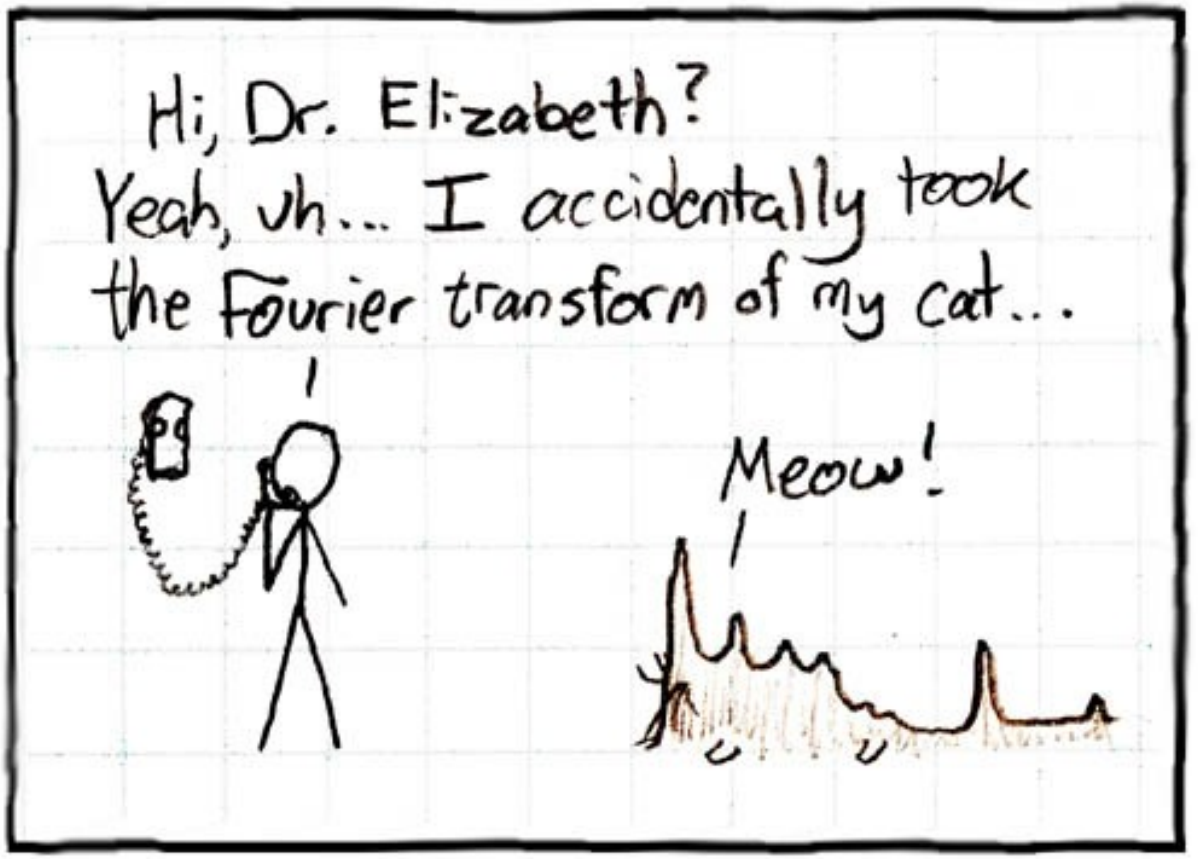
# Numerical Methods for Astrophysics:

## FOURIER TRANSFORMS

**Michela Mapelli**

# Fourier transforms. Concept

**Fourier transforms are ubiquitous in physics, astrophysics and in every-day life (e.g. the mp3 and jpg file format)**




**Very nice Fourier summary here (plus, you learn how to take the Fourier transform of your cat):**

**https://www.youtube.com/watch?v=UGRhqZ2ooIw**

# Fourier transforms. Concept

**A periodic function f(x)** defined on a finite interval 0, L
can be written as Fourier series provided that it is bounded and has at most a finite number of discontinuities and extrema

**EVEN (i.e. symmetric)** function about the midpoint x=L/2

$$f(x) = \sum_{k=0}^{\infty} \alpha_k \cos\left(\frac{2\,\pi\,k\,x}{L}\right)$$

**ODD (i.e. antisymmetric)** function about the midpoint

$$f(x) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{2\,\pi\,k\,x}{L}\right)$$

**GENERAL PERIODIC FUNCTION**

$$f(x) = \sum_{k=0}^{\infty} \alpha_k \cos\left(\frac{2\,\pi\,k\,x}{L}\right) + \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{2\,\pi\,k\,x}{L}\right)$$

# Fourier transforms. Concept

Using the complex notation:
(from Euler's formulas)

$$\cos\theta = \frac{1}{2}\left(\exp\left(-i\,\theta\right) + \exp\left(i\,\theta\right)\right)$$

$$\sin\theta = \frac{1}{2}\,i\,\left(\exp\left(-i\,\theta\right) - \exp\left(i\,\theta\right)\right)$$

we get

$$f(x) = \frac{1}{2}\sum_{k=0}^{\infty}\alpha_k\left[\exp\left(-i\,\frac{2\,\pi\,k\,x}{L}\right) + \exp\left(i\,\frac{2\,\pi\,k\,x}{L}\right)\right]$$
$$+\frac{i}{2}\sum_{k=1}^{\infty}\beta_k\left[\exp\left(-i\,\frac{2\,\pi\,k\,x}{L}\right) - \exp\left(i\,\frac{2\,\pi\,k\,x}{L}\right)\right]$$

Collecting terms and generalizing for whatever k

$$f(x) = \sum_{k=-\infty}^{\infty}\gamma_k\,\exp\left(i\,\frac{2\,\pi\,k\,x}{L}\right)$$

where

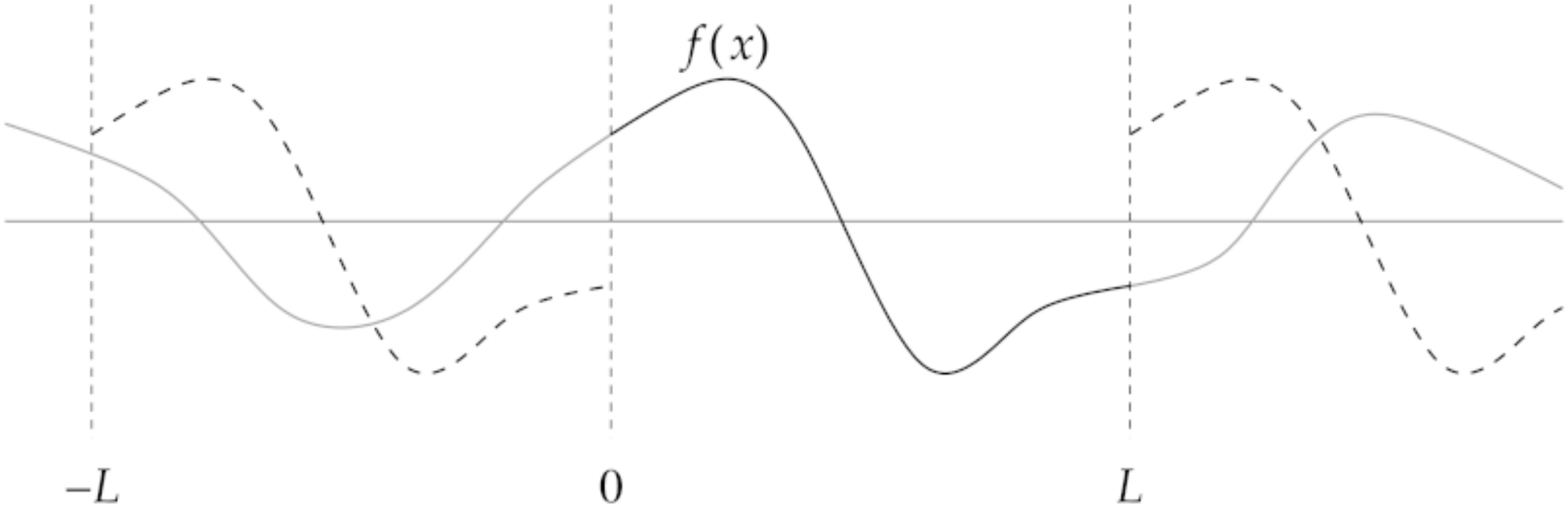
$$\gamma_k = \begin{cases} \frac{1}{2}\left(\alpha_{-k} + i\,\beta_{-k}\right) & \text{if } k < 0 \\ \alpha_0 & \text{if } k = 0 \\ \frac{1}{2}\left(\alpha_k - i\,\beta_k\right) & \text{if } k > 0 \end{cases}$$

# Fourier transforms. Concept

ONLY FOR PERIODIC FUNCTIONS

MOST FUNCTIONS are NOT PERIODIC

**If we are interested only in a portion of a non-periodic function in the interval 0, L, it is always possible to take that portion and just repeat it to create a periodic function**



# Fourier transforms. Concept

**The coefficients $\gamma_k$ are COMPLEX numbers**

**To calculate them** take $$f(x) = \sum_{k=-\infty}^{\infty} \gamma_k \exp\left(i \frac{2\,\pi\,k\,x}{L}\right)$$

and integrate the two terms with a “trick”

i.e. multiply both sides by $$\exp\left(-i\frac{2\,\pi k\,x}{L}\right)$$

$$\int_0^L f(x) \exp\left(-i \frac{2\,\pi\,k\,x}{L}\right) \mathrm{d}x = \sum_{k'=-\infty}^{\infty} \gamma_{k'} \int_0^L \exp\left(i\frac{2\,\pi\,(k'-k)\,x}{L}\right) \mathrm{d}x$$

if k == k’ then $$\int_0^L \exp\left(i\frac{2\,\pi\,(k'-k)\,x}{L}\right) \mathrm{d}x = \int_0^L e^0\,\mathrm{d}x = L$$

if k != k’ then $$\int_0^L \exp\left(i\frac{2\,\pi\,(k'-k)\,x}{L}\right) \mathrm{d}x = 0$$

# Fourier transforms. Concept

if k != k' then $$\int_0^L \exp\left(i\frac{2\,\pi\,(k'-k)\,x}{L}\right)\mathrm{d}x = 0$$

**WHY?**

You know from your analysis courses (otherwise ask me) that

$$\int e^{z\,x}\,\mathrm{d}x = \frac{e^{z\,x}}{z} \qquad\qquad z = a + i\,b$$

here $$z = \frac{i\,2\,\pi\,(k'-k)}{L}$$

Hence $$\int_0^L \exp\left(i\,\frac{2\,\pi\,(k'-k)\,x}{L}\right)\mathrm{d}x = \frac{L}{i\,2\,\pi\,(k'-k)}\left[\exp\left(i\,\frac{2\,\pi\,(k'-k)\,x}{L}\right)\right]_0^L =$$

$$\frac{L}{i\,2\,\pi\,(k'-k)}\left\{\exp\left[i\,2\,\pi\,(k'-k)\right]-1\right\} = 0$$

because $\exp\left(i\,2\,\pi n\right) = 1$ if *n* integer

# Fourier transforms. Concept

Hence the result of $$\int_0^L f(x)\,\exp\left(-i\,\frac{2\,\pi\,k\,x}{L}\right)\mathrm{d}x = \sum_{k'=-\infty}^{\infty}\gamma_{k'}\int_0^L \exp\left(i\frac{2\,\pi\,(k'-k)\,x}{L}\right)\mathrm{d}x$$

is $$\int_0^L f(x)\,\exp\left(-i\,\frac{2\,\pi\,k\,x}{L}\right)\mathrm{d}x = \gamma_k\,L$$

Or, re-writing in terms of $\gamma k$

$$\gamma_k = \frac{1}{L}\int_0^L f(x)\,\exp\left(-i\,\frac{2\,\pi\,k\,x}{L}\right)\mathrm{d}x$$

**Given f(x) we know the $\gamma k$ and**
**vice versa from the $\gamma k$ we can reconstruct f(x)**

# Fourier transforms. Discrete Fourier Transform (DFT)

**1. The integral** $\gamma_k = \frac{1}{L}\int_0^L f(x)\,\exp\left(-i\,\frac{2\,\pi\,k\,x}{L}\right)\mathrm{d}x$

**can be calculated analytically only for some f(x)**

**2. in many cases we do not know f(x) but just a sample $y_n = f(x_n)$**
**e.g. we have experimental data for $y_n$, not a functional form**

**→ it is important to calculate the $\gamma_k$ numerically**

**USE TRAPEZOIDAL RULE**

For $N$ slices of width $h = L/N$

$$\gamma_k = \frac{1}{L}\frac{L}{N}\left[\frac{1}{2}f(0) + \frac{1}{2}f(L) + \sum_{n=1}^{N-1} f(x_n)\,\exp\left(-i\,\frac{2\,\pi\,k\,x_n}{L}\right)\right]$$

where $x_n = \frac{n}{N}\,L$

Using f(0) = f(L) for periodicity, we simplify the sum

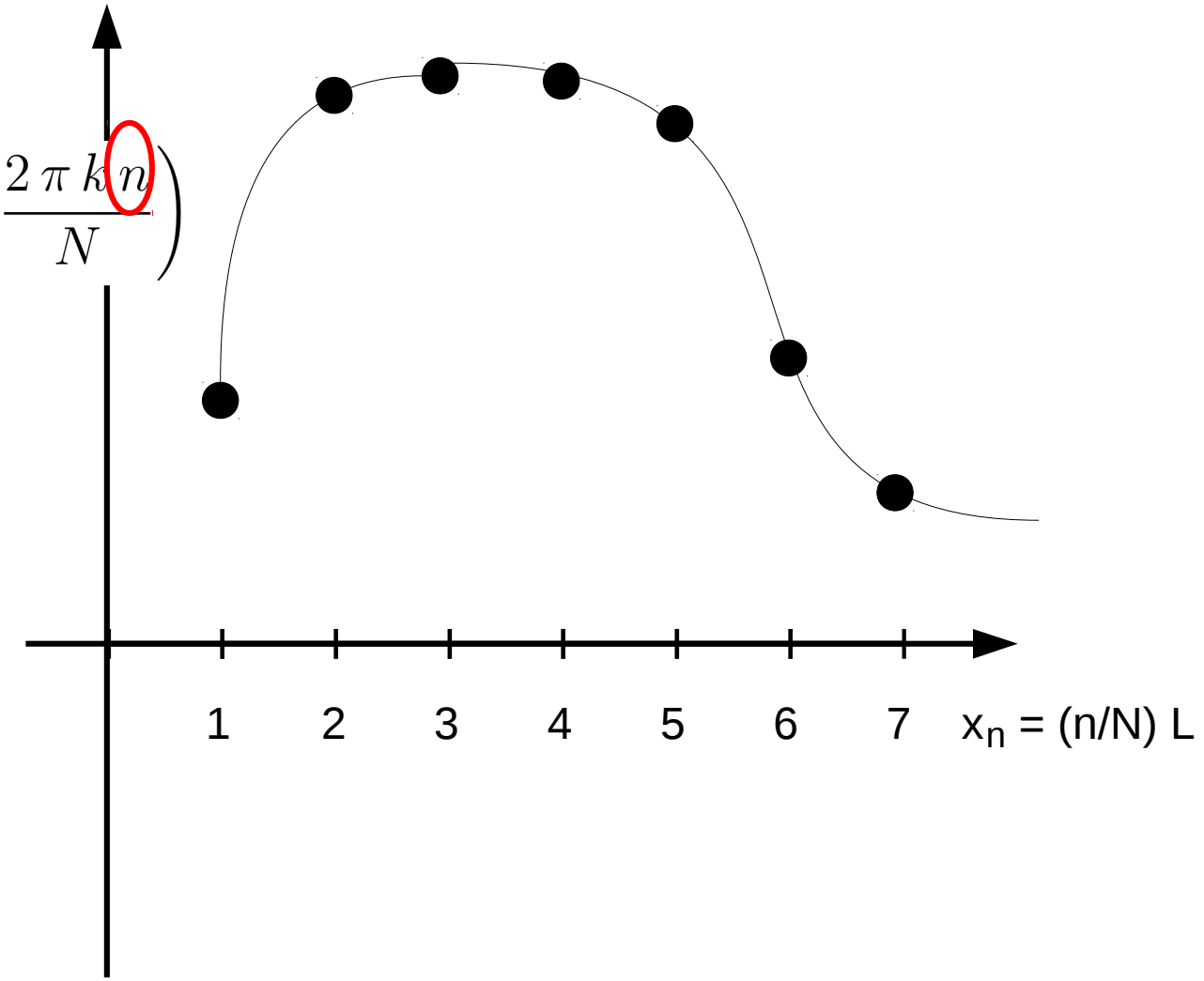
$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} f(x_n)\,\exp\left(-i\,\frac{2\,\pi\,k\,x_n}{L}\right)$$

# Fourier transforms. Discrete Fourier Transform (DFT)

$$\gamma_k = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \exp\left(-i \frac{2\,\pi\,k\,x_n}{L}\right)$$

**If function f(x) is equally sampled over the interval,**
**i.e. the $x_n$ are equally spaced, we simplify $y_n = f(x_n)$ and $x_n = (n / N)\, L$**

$$\gamma_k = \frac{1}{N} \sum_{n=0}^{N-1} y_n \exp\left(-i \frac{2\,\pi\,k\,n}{N}\right)$$



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$$\gamma_k = \frac{1}{N} \sum_{n=0}^{N-1} y_n \exp\left(-i \frac{2\,\pi\,k\,n}{N}\right)$$

**By convention, we use the following formula**

$$c_k = \sum_{n=0}^{N-1} y_n \exp\left(-i \frac{2\,\pi\,k\,n}{N}\right)$$

**where $c_k := \gamma_k N$**

**discrete Fourier transform (DFT) formula**

# Fourier transforms. Discrete Fourier Transform (DFT)

We derived this with trapezoidal rule: simplified method

However, with few mathematical steps we can show that
the DFT is, in a certain sense, exact:

with some math tricks we see that

$$y_n = \frac{1}{N} \sum_{k=0}^{N-1} c_k \, \exp\left( i \, \frac{2 \, \pi \, k \, n}{N} \right)$$

**Inverse discrete Fourier transform (inverse DFT)**

**given the coefficients $c_k$ we can recover the values of the samples $y_n$ that they come from exactly (except for rounding errors).**

# Fourier transforms. Discrete Fourier Transform (DFT)

**Even though Fourier coefficients are only approximate, they are actually exact in the sense that**
**we can completely recover the original samples from them**

**We can freely move back and fourth from the coefficients to the samples and vice versa WITHOUT LOSING ANY INFORMATION in our data by using**

$$c_k = \sum_{n=0}^{N-1} y_n \exp\left(-i \frac{2\pi k n}{N}\right)$$

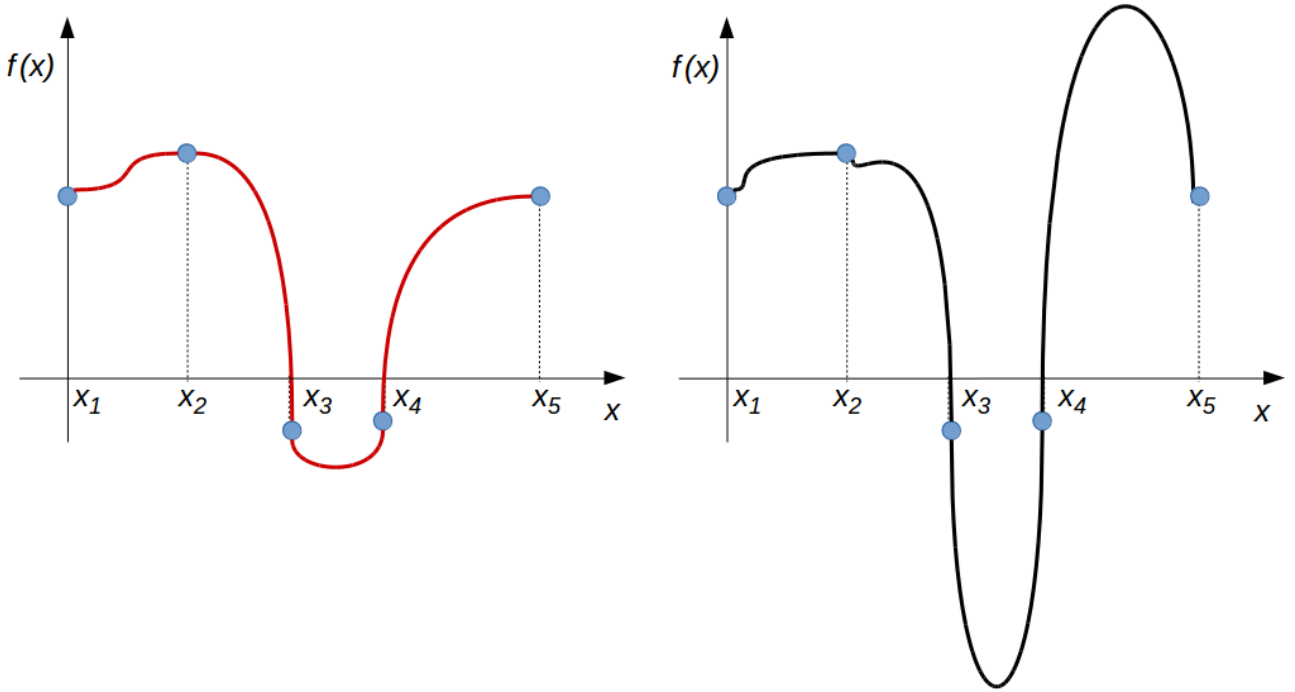
$$y_n = \frac{1}{N} \sum_{k=0}^{N-1} c_k \exp\left(i \frac{2\pi k n}{N}\right)$$

Notice that we only need the Fourier coefficients $c_k$ up to k = N – 1

to recover the samples, so 0<= k < N : no need to calculate an infinite sum

# Fourier transforms. Discrete Fourier Transform (DFT)

The inverse DFT equation only gives us the sample values $y_n = f(x_n)$:
it tells us nothing about the value of f(x) between sample points.




**Any two functions that have the same values $y_n$ at the sample points $x_n$ will have the same DFT,**
**no matter what they do between the sample points.**

**We cannot do better than this when we do not know the functional form of the function, but we have only a sample of discrete experimental data.**

# Fourier transforms. Discrete Fourier Transform (DFT)

**If REAL FUNCTIONS, further simplifications.**

Suppose all the $y_n$ are real and consider the value $c_k$ for *N/2 <k <N,*

which we write as *k = N – r* where *1<= r <= N/2*

$$c_{N-r} = \sum_{n=0}^{N-1} y_n \exp\left(-i\,\frac{2\,\pi\,(N-r)\,n}{N}\right)$$

$$= \sum_{n=0}^{N-1} y_n \exp\left(-i\,2\,\pi\,n\right) \exp\left(i\,\frac{2\,\pi\,r\,n}{N}\right)$$

$$= \sum_{n=0}^{N-1} y_n \exp\left(i\,\frac{2\,\pi\,r\,n}{N}\right) = c_r^*$$

where c* is the complex conjugate of c

This implies $c_{N-1} = c_1^*$

$c_{N-2} = c_2^*$

$c_{N-3} = c_3^*$

if f(x) is real, we calculate only the coefficients $c_k$ with *0 <= k <= N/2*

If N is even, we must calculate N/2+1 coefficients
if N is odd, we must calculate (N+1)/2

In python $N//2+1$

# Fourier transforms. Fast Fourier Transform (FFT)

**Computational cost of the DFT method: $N^2$,**
because we calculate *N/2+1* or (N+1)/2 coefficients and
for each coefficient we loop over *N* sampled points

→ **Scaling as $N\ (N//2+1) \sim N^2$**

**Fast Fourier transform (FFT) is faster**

Discovered in 1805 by Carl F. Gauss
Re-discovered independently in 1965 by James Cooley and John Tukey

**Assumes that the samples are a power of two, hence $N = 2^m$**
**with m integer.**

# Fourier transforms. Fast Fourier Transform (FFT)

**Divide the terms of the sum in the DFT formula into two equally sized groups.**

**- First group consists of the terms with *n* even, i.e. the terms with *n = 2 r* with *r = 0, 1,...N/2 – 1***

$$E_k = \sum_{r=0}^{\frac{N}{2}-1} y_{2\,r}\, \exp\left(-i\,\frac{2\,\pi k\,(2\,r)}{N}\right) = \sum_{r=0}^{\frac{N}{2}-1} y_{2\,r}\, \exp\left(-i\,\frac{2\,\pi k\,r}{\frac{1}{2}\,N}\right)$$

**is a Fourier transform of *N/2* samples**

**- Second group consists of the terms with *n* odd, i.e. the terms with *n = 2 r + 1***

$$\sum_{r=0}^{\frac{N}{2}-1} y_{2\,r+1}\, \exp\left(-i\,\frac{2\,\pi k\,(2\,r+1)}{N}\right) = \exp\left(-i\,\frac{2\,\pi k}{N}\right) \sum_{r=0}^{\frac{N}{2}-1} y_{2\,r+1}\, \exp\left(-i\,\frac{2\,\pi k\,r}{\frac{1}{2}\,N}\right) = \exp\left(-i\,\frac{2\,\pi k}{N}\right) O_k$$

**is another Fourier transform of *N/2* samples**

$$\longrightarrow \qquad c_k = E_k + \exp\left(-i\,\frac{2\,\pi k}{N}\right) O_k$$

# Fourier transforms. Fast Fourier Transform (FFT)

$$c_k = E_k + \exp\left(-i\,\frac{2\,\pi k}{N}\right) O_k$$

**The $c_k$ can be written as sum of $E_k$ and $O_k$,**
**both DFTs of f(x) but with half as many points**

**plus an extra-factor (TWIDDLE FACTOR):** $\exp\left(-i\,\frac{2\,\pi k}{N}\right)$

Repeat the splitting process onto each of the two Fourier transforms
Each of them can be divided in its even and its odd terms

**We repeat the splitting, until eventually we get to the point where each transform is the transform of just a single sample:**

$$c_0 = \sum_{n=0}^{0} y_n \, \exp\left(0\right) = y_0$$

# Fourier transforms. Fast Fourier Transform (FFT)

**The actual calculation of the FFT is the REVERSE of this reasoning:**

**We start from the individual samples and we combine them in pairs,**

**then we combine the pairs into fours,**

**the fours into eights, and so on,**

**creating larger and larger Fourier transforms, until we have reconstructed the full transform of the complete set of samples.**

## VERY FAST:

First round of the calculation: *N* samples,
Second round: *N/2* transforms with 2 coefficients each
Third round: *N/4* transforms with 4 coefficients each

→ ***N* operations per each level and *m* levels, where $m = \log_2(N)$,**

→ **FFT method scales as $N \log_2(N)$**

**DFT method scales as $N^2$**

# Fourier transforms. FFT in python

**module for FFTs in numpy: numpy.fft**

**1. numpy.fft.rfft:** FFT for real values

```
from numpy.fft import rfft
c=rfft(y)
```

returns N//2+1 values

**2. numpy.fft.irfft:** inverse FFT for real values

```
from numpy.fft import rfft, irfft
c=rfft(y)
y2 =ifft(c)
```

returns N values

**3. numpy.fft.rfft2:** FFT for real values in 2D

**4. numpy.fft.irfft2:** inverse FFT for real values in 2D

# Fourier transforms. FFT in python

**module for FFTs in numpy: numpy.fft**

**5. numpy.fft.fft:** FFT for complex values

**6. numpy.fft.ifft:** inverse FFT for complex values

**7. numpy.fft.fft2:** FFT for complex values in 2D

**8. numpy.fft.ifft2:** inverse FFT for complex values in 2D

# Fourier transforms. Physical interpretation

**Fourier transform represents a function via a set of real or complex sinusoidal waves.**

**Each term in the sum is a single wave with its own well-defined frequency $\nu = k/N$ and period $T = 1 / \nu$**

If f(x) is a function in space, then spatial frequencies,
if f(x) is a function in time, then temporal frequencies.

Saying that any function can be expressed as a Fourier transform is equivalent to saying that
**any function can be represented as a sum of waves of given frequencies**

**and the coefficients of the Fourier transform tell us how much power is associated with each frequency, i.e. how big is the contribution of each frequency to the sum.**

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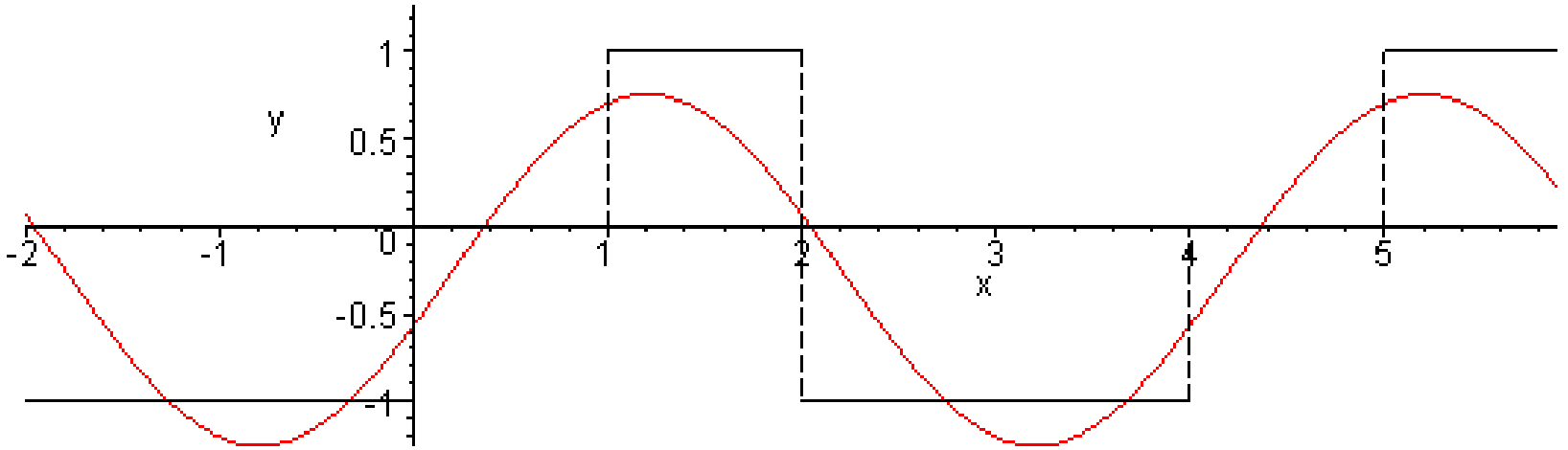
**and the coefficients of the Fourier transform tell us**
**how much power is associated with each frequency,**
**i.e. how big is the contribution of each frequency to the sum.**

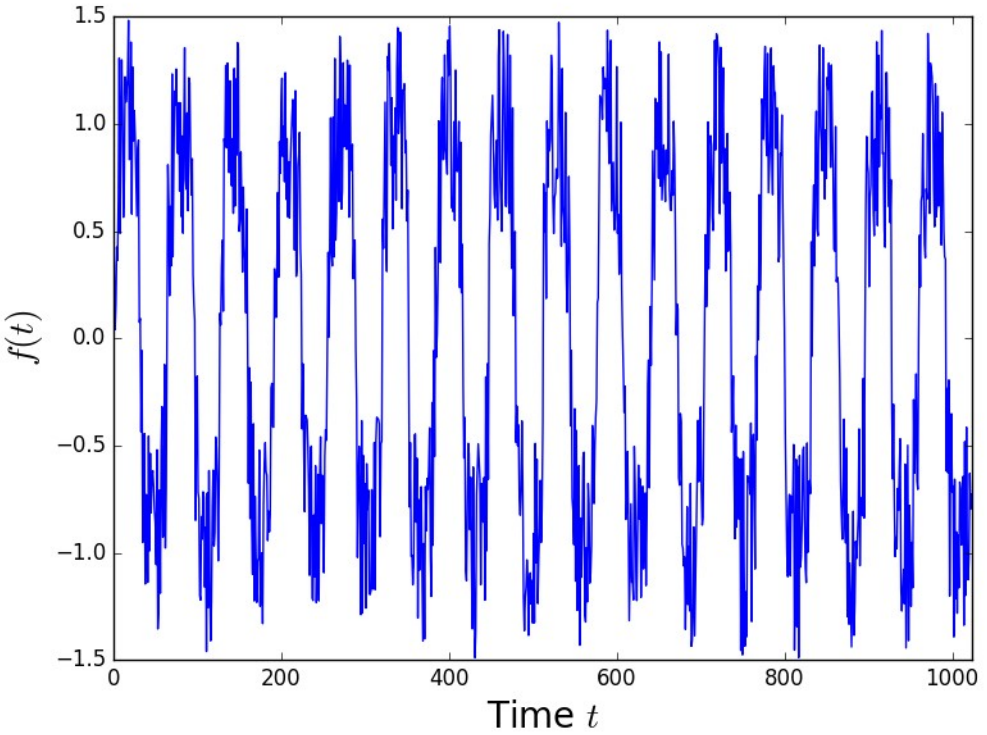
Example: Fourier representation of the piecewise function



# Fourier transforms. Physical interpretation

**Example:**
**examples/fourier/fft.py**

**signal consists of**
**a wave with**
**a well-defined frequency**
**and period,**
**but also some noise,**
**visible as smaller wiggles**
**in the line**




**We can calculate the Fourier transform of this signal as shown in examples/fourier/fft.py**

1. reads the file pitch.txt and plots
2. calculates the Fourier transform (numpy.fft.rfft)
3. plots the value of the absolute value of the coefficients $|c_k|$ versus k
4. calculates what is the frequency associated with the largest $|c_k|$

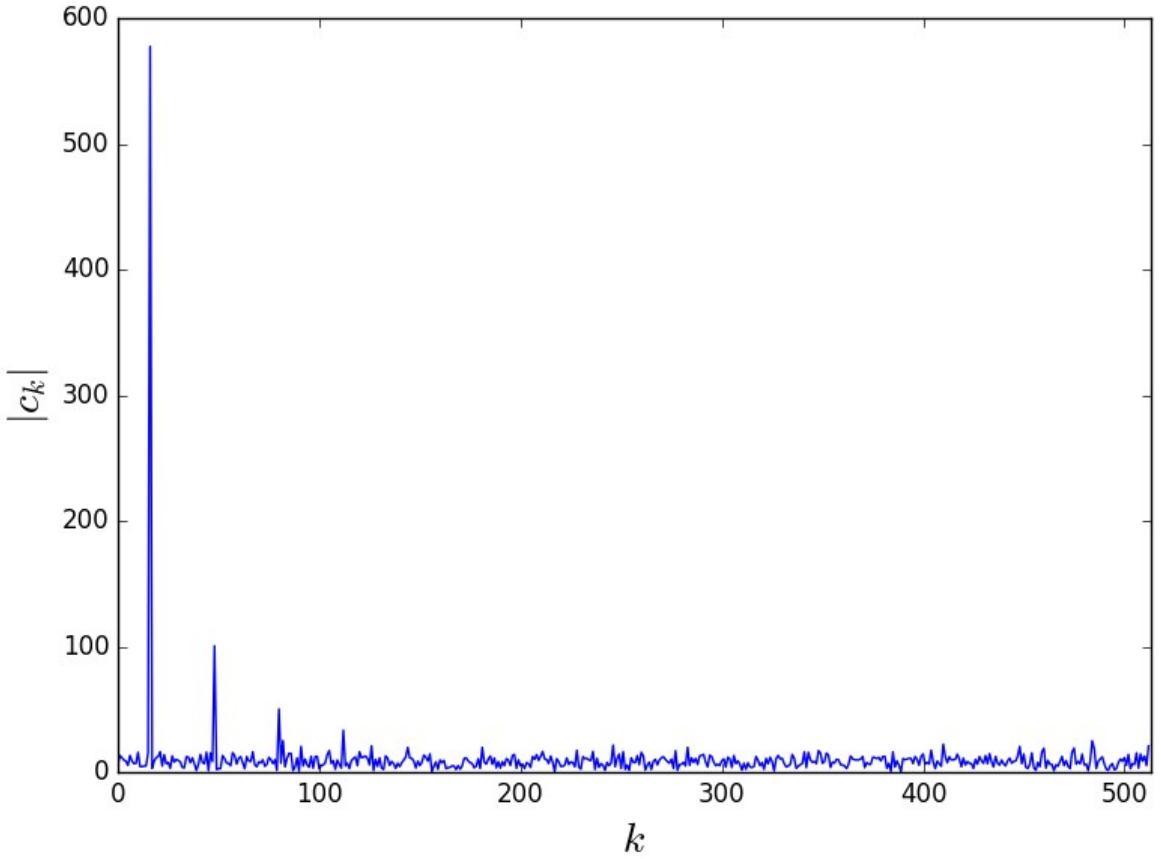
# Fourier transforms. Physical interpretation

**$|c_k|$ versus k**

**k proportional to the frequency of the waves $\nu = k/N$**

**Spikes in the plot**

**Main SPIKE**
**k = 16, $\nu \sim 0.0156$**
**$T = 1/\nu = 64$.**




**Main spike = main frequency and periodicity visible in Figure**

**Secondary spikes: HARMONICS**

**LOW-LEVEL BACKGROUND:** some small, random values of $|c_k|$.

Non-zero but very small $|c_k|$ are produced by some noise in the signal,

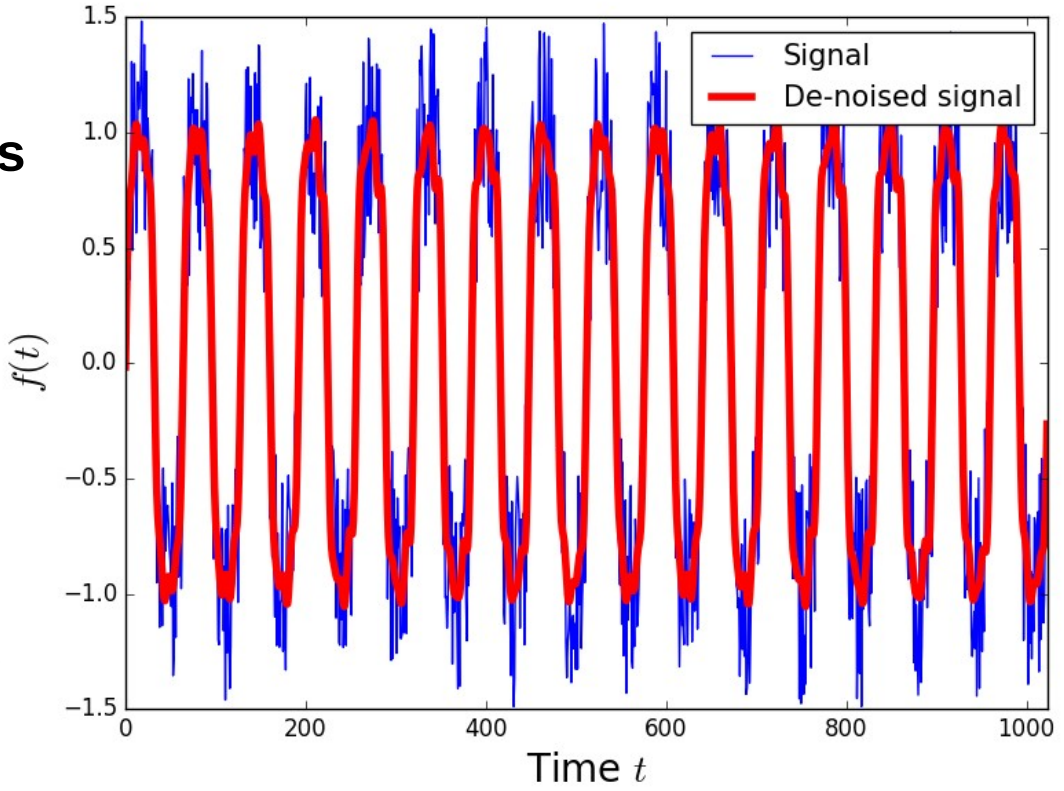
**WHITE NOISE (= does not show any dependence with k)**

# Fourier transforms. De-noising

**To get rid of the noise, reconstruct the initial samples without considering the smallest $|c_k|$ :**

cut the small $|c_k|$

**equivalent to applying a FILTER**




**Reconstruct original samples by doing the inverse FFT**
**after zeroing all $c_k$**
**with $|c_k| <= threshold \sim 25$**

Red: reconstructed signal after removing the noise-like coefficients.

Procedure works only if WHITE NOISE and
final result is sensitive to the choice of the threshold.

Not a tutorial on de-noising but very simplified example.

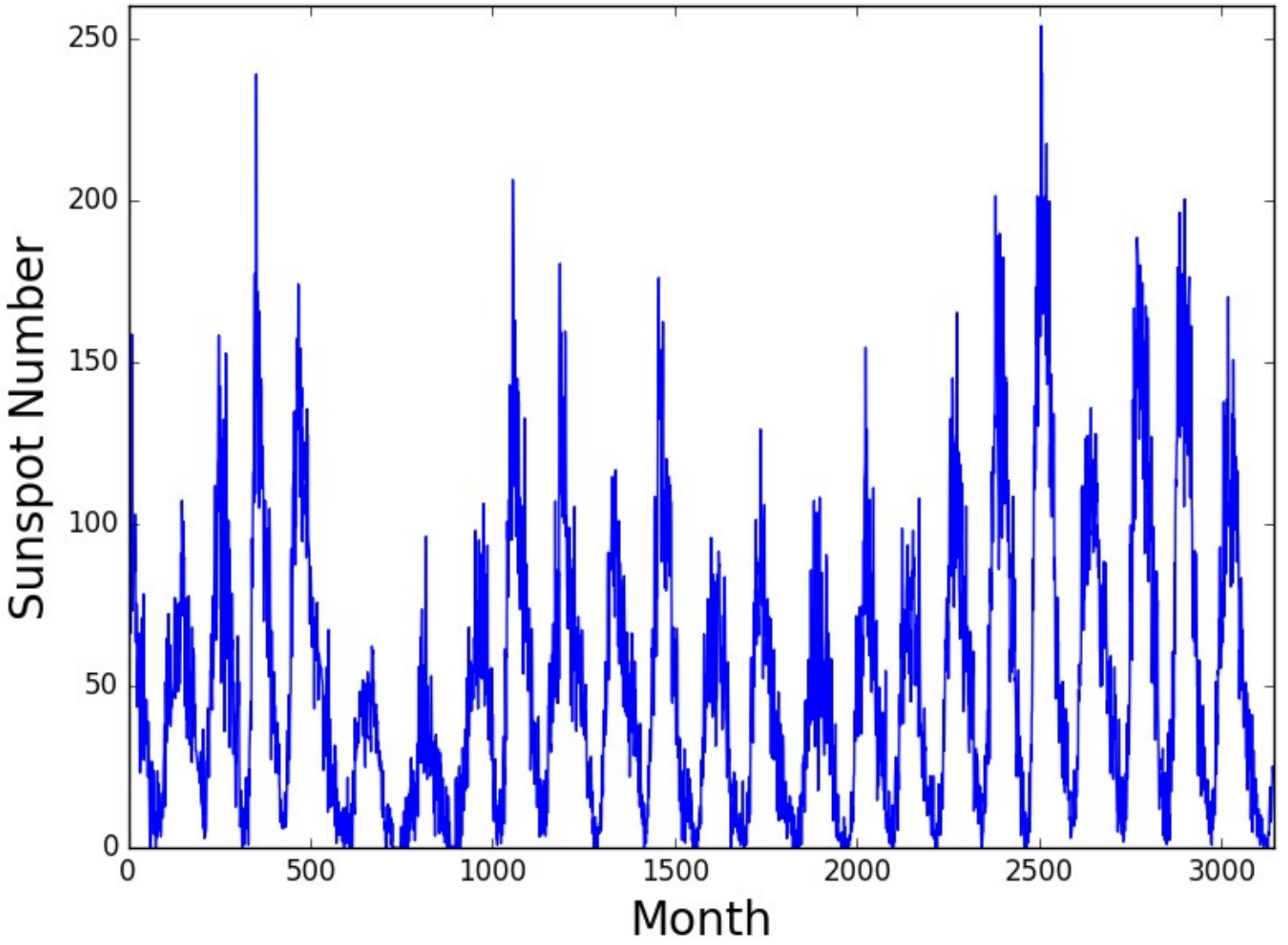
# Fourier transforms. Exercise

**EXERCISE:**

The file exercises/fourier/sunspots.txt (or, if you prefer to use gitlab: the file exercises/fourier/sunspots.txt) contains the observed number of sunspots on the Sun for each month since January 1749. The file contains two columns: column 0 is the month and column 1 is the number of sunspots per each month.

1. Plot the number of sunspots as a function of the month. The result should look like Figure 76.

# Fourier transforms. Exercise



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1. Plot the number of sunspots as a function of the month. The result should look like Figure 76.
2. Write a script to perform the discrete Fourier transform (DFT) and the fast Fourier transform (FFT) of the number of sunspots as a function of the month. For the DFT, use equation 314. For the FFT, use the numpy.fft.rfft function. Compare the different performances. The DFT should be significantly slower than the FFT.

$$c_k = \sum_{n=0}^{N-1} y_n \exp\left(-i \frac{2\pi k n}{N}\right)$$

Note:

| | |
|---|---|
| Work with complex numbers | **from cmath import exp** |
| Define complex array | **ck =numpy.zeros(N//2+1,complex)** |
| *2 i* should be written as | **2j** |

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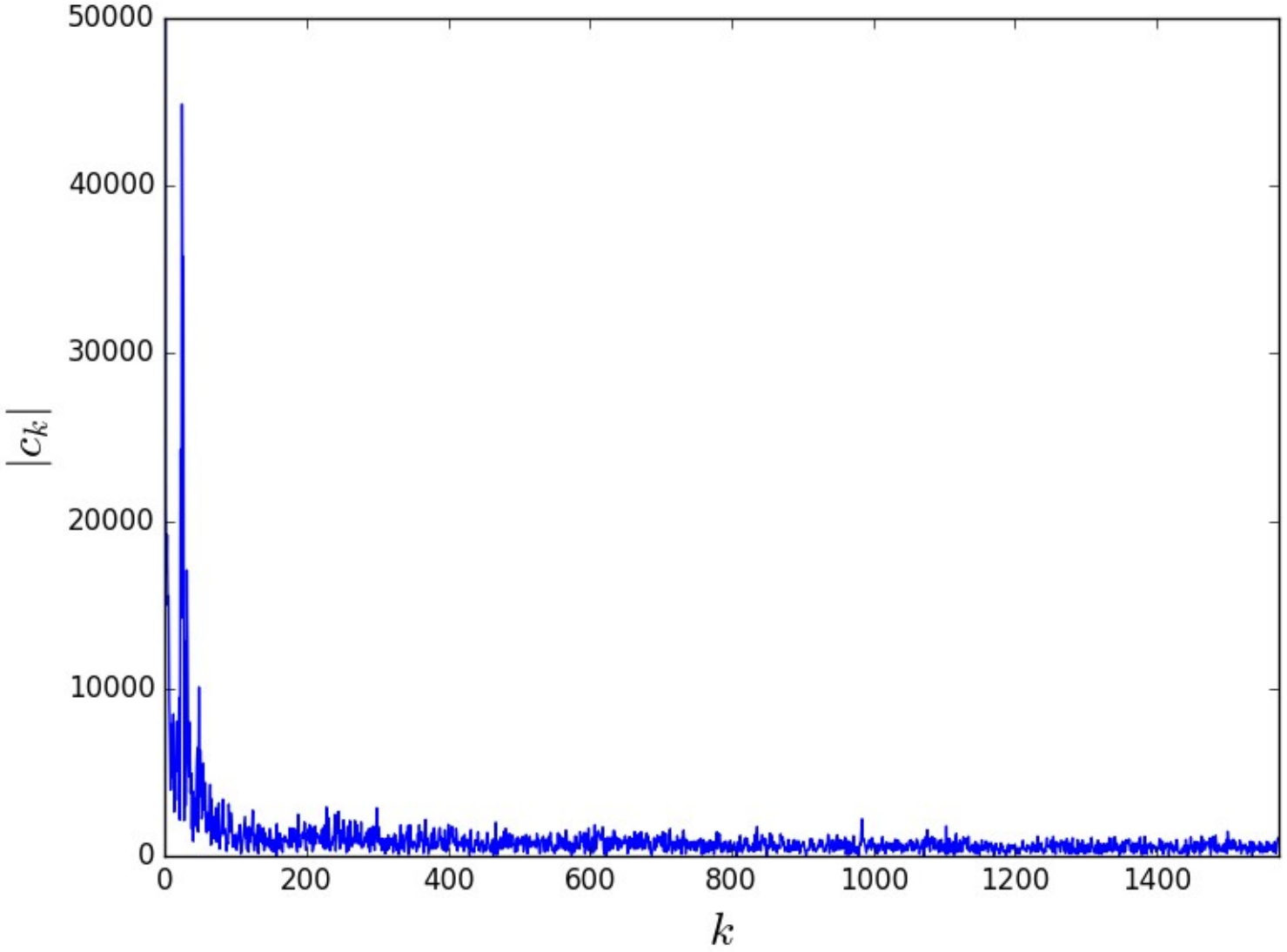

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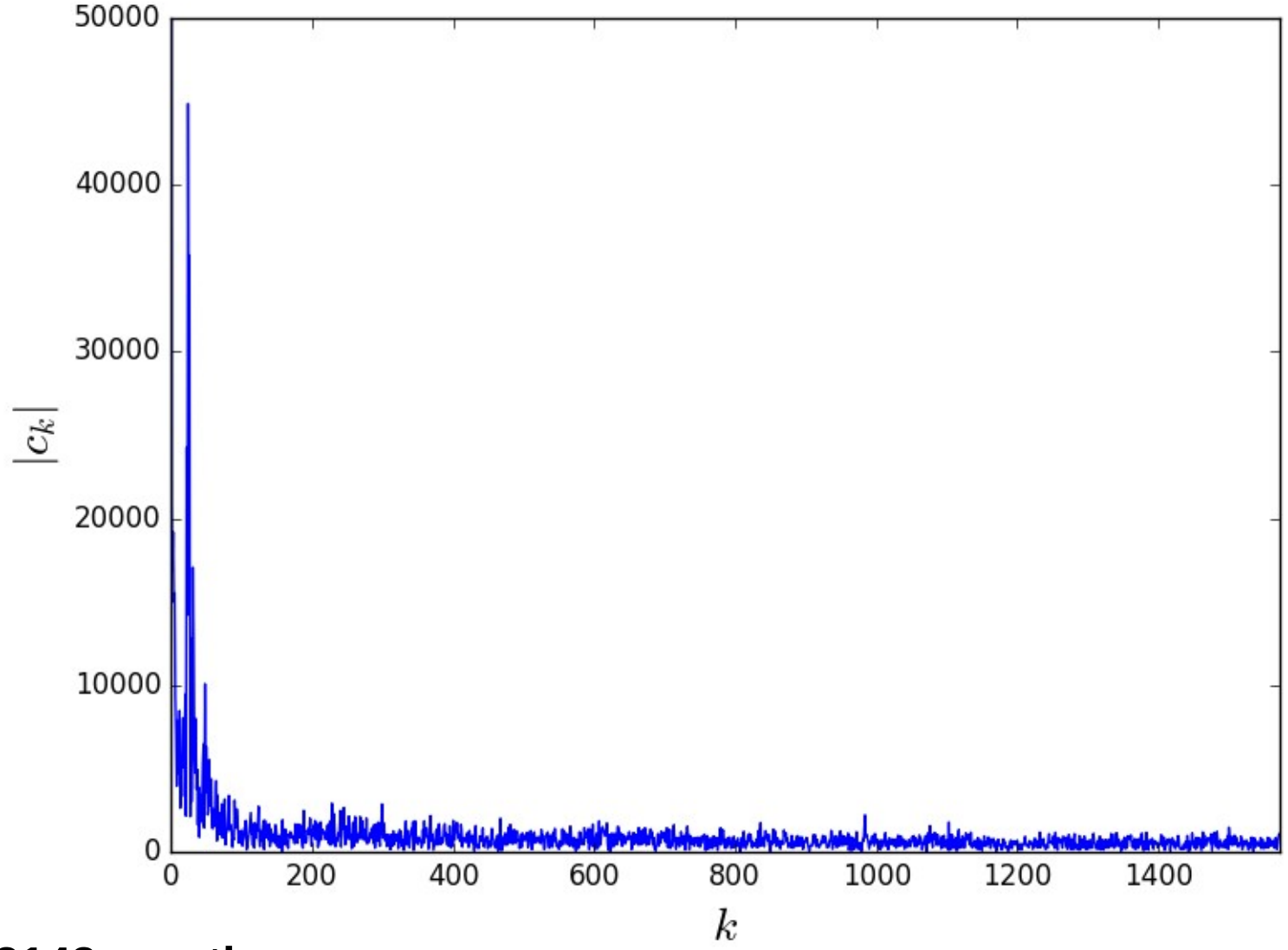
4. Calculate the period $T = N/k$ associated with the largest $|c_k|$. This gives you the sunspot periodicity ($t \sim 10.9$ years).

# Fourier transforms. Exercise




**N = 3143 months**
**Frequency : k/N ~ 0.0076 months$^{-1}$**
**Period: T = N/K ~ 10.9 yr**

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3. Plot the Fourier coefficients $|c_k|$ as a function of $k$. The result should look like Figure 77.
4. Calculate the period $T = N/k$ associated with the largest $|c_k|$. This gives you the sunspot periodicity ($t \sim 10.9$ years).
5. Try to denoise the data assuming that there is just white noise. Plot the denoised signal. The result should look like Figure 78. Note: for the inverse Fourier transform it is sufficient that you use the corresponding numpy function (no need to write an indirect DFT).

# Fourier transforms. Exercise