|^3}$$

$$\vec{x}_i(t+h) = \vec{x}_i(t) + \vec{v}_i(t)\, h$$

$$\vec{v}_i(t+h) = \vec{v}_i(t) + \vec{a}_i(t)\, h$$

ODEs. EXERCISE on binary star with Euler

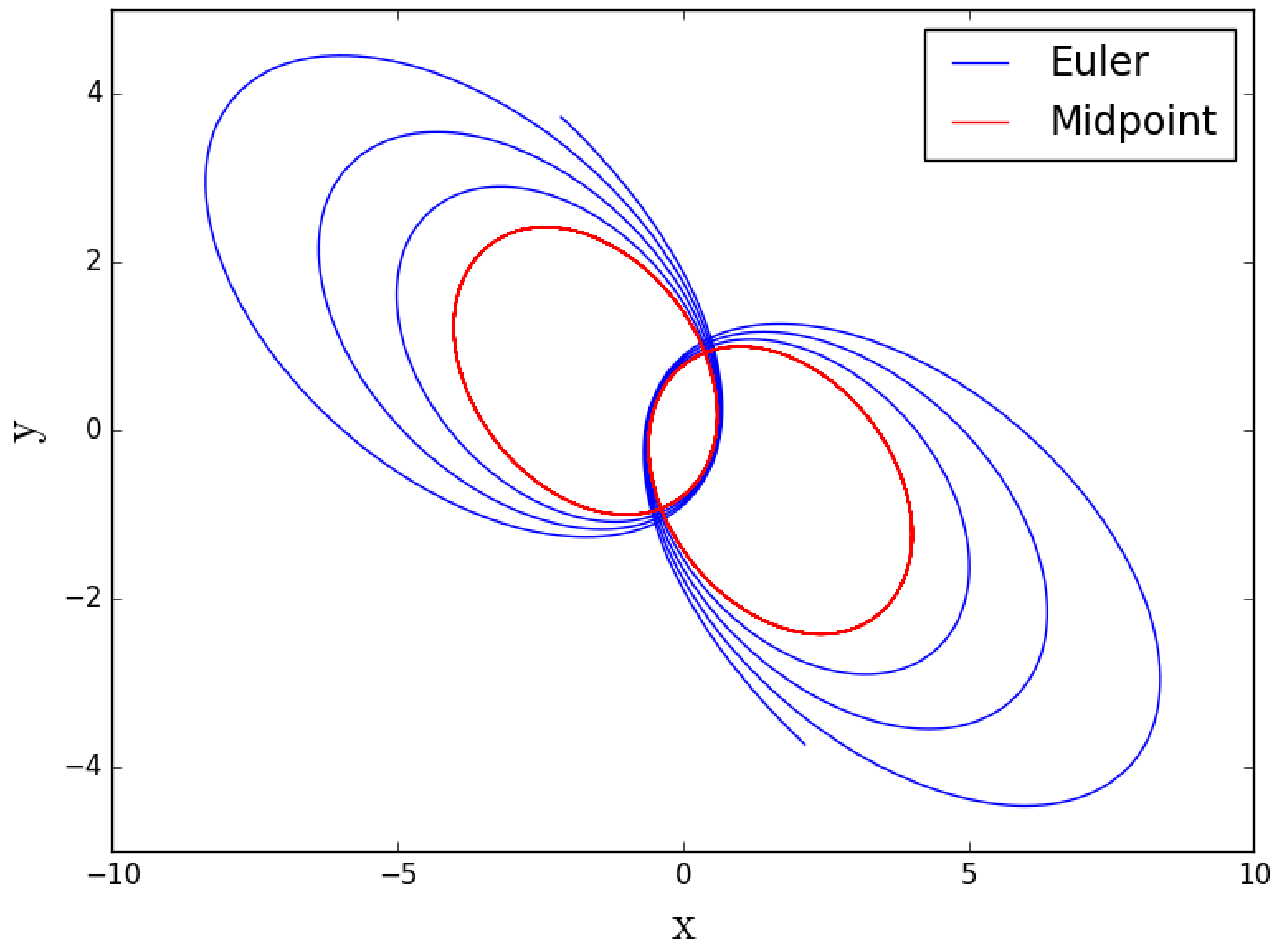
Newton's equation can be solved with Runge-Kutta methods.
For example, the Euler scheme:

EXERCISE:

Write a new script to implement Euler's method to evolve a system of two points in two dimensions (xy plane), subject to gravity forces, with the following initial conditions. Initial positions of particles 1 and 2 (in the plane xy): $\mathbf{x} = (1.0, -1.0)$, $\mathbf{y} = (1.0, -1.0)$.
Initial velocities of particles 1 and 2 (in the plane xy): $\mathbf{v}_x = (-0.5, 0.5)$, $\mathbf{v}_y = (0.0, 0.0)$.
Let us assume that the masses are $m_1 = m_2 = 1$, and the gravity constant in our units is G = 1.
Let us assume $t_0 = 0$, $t_{\text{fin}} = 300$ and $h = 0.01$. The result should look like the blue line in Figure 42.

ODEs. EXERCISE on binary star with Euler

Result of previous exercise is the blue line:



ODEs. Midpoint & the astrophysical N-body problem

General expression of the midpoint scheme

$$k_1 = \frac{h}{2}\, f(x(t), t)$$
$$k_2 = h\, f\left(x(t) + k_1,\, t + \frac{h}{2}\right)$$
$$x(t+h) = x(t) + k_2$$

How does it look like when applied to the astrophysical N-body problem?

$$\boxed{k_1 = \frac{h}{2}\, f(x(t), t)} \longrightarrow \boxed{\begin{aligned} k_{1,x} &= \frac{h}{2}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} \\ k_{1,v} &= \frac{h}{2}\frac{\mathrm{d}v_i(x_i(t), t)}{\mathrm{d}t} \end{aligned}}$$

$$\boxed{k_2 = h\, f\left(x(t) + k_1,\, t + \frac{h}{2}\right)} \longrightarrow \boxed{\begin{aligned} k_{2,x} &= h\,\frac{\mathrm{d}(x_i(t) + k_{1,x}, t + h/2)}{\mathrm{d}t} \\ k_{2,v} &= h\,\frac{\mathrm{d}(v_i(t) + k_{1,v}, t + h/2)}{\mathrm{d}t} \end{aligned}}$$

$$\boxed{x(t+h) = x(t) + k_2} \longrightarrow \boxed{\begin{aligned} x_i(t+h) &= x_i(t) + k_{2,x} \\ v_i(t+h) &= v_i(t) + k_{2,v} \end{aligned}}$$

ODEs. Midpoint & the astrophysical N-body problem

$$k_{1,x} = \frac{h}{2}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}$$
$$k_{1,v} = \frac{h}{2}\frac{\mathrm{d}v_i(x_i(t),t)}{\mathrm{d}t}$$

$$k_{1,x} = \frac{h}{2}\,v_i(t)$$
$$k_{1,v} = \frac{h}{2}\,a_i(t)$$

$$k_{2,x} = h\,\frac{\mathrm{d}(x_i(t)+k_{1,x},t+h/2)}{\mathrm{d}t}$$
$$k_{2,v} = h\,\frac{\mathrm{d}(v_i(t)+k_{1,v},t+h/2)}{\mathrm{d}t}$$

$$k_{2,x} = h\,\frac{\mathrm{d}}{\mathrm{d}t}\left(x_i(t) + \overbrace{\frac{h}{2}v_i(t)}^{k_{1,x} = \frac{h}{2}\,v_i(t)}\right) = h\,\left(v_i(t)+\frac{h}{2}a_i(t)\right)$$
$$k_{2,v} = h\,\frac{\mathrm{d}}{\mathrm{d}t}\left(v_i(t)+\frac{h}{2}\,a_i(t)\right)$$

$$x_i(t+h) = x_i(t) + k_{2,x}$$
$$v_i(t+h) = v_i(t) + k_{2,v}$$

ODEs. Midpoint & the astrophysical N-body problem

$$k_{1,x} = \frac{h}{2}\, v_i(t)$$
$$k_{1,v} = \frac{h}{2}\, a_i(t)$$

$$k_{2,x} = h\, \left(v_i(t) + \frac{h}{2} a_i(t)\right)$$
$$k_{2,v} = h\, \frac{\mathrm{d}}{\mathrm{d}t} \left(v_i(t) + \frac{h}{2}\, a_i(t)\right)$$

$$x_i(t+h) = x_i(t) + k_{2,x}$$
$$v_i(t+h) = v_i(t) + k_{2,v}$$

ODEs. Midpoint & the astrophysical N-body problem

$$k_{1,x} = \frac{h}{2}\, v_i(t)$$
$$k_{1,v} = \frac{h}{2}\, a_i(t)$$

$$k_{2,x} = h\, \left(v_i(t) + \frac{h}{2} a_i(t) \right)$$
$$\cancel{k_{2,v} = h\, \frac{\mathrm{d}}{\mathrm{d}t} \left(v_i(t) + \frac{h}{2} a_i(t) \right)}$$

$$x_i(t+h) = x_i(t) + k_{2,x}$$
$$v_i(t+h) = v_i(t) + k_{2,v}$$

Remember that the acceleration in Newton eqs depends only on positions ($\boldsymbol{a}$ does not depend on $\boldsymbol{v}$)

$$k_{2,v} = h\, \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\Big|_{t+\frac{h}{2}} = h\, a_i(x_i(t+h/2), t+h/2)$$

Writing Euler explicitly

$$k_{2,v} = h\, a_i(x_i(t) + h/2\, v_i(t))$$

ODEs. Midpoint & the astrophysical N-body problem

$$k_{1,x} = \frac{h}{2}\,v_i(t)$$
$$k_{1,v} = \frac{h}{2}\,a_i(t)$$

$$k_{2,x} = h\,\left(v_i(t) + \frac{h}{2}a_i(t)\right)$$
$$k_{2,v} = h\,a_i(x_i(t) + h/2\,v_i(t))$$

$$x_i(t+h) = x_i(t) + k_{2,x}$$
$$v_i(t+h) = v_i(t) + k_{2,v}$$

$$k_{2,x} = h\,\left(v_i(t) + k_{1,v}\right)$$

$$k_{2,v} = h\,a_i(x_i(t) + k_{1,x})$$

ODEs. Midpoint & the astrophysical N-body problem

$$k_{1,x} = \frac{h}{2}\, v_i(t)$$
$$k_{1,v} = \frac{h}{2}\, a_i(t)$$

$$k_{2,x} = h\, \left[v_i(t) + k_{1,v}\right]$$
$$k_{2,v} = h\, a_i(x_i(t) + k_{1,x})$$

$$x_i(t+h) = x_i(t) + k_{2,x}$$
$$v_i(t+h) = v_i(t) + k_{2,v}$$

This is the most elegant form of the midpoint scheme for the N-body problem

ODEs. Midpoint & the astrophysical N-body problem

In practice,

$$k_{1,x} = \frac{h}{2}\, v_i(t)$$
$$k_{1,v} = \frac{h}{2}\, a_i(t)$$

$$a_i(t) = -G \sum_{j=1, j\neq i}^{n} \frac{m_j\,(x_i(t) - x_j(t))}{|x_i(t) - x_j(t)|^3}$$
$$k_{1,x} = \frac{h}{2}\, v_i(t)$$
$$k_{1,v} = \frac{h}{2}\, a_i(t)$$

$$k_{2,x} = h\, [v_i(t) + k_{1,v}]$$
$$k_{2,v} = h\, a_i(x_i(t) + k_{1,x})$$

$$x_i(t + h/2) = x_i(t) + k_{1,x}$$
$$a_i(t + h/2) = -G \sum_{j=1, j\neq i}^{n} \frac{m_j\,(x_i(t + h/2) - x_j(t + h/2))}{|x_i(t + h/2) - x_j(t + h/2)|^3}$$
$$k_{2,x} = h\,(v_i(t) + k_{1,v})$$
$$k_{2,v} = h\, a_i(t + h/2)$$

$$x_i(t + h) = x_i(t) + k_{2,x}$$
$$v_i(t + h) = v_i(t) + k_{2,v}$$

$$x_i(t + h) = x_i(t) + k_{2,x}$$
$$v_i(t + h) = v_i(t) + k_{2,v}$$

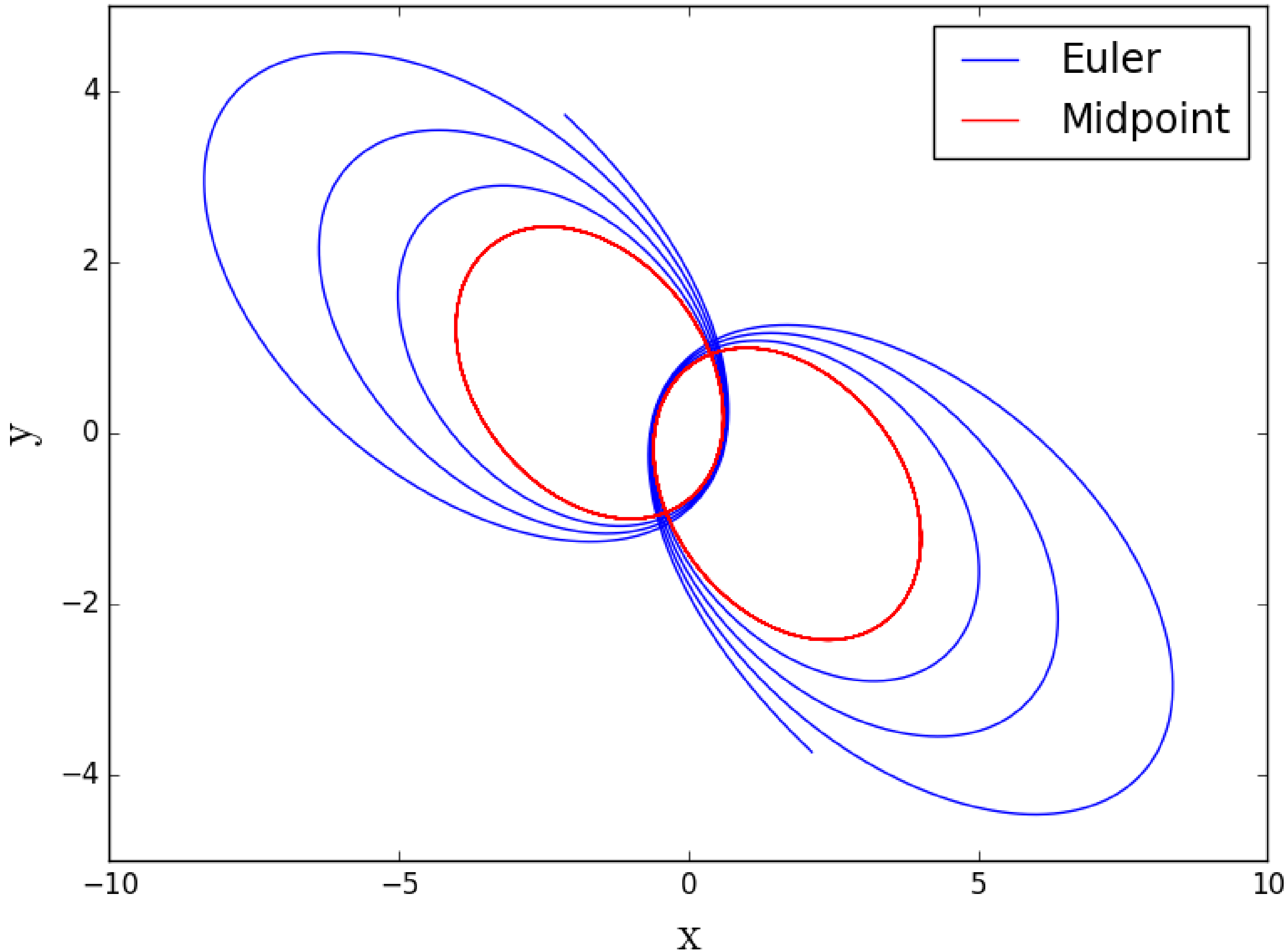
ODEs. EXERCISE on binary star with midpoint/ RK 4

EXERCISE:

Write a new script to implement the Midpoint method and/or the Runge-Kutta 4th order method to evolve a system of two points in two dimensions (xy plane) described in the pevious exercise.
Let us assume $t_0 = 0$, $t_{\text{fin}} = 300$ and $h = 0.01$. The result should look like the red line in Figure 42 (Midpoint and Runge-Kutta 4th order cannot be distinguished by eye in this case).

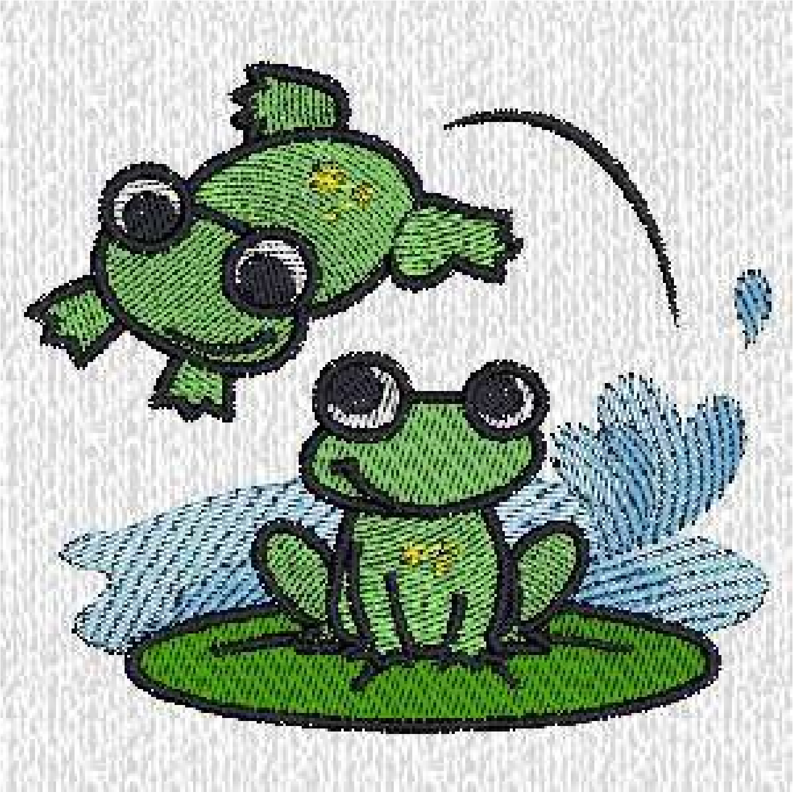
ODEs. EXERCISE on binary star with midpoint/ RK 4

Result of Euler is the blue line
Result of Midpoint is the red line



ODEs. Leapfrog scheme

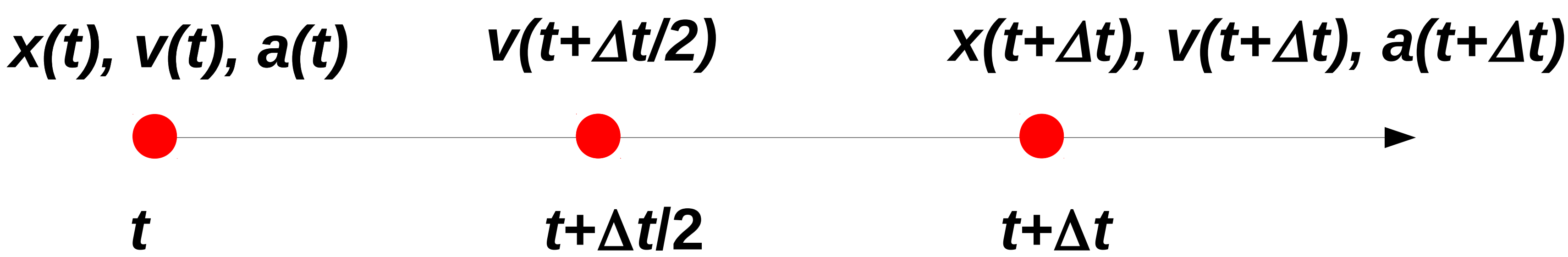
- a particular version of the midpoint method
- leapfrog play (Italian: la cavallina)
- similar to Euler's method but evaluated in between a time-step



ODEs. Leapfrog scheme

Most common version of leapfrog scheme is

Kick – Drift – Kick (KDK) algorithm



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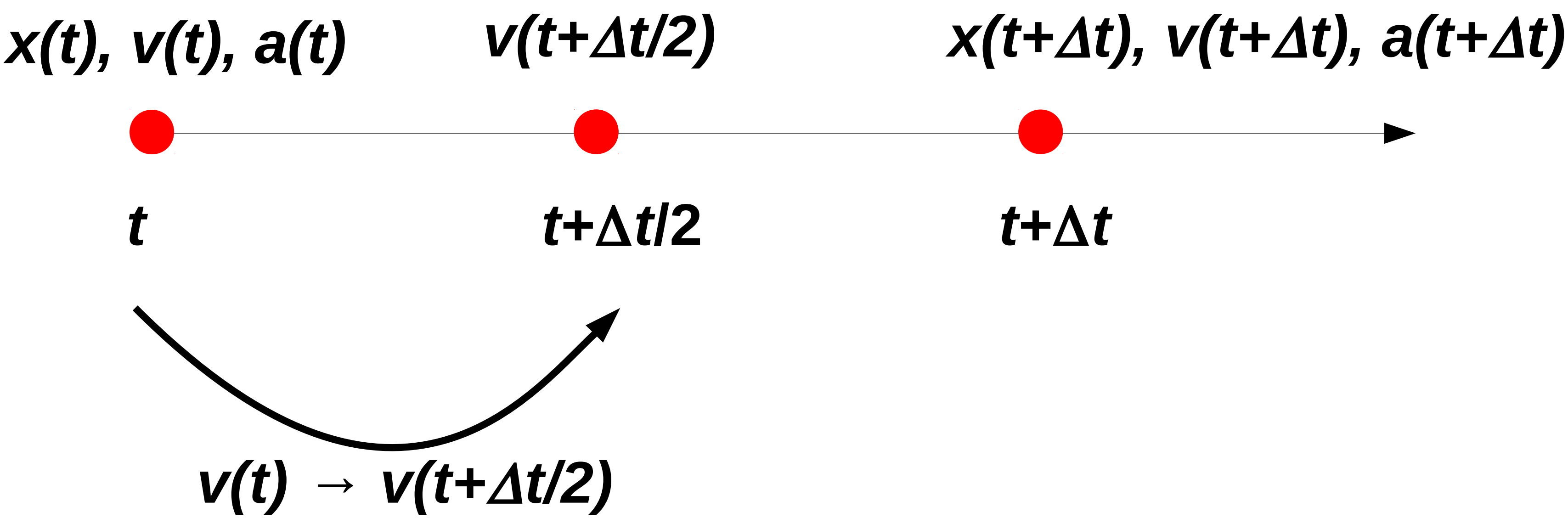

ODEs. Leapfrog scheme

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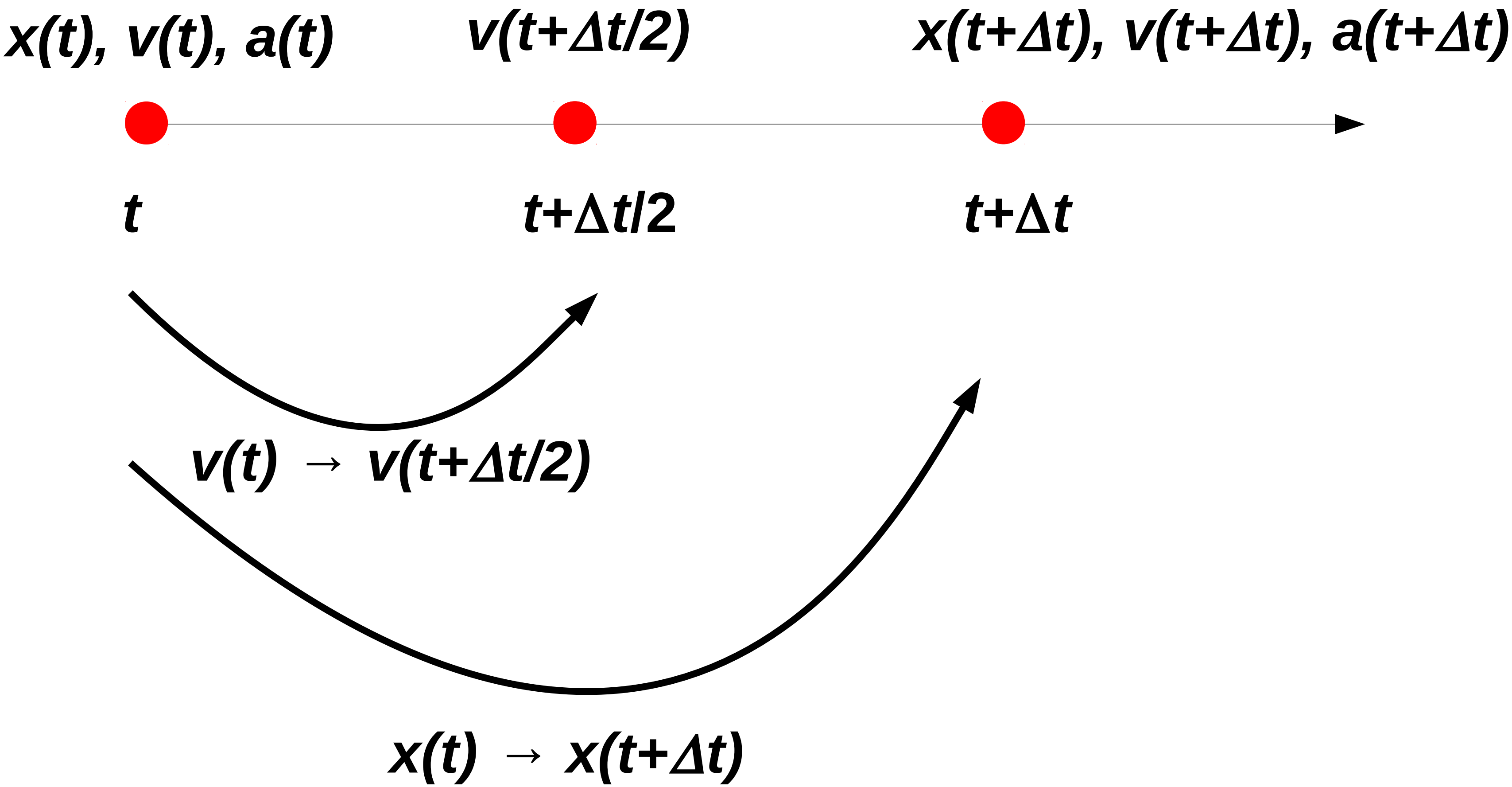

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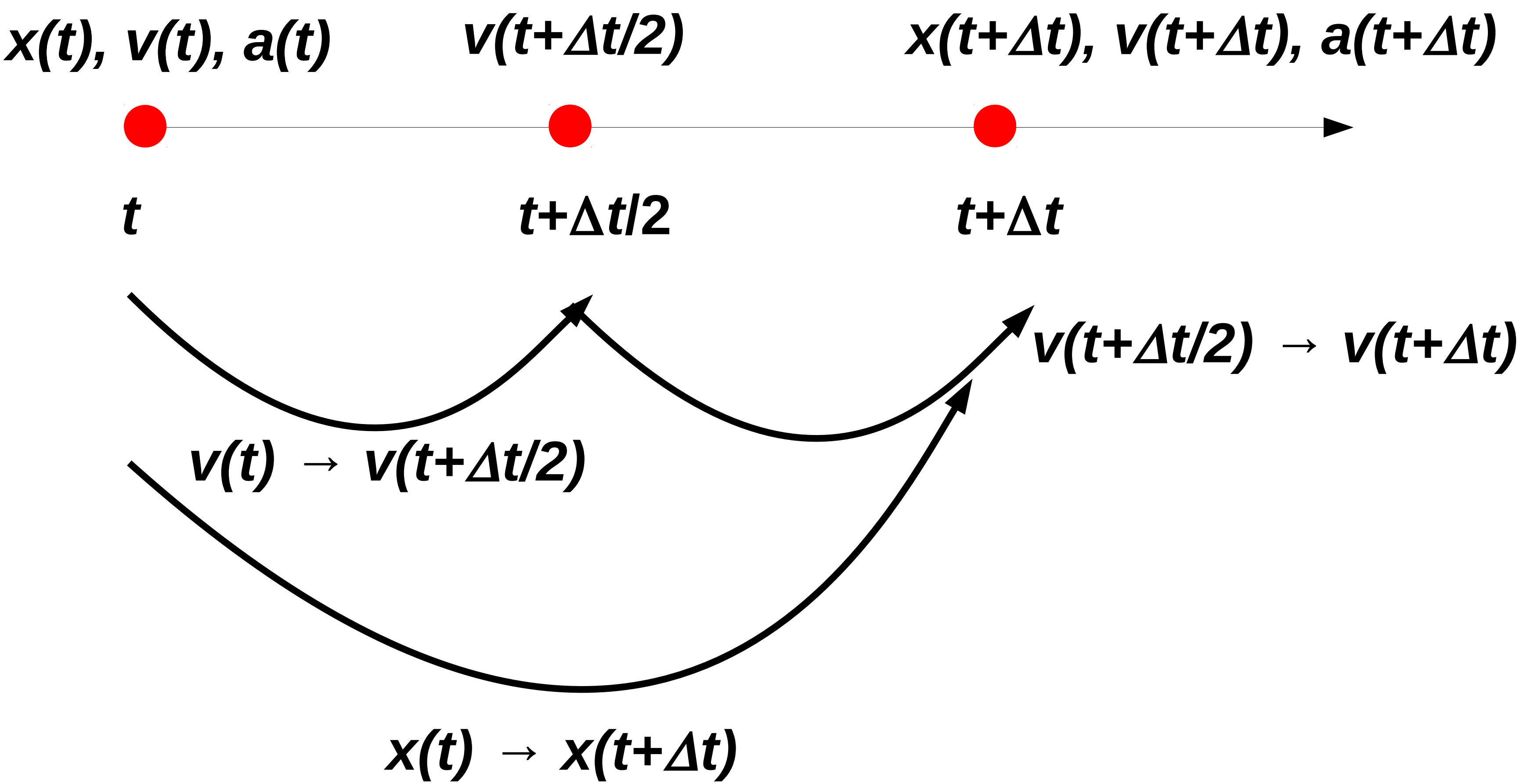

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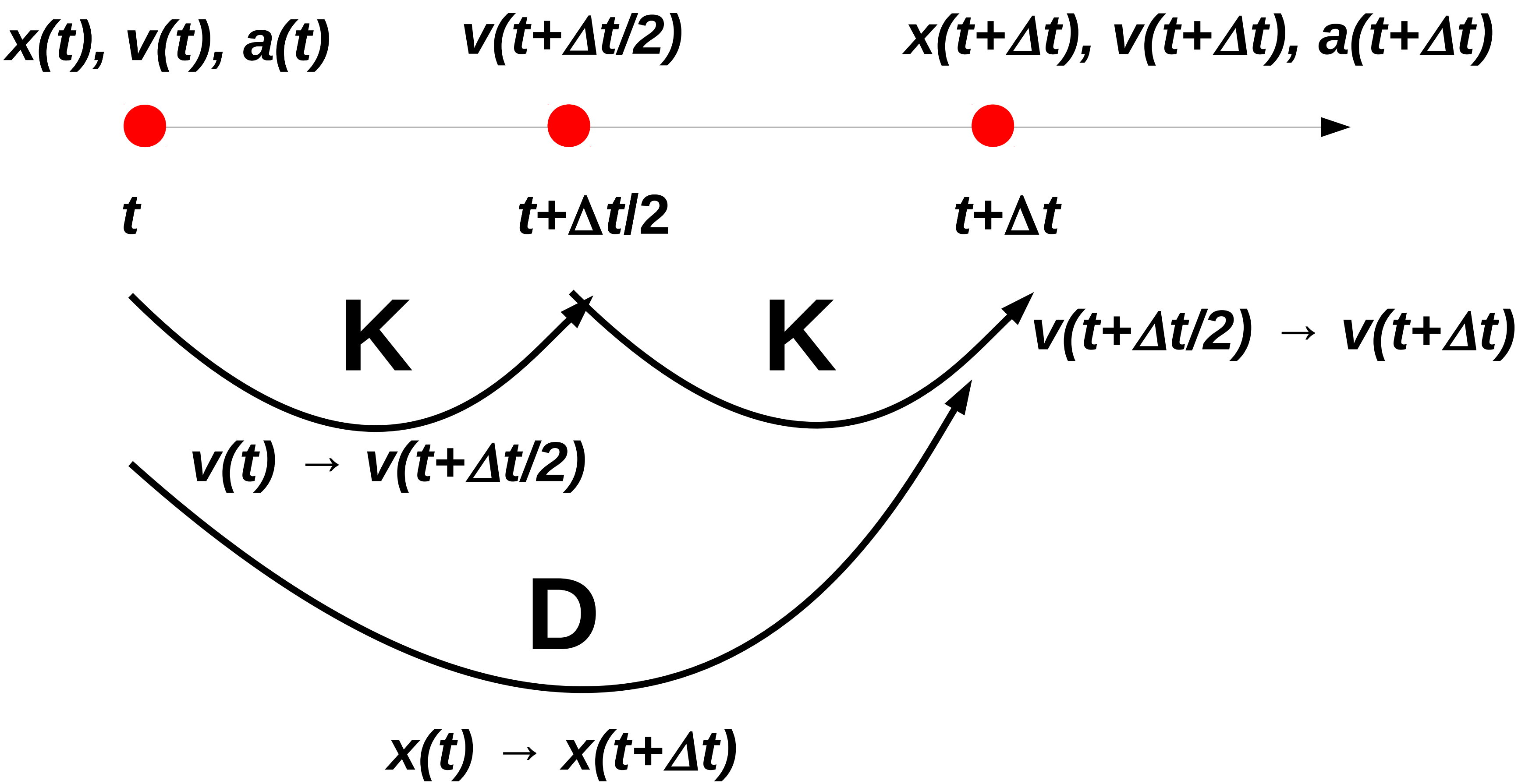
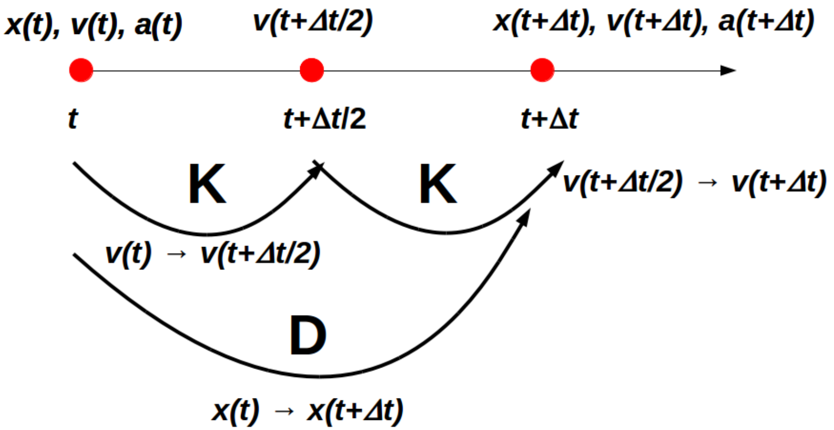


Kick + Drift + Kick (KDK) scheme

ODEs. Leapfrog scheme



Kick + Drift + Kick (KDK) scheme

Mathematically

$$\vec{a}_i(t) = -G \sum_{j=1,\, j\neq i}^{N} m_j \frac{\vec{x}_i(t) - \vec{x}_j(t)}{\left|\vec{x}_i(t) - \vec{x}_j(t)\right|^3},$$

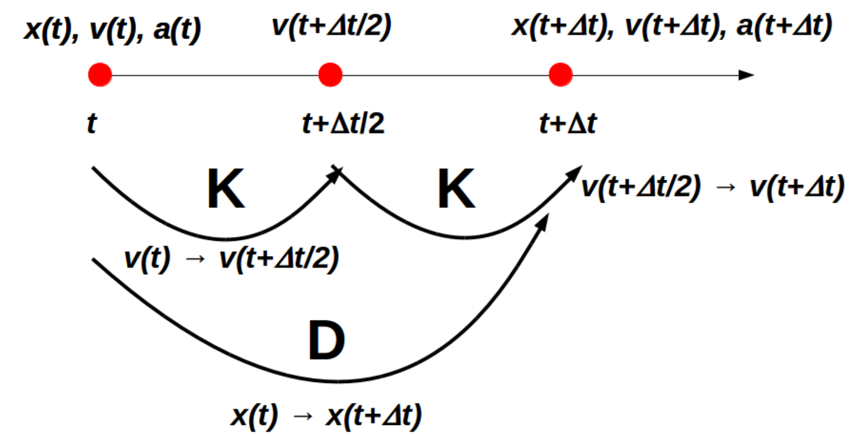
$$\vec{v}_i\left(t + \frac{h}{2}\right) = \vec{v}_i(t) + \frac{h}{2}\vec{a}_i(t)$$

$$\vec{x}_i(t+h) = \vec{x}_i(t) + h\,\vec{v}_i\left(t + \frac{h}{2}\right)$$

$$\vec{a}_i(t+h) = -G \sum_{j=1,\, j\neq i}^{N} m_j \frac{\vec{x}_i(t+h) - \vec{x}_j(t+h)}{\left|\vec{x}_i(t+h) - \vec{x}_j(t+h)\right|^3},$$

$$\vec{v}_i(t+h) = \vec{v}_i\left(t + \frac{h}{2}\right) + \frac{h}{2}\vec{a}_i(t+h)$$

ODEs. Leapfrog scheme



Kick + Drift + Kick (KDK) scheme

In more compact form:

$$\vec{a}_i(t) = -G \sum_{j=1,\, j\neq i}^{N} m_j \frac{\vec{x}_i(t) - \vec{x}_j(t)}{|\vec{x}_i(t) - \vec{x}_j(t)|^3},$$

$$\vec{x}_i(t+h) = \vec{x}_i(t) + h\,\vec{v}_i(t) + \frac{h^2}{2}\vec{a}_i(t)$$

$$\vec{a}_i(t+h) = -G \sum_{j=1,\, j\neq i}^{N} m_j \frac{\vec{x}_i(t+h) - \vec{x}_j(t+h)}{|\vec{x}_i(t+h) - \vec{x}_j(t+h)|^3},$$

$$\vec{v}_i(t+h) = \vec{v}_i(t) + \frac{h}{2}\left[\vec{a}_i(t) + \vec{a}_i(t+h)\right]$$

ODEs. Leapfrog scheme

- second-order scheme (barely)

- surprisingly accurate

- alternative version: drift-kick-drift (DKD) leapfrog scheme,
  in which position is evaluated at the midpoint (t+h/2),
  then velocity is advanced to the end and
  finally position is recalculated to the end of the step.
  You can try to derive this one by yourself

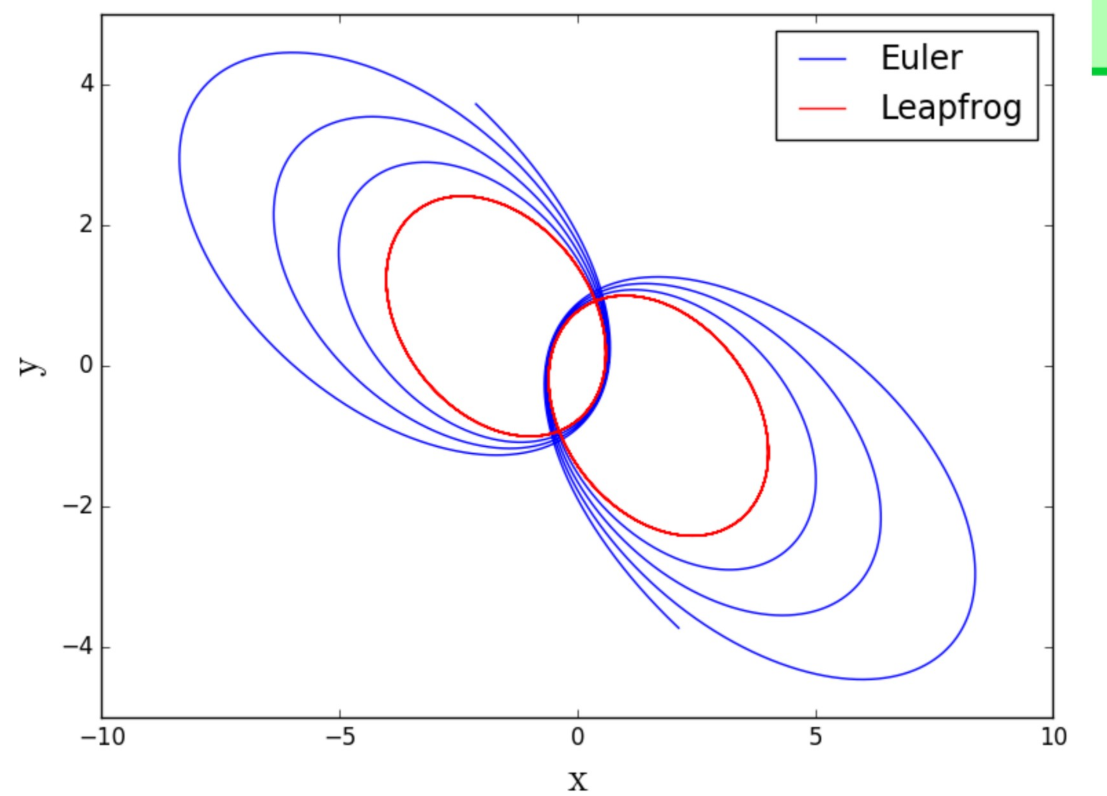
- (unlike Runge-Kutta) leapfrog is time-reversal symmetric
→ the error on energy conservation does not grow with time

NOTE: A nice way to estimate how well an integrator of celestial dynamics works is to calculate the conservation of total energy and total angular momentum as a function of time during the integration

ODEs. Exercise on binary star with leapfrog

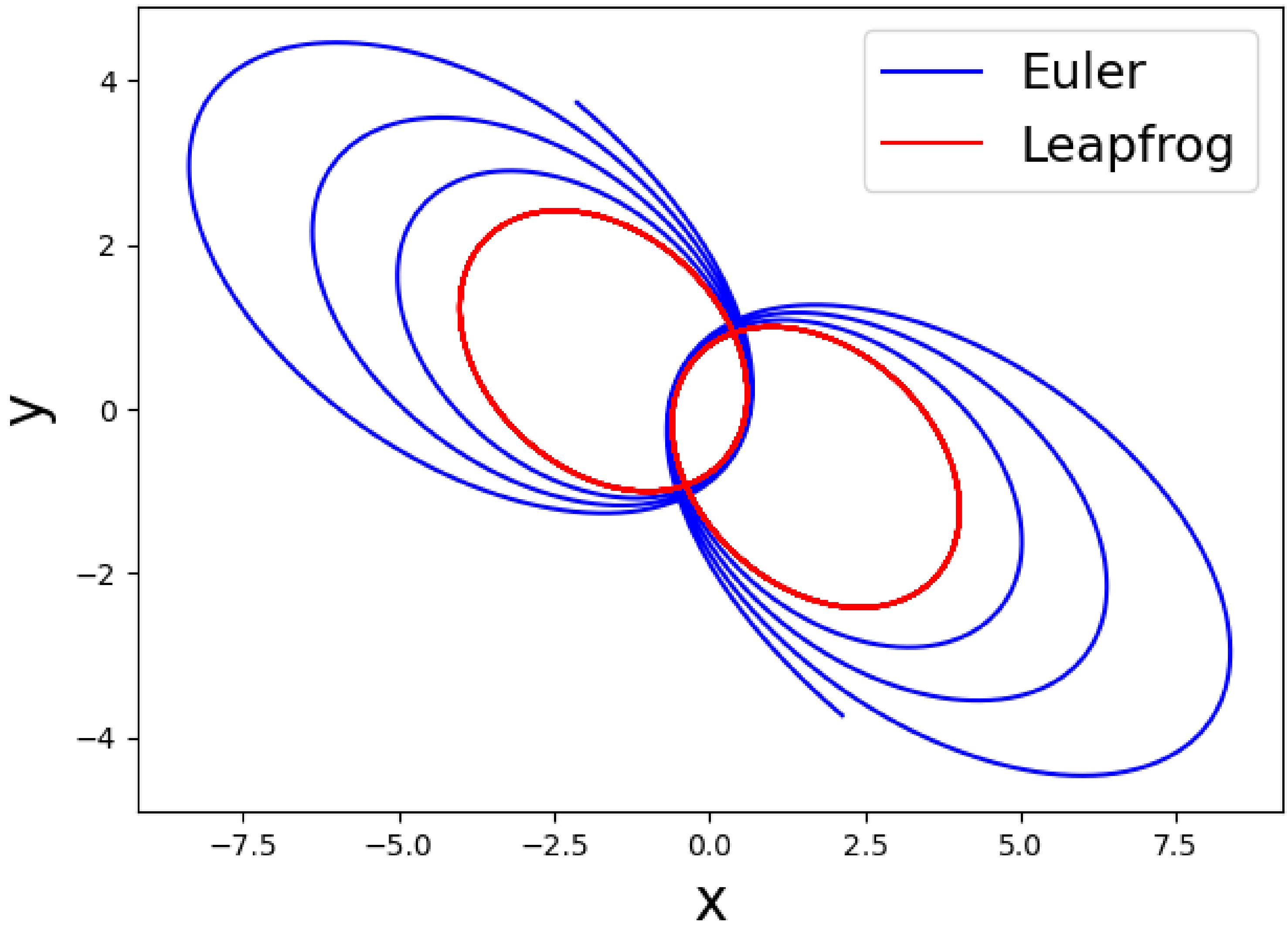
EXERCISE:

Write a code to implement the leapfrog scheme. Integrate the binary star in the previous exercises with the leapfrog scheme. Compare Euler's method with the leapfrog scheme. Choose $t_0 = 0$, $t_{\text{fin}} = 300$ and $h = 0.01$. The result should look like Figure 44. Leapfrog is much better, isn't it?



ODEs. Exercise on binary star with leapfrog

Euler versus Leapfrog



Same initial conditions: integration of a Keplerian binary

ODEs. Euler vs Leapfrog: a simple test

Energy of an N-body system

$$E = \sum_{i=1}^{N} \frac{1}{2} m_i \, v_i^2 - G \sum_{i=1}^{N} \sum_{j>i}^{N} \frac{m_i \, m_j}{|r_i - r_j|}$$

For a binary star, energy in the center of mass of the system

$$E = \frac{1}{2} \frac{m_1 \, m_2}{(m_1 + m_2)} \left| v_1 - v_2 \right|^2 - G \, \frac{m_1 \, m_2}{|r_1 - r_2|}$$

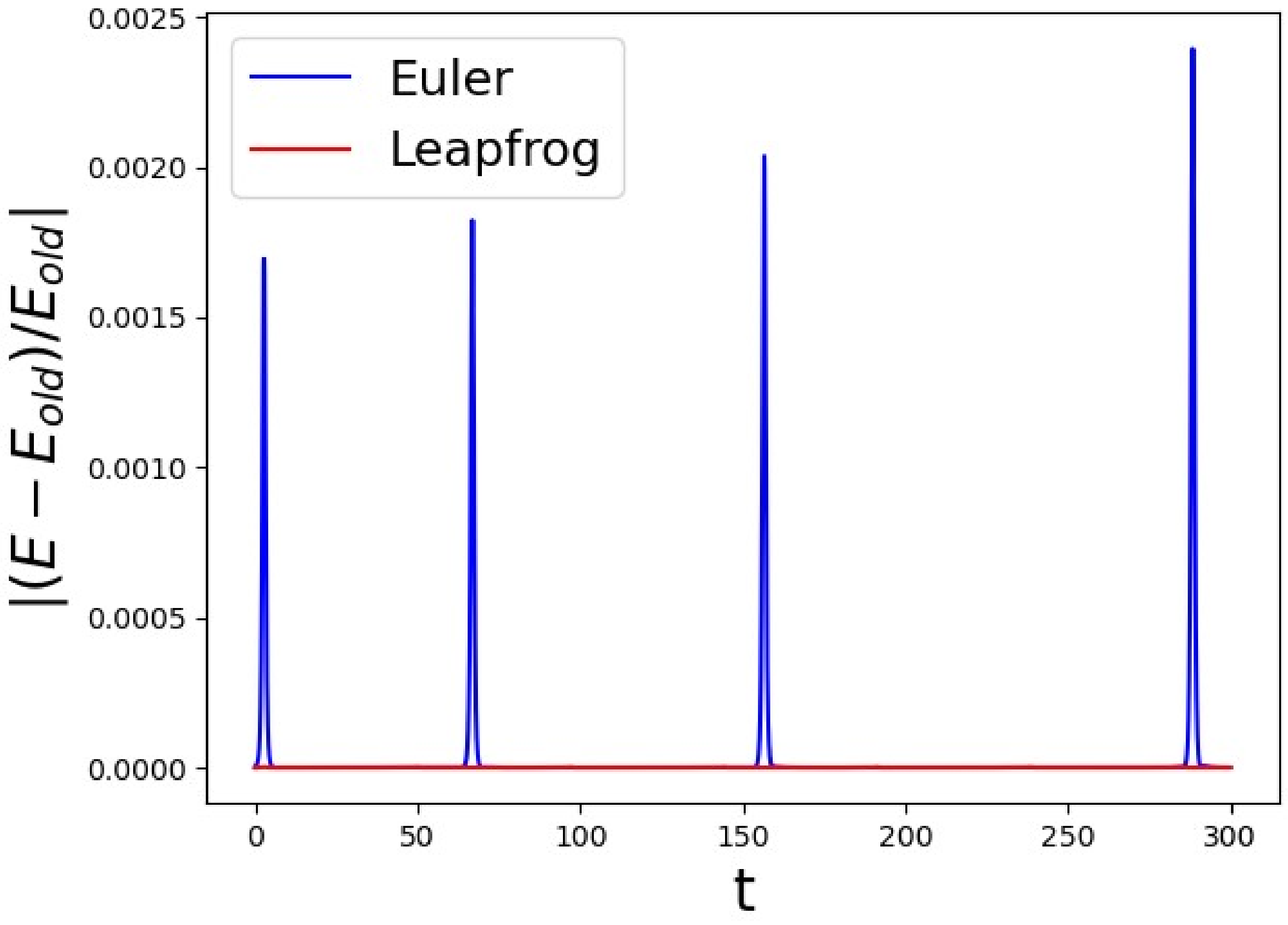
Modulus of angular momentum

$$L = \sum_{i=1}^{N} \left| m_i \, v_i \times r_i \right|$$

If energy and angular momentum are supposed to be conserved in the system we simulate,
the level of energy / angular momentum conservation between previous and next step is a good indicator of the accuracy of the integrator

ODEs. Euler vs Leapfrog: a simple test

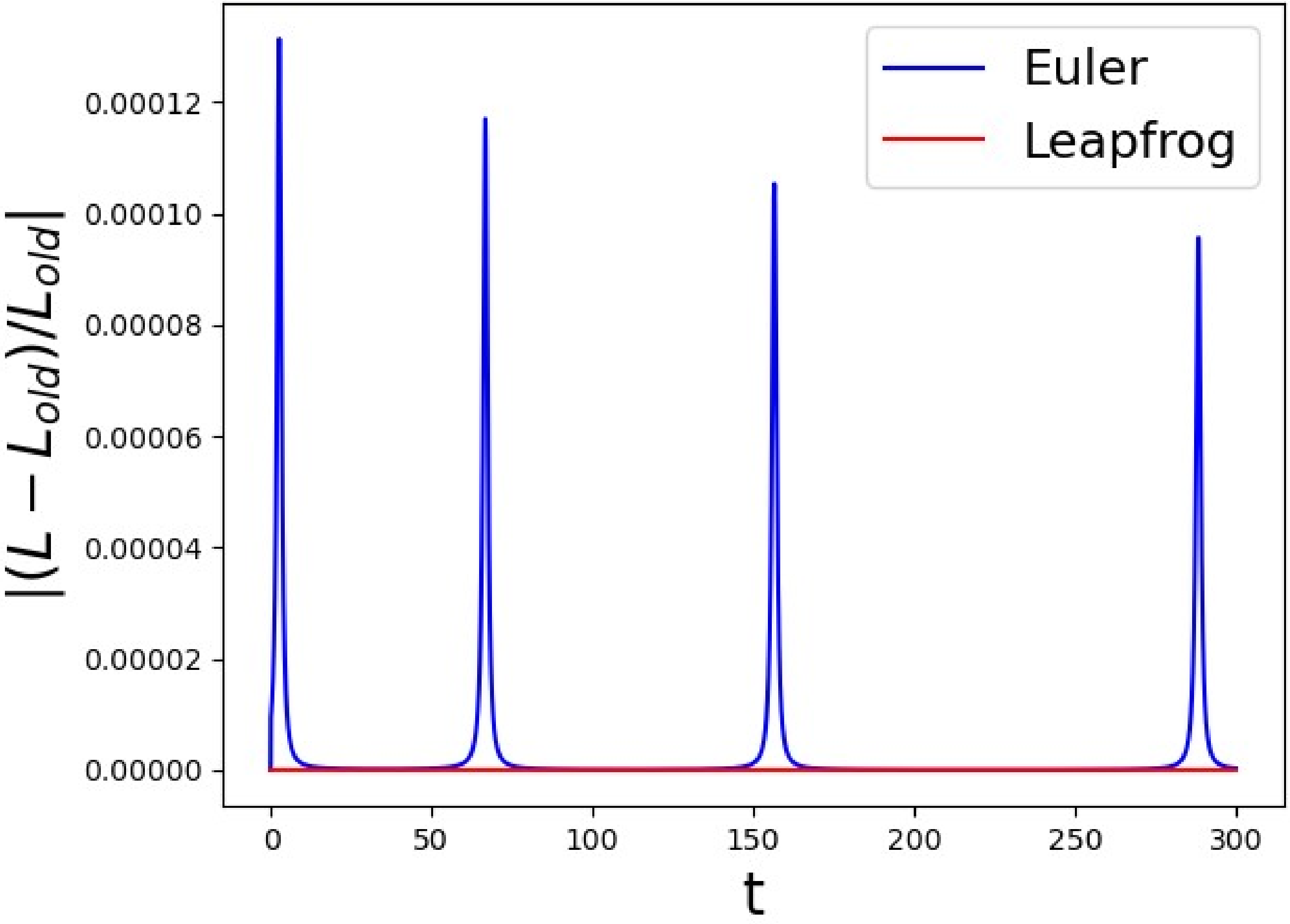
Energy conservation test



Leapfrog Delta E / E ~ 2.1e-06 Euler Delta E/E = 0.0024

ODEs. Euler vs Leapfrog: a simple test

Angular momentum conservation test



Leapfrog Delta L / L ~ 5.6e-16 Euler Delta L/L = 0.00013