Linear Programming and the Simplex method

Luigi De Giovanni

Dipartimento di Matematica, Università di Padova

Mathematical Programming models

$$\begin{aligned} \min(\max) \quad & f(x) \\ \text{s.t.} \quad & g_i(x) = b_i \quad (i = 1 \dots k) \\ & g_i(x) \leq b_i \quad (i = k+1 \dots k') \\ & g_i(x) \geq b_i \quad (i = k'+1 \dots m) \\ & x \in \mathbb{R}^n \end{aligned}$$

- $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a vector (column) of n **REAL** variables;
- f e g_i are functions $\mathbb{R}^n \rightarrow \mathbb{R}$
- $b_i \in \mathbb{R}$

Linear Programming (LP) models

f e g_i are **linear** functions of x

$$
\begin{array}{lllr}
\min(\max) & c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\
\text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n & = b_i & (i = 1 \ldots k) \\
& a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n & \leq b_i & (i = k+1 \ldots k') \\
& a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n & \geq b_i & (i = k'+1 \ldots m) \\
& x_i \in \mathbb{R} & & (i = 1 \ldots n)
\end{array}
$$

Notice: for the moment, just **CONTINUOUS variables are considered**!!!

We need different methods for models with integer or binary variables.

Resolution of an LP model

- *Feasible solution*: $x \in \mathbb{R}^n$ satisfying all the constraints
- *Feasible region*: set of all the feasible solutions x
- *Optimal solution* x^* [min]: $c^T x^* \leq c^T x, \forall x \in \mathbb{R}^n, x$ feasible.

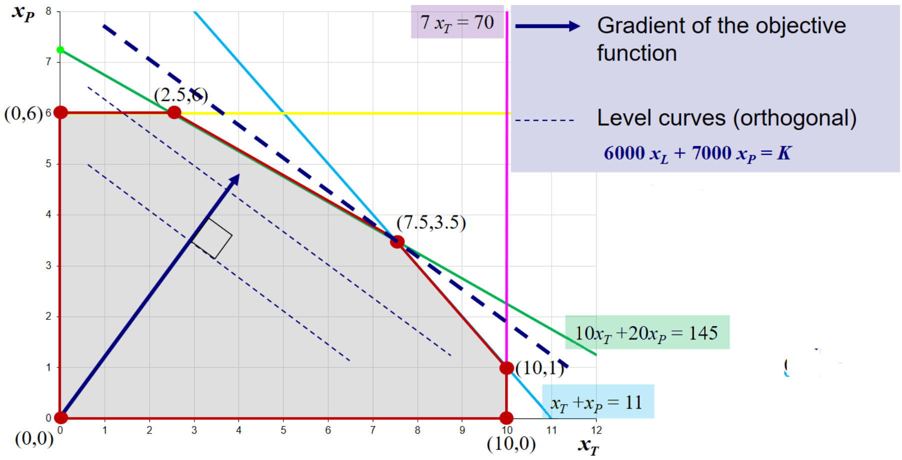
Solving a LP model is determining if it:

- is unfeasible
- is unlimited
- has a (finite) optimal solution

Solving an LP: example

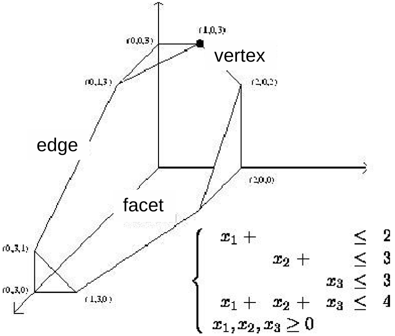
The farmer problem:

$$\begin{array}{llll} \max & 6\,000\,x_T + 7\,000\,x_P & & \\ \text{s.t.} & x_T + x_P \leq 11 & 7\,x_T \leq 70 & 3\,x_P \leq 18 \\ & 10\,x_T + 20\,x_P \leq 145 & x_T \geq 0 & x_P \geq 0 \end{array}$$



Geometry of LP

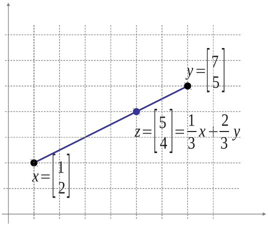
The feasible region is a **polyedron** (intersection of a finite number of closed half-spaces and hyperplanes in $\mathbb{R}^n$)



LP problem: $\min(\max)\{c^T x : x \in P\}$, P is a polyhedron in $\mathbb{R}^n$.

Vertex of a polyhedron: definition

- $z \in \mathbb{R}^n$ is a **convex combination** of two points x and y if $\exists\ \lambda \in [0, 1]$: $z = \lambda x + (1 - \lambda)y$

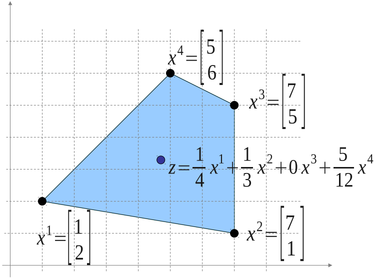


- $z \in \mathbb{R}^n$ is a **strict convex combination** of two points x and y if $\exists\ \lambda \in < (0, 1) >$: $z = \lambda x + (1 - \lambda)y$.

- $v \in P$ is **vertex of a polyhedron** P if it **is not** a **strict convex combination** of two *distinct* points of P: $\nexists\ x, y \in P, \lambda \in (0, 1) : x \neq y, v = \lambda x + (1 - \lambda)y$

Representation of polyhedra

$z \in \mathbb{R}^n$ is **convex combination** of $x^1, x^2 \dots x^k$ if $\exists\ \lambda_1, \lambda_2 \dots \lambda_k \geq 0$:
$$\sum_{i=1}^{k} \lambda_i = 1 \text{ and } z = \sum_{i=1}^{k} \lambda_i x^i$$



Theorem: representation of polyhedra [Minkowski-Weyl] - case 'limited'

Polydron *limited* $P \subseteq \mathbb{R}^n$, $v^1, v^2, ..., v^k$ ($v^i \in \mathbb{R}^n$) vertices of P

if $x \in P$ then $x = \sum_{i=1}^{k} \lambda_i v^i$ with $\lambda_i \geq 0, \forall i = 1..k$ and $\sum_{i=1}^{k} \lambda_i = 1$

(x is convex combination of the vertices of P)

Optimal vertex: from graphical intuition to proof

Theorem: optimal vertex(fix *min* objective function)

LP problem $\min\{c^T x : x \in P\}$, P non empty and limited

- LP ha optimal solution
- **one of the optimal solution of LP is a vertex of** P

Proof:

$$V = \{v^1, v^2 \dots v^k\} \qquad v^* = \arg\min_{v \in V} c^T v$$

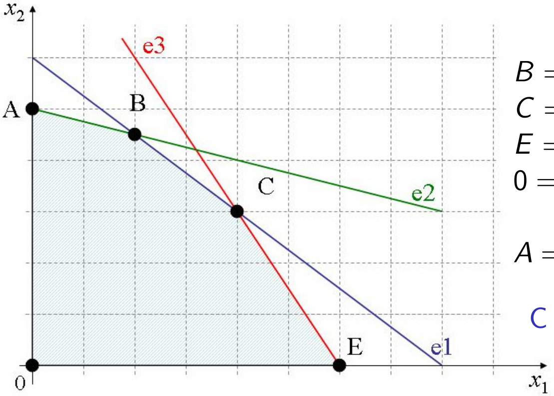
$$c^T x = c^T \sum_{i=1}^{k} \lambda_i v^i = \sum_{i=1}^{k} \lambda_i c^T v^i \geq \sum_{i=1}^{k} \lambda_i c^T v^* = c^T v^* \sum_{i=1}^{k} \lambda_i = c^T v^*$$

Summarizing: $\forall x \in P, \;\; c^T v^* \leq c^T x$ ■

We can limit the search of an optimal solution to the vertices of P**!**

Vertex comes from intersection of generating hyperplanes

$$\begin{array}{rrcrcrl}
\max & 13x_1 & + & 10x_2 & & & \\
s.t. & 3x_1 & + & 4x_2 & \leq & 24 & \text{(e1)} \\
& x_1 & + & 4x_2 & \leq & 20 & \text{(e2)} \\
& 3x_1 & + & 2x_2 & \leq & 18 & \text{(e3)} \\
& x_1 & , & x_2 & \geq & 0 &
\end{array}$$



$B = e1 \cap e2$	$(2, 9/2)$	71
$C = e1 \cap e3$	$(4, 3)$	82
$E = e3 \cap (x_2 = 0)$	$(6, 0)$	78
$0 = (x_1 = 0) \cap (x_2 = 0)$	$(0, 0)$	0
$A = e2 \cap (x_1 = 0)$	$(0, 5)$	50

C optimum!

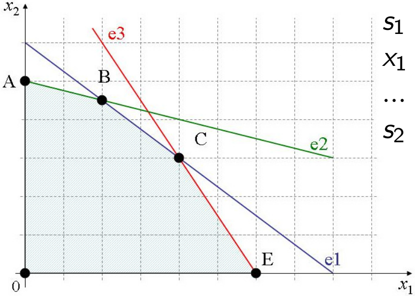
Algebraic representation of vertices

Write the constraints as **equations**

$$\begin{array}{rcrcrcrcrcl} 3x_1 & + & 4x_2 & + & s_1 & & & & & = & 24 \\ x_1 & + & 4x_2 & & & + & s_2 & & & = & 20 \\ 3x_1 & + & 2x_2 & & & & & + & s_3 & = & 18 \end{array}$$

$5 - 3 = 2$ degrees of freedom:

we can set (any) two variables to 0 and obtain a unique solution!



$s_1 = s_2 = 0$ $(2, 9/2, 0, 0, 3)$ B

$x_1 = s_2 = 0$ $(0, 5, 4, 0, 8)$ A

...

$s_2 = s_3 = 0$ $(3.2, 4.2, -2.4, 0, 0)$

not feasible!

Standard form for LP problems

$$
\begin{array}{lllr}
\min & c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\
\text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n & = b_i & (i = 1 \ldots m) \\
 & x_i \in \mathbb{R}_+ & & (i = 1 \ldots n)
\end{array}
$$

- **minimizing** objective function (if not, multiply by -1);
- variables ≥ 0; (if not, substitution)
- all constraints are equalities; ($+/-$ slack/surplus variables)
- $b_i \geq 0$. (if not, multiply by -1)

Standard form: example

$$\begin{aligned} \max \quad & 5(-3x_1 + 5x_2 - 7x_3) + 34 \\ s.t. \quad & -2x_1 + 7x_2 + 6x_3 - 2x_1 \leq 5 \\ & -3x_1 + x_3 + 12 \geq 13 \\ & x_1 + x_2 \leq -2 \\ & x_1 \leq 0 \\ & x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} & \hat{x}_1 = -x_1 && (\hat{x}_1 \geq 0) \\ & x_3 = x_3^+ - x_3^- && (x_3^+ \geq 0 \ , \ \ x_3^- \geq 0) \end{aligned}$$

$$\begin{aligned} \min \quad & -3\hat{x}_1 - 5x_2 + 7x_3^+ - 7x_3^- \\ s.t. \quad & 4\hat{x}_1 + 7x_2 + 6x_3^+ - 6x_3^- + s_1 = 5 \\ & 3\hat{x}_1 + x_3^+ - x_3^- - s_2 = 1 \\ & \hat{x}_1 - x_2 - s_3 = 2 \\ & \hat{x}_1 \geq 0 \ , \ x_2 \geq 0 \ , \ x_3^+ \geq 0 \ , \ x_3^- \geq 0 \ , \ s_1 \geq 0 \ , \ s_2 \geq 0 \ , \ s_3 \geq 0. \end{aligned}$$

Linear algebra: definitions

- column vector $v \in \mathbb{R}^{n \times 1}$: $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$
- row vector $v^T \in \mathbb{R}^{1 \times n}$: $v^T = [v_1, v_2, ..., v_n]$
- matrix $A \in \mathbb{R}^{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$
- $v, w \in \mathbb{R}^n$, scalar product $v \cdot w = \sum_{i=1}^{n} v_i w_i = v^T w = w^T v$
- Rank of $A \in \mathbb{R}^{m \times n}$, $\rho(A)$, max linearly independent rows/columns
- $B \in \mathbb{R}^{m \times m}$ *invertible* $\iff \rho(B) = m \iff det(B) \neq 0$

Systems of linear equations

- *Systems of equations in matrix form*: a system of m equations in n variables can be written as:
  $Ax = b$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ e $x \in \mathbb{R}^n$.
- *Theorem of Rouché-Capelli*:
  $Ax = b$ has solutions $\iff \rho(A) = \rho(A|b) = r$ (∞^{n-r} solutions).
- *Elementary row operations*:
  - swap row i and row j;
  - multiply row i by a non-zero scalar;
  - substitute row i by row i plus α times row j ($\alpha \in \mathbb{R}$).

  Elementary operations on (augmented) matrix $[A|b]$ leave the same solutions as $Ax = b$.
- *Gauss-Jordan method* for solving $Ax = b$: make elementary row operations on $[A|b]$ so that A contains an identity matrix of dimension $\rho(A) = \rho(A|b)$.

Basic solutions

- **Assumptions**: system $Ax = b$, $A \in \mathbb{R}^{m \times n}$, $\rho(A) = m$, $m < n$
- **Basis of** A: square submatrix with maximum rank, $B \in \mathbb{R}^{m \times m}$
- $A = [B|N] \quad B \in \mathbb{R}^{m \times m}, det(B) \neq 0$
$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}, x_B \in \mathbb{R}^m, x_N \in \mathbb{R}^{n-m}$$
- $Ax = b \Longrightarrow [B|N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = Bx_B + Nx_N = b$
- $x_B = B^{-1}b - B^{-1}Nx_N$
- imposing $x_N = 0$, we obtain a so called **basic solution**:
$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$
- many different basic solutions by choosing a **different basis** of A
- **variables equal to 0 are** $n - m$ (or more: *degenerate* basic solutions)

Basic solutions and LP in standard form

$$\begin{array}{lllr} \min & c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\ \text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n & = b_i & (i = 1 \ldots m) \\ & x_i \in \mathbb{R}_+ & & (i = 1 \ldots n) \end{array}$$

$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & Ax & = b \\ & x & \geq 0 \end{array}$$

- basis B gives a **feasible basic solution** if $x_B = B^{-1}b \geq 0$

$$\begin{array}{llllll} 3x_1 & +4x_2 & +s_1 & & & = & 24 \\ x_1 & +4x_2 & & +s_2 & & = & 20 \\ 3x_1 & +2x_2 & & & +s_3 & = & 18 \end{array}$$

$$A = \begin{bmatrix} 3 & 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 24 \\ 20 \\ 18 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = B_1^{-1}b = \begin{bmatrix} 2 \\ 4,5 \\ 3 \end{bmatrix}$$

$$x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x^T = (2 \;\; 9/2 \;\; 0 \;\; 0 \;\; 3)$ $\rightarrow$ **vertex** B

$$B_2 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

Vertices and basic solution

Feasible basic solution $\rightsquigarrow$ $n - m$ variables are 0 $\rightsquigarrow$ intersection of the right number of hyperplanes $\rightsquigarrow$ vertex!

PL $\min\{c^T x : Ax = b, x \geq 0\}$ $\qquad$ $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$

Theorem: **vertices correspond to feasible basic solutions** *(algebraic characterization of the vertices of a polyhedron)*

x **feasible basic solution of** $Ax = b$ $\iff$ x **is a vertex of** P

Corollary: **optimal basic solution**

If P non empty and limited, then **there exists at least an optimal solution which is a basic feasible solution**

Algorithm for LP (case limited): sketch

Consider **all** the feasible basic solutions:

1. put the LP in standard form $\min\{c^T x : Ax = b, x \geq 0\}$
2. *incumbent* $= +\infty$
3. **repeat**
4. generate a combination of m columns of A
5. let B be the corresponding submatrix of A
6. **if** $det(B) == 0$ **then continue else** compute $x_B = B^{-1}b$
7. **if** $x_B \geq 0$ **and** $c_B^T x_B <$ *incumbent* **then** update *incumbent*
8. **until**(no other column combinations)

Complexity: up to $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ basic solution!!!

$\Rightarrow$ **Symplex method**: more efficient exploration of the basic solutions (only **feasible** and **improving**)

Example

LP problem in **standard form**:

$$
\begin{array}{llrcrcrcrcrcl}
\min & z = & -13x_1 & - & 10x_2 & & & & & & & & \\
s.t. & & 3x_1 & + & 4x_2 & + & s_1 & & & & & = & 24 \\
 & & x_1 & + & 4x_2 & & & + & s_2 & & & = & 20 \\
 & & 3x_1 & + & 2x_2 & & & & & + & s_3 & = & 18 \\
 & & x_1 & , & x_2 & , & s_1 & , & s_2 & , & s_3 & \geq & 0
\end{array}
$$

an initial **basic feasible solution** (vertex B):

- $B = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
- $x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 9/2 \\ 3 \end{bmatrix} \quad x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $z_B = c^T x = c_B^T x_B + c_N^T x_N = -71$

Example

Change basis: **New basic solution** $\Rightarrow$ one non-basic variable increases **affecting x_B and z_B**

$$
\begin{aligned}
x_B &= B^{-1}b - B^{-1}N\, x_N \\
z &= c^T x = c_B^T x_B + c_N^T x_N = c_B^T (B^{-1}b - B^{-1}N\, x_N) + c_N^T x_N \\
&\qquad\qquad = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N)\, x_N
\end{aligned}
$$

Write x_B and z as functions of only **non-basic** variables

For the sake of manual computation, use **Gauss-Jordan**:

$$
Ax = b \quad \rightsquigarrow \quad [\, B\ N \mid b\,] \quad \rightsquigarrow \quad [\, B^{-1}B = I\ \ B^{-1}N = \bar{N} \mid B^{-1}b = \bar{b}\,]
$$

$$
x_B = \bar{b} - \bar{N} x_N \qquad z = \ldots
$$

Example

	x_1	x_2	s_3	s_1	s_2	$\bar{b}$
	3	4	0	1	0	24
	1	4	0	0	1	20
	3	2	1	0	0	18
$(R_1/3)$	1	4/3	0	1/3	0	8
$(R_2 - R_1/3)$	0	8/3	0	−1/3	1	12
$(R_3 - R_1)$	0	−2	1	−1	0	−6
$(R_1 - 1/2\ R_2)$	1	0	0	1/2	−1/2	2
$(3/8\ R_2)$	0	1	0	−1/8	3/8	9/2
$(R_3 + 3/4\ R_2)$	0	0	1	−5/4	3/4	3

$$
\begin{aligned}
x_1 &= 2 - 1/2\ s_1 + 1/2\ s_2 \\
x_2 &= 9/2 + 1/8\ s_1 - 3/8\ s_2 \\
s_3 &= 3 + 5/4\ s_1 - 3/4\ s_2 \\
z = -13x_1 - 10x_2 &= -71 + 21/4\ s_1 - 11/4\ s_2
\end{aligned}
$$

Example

$$z = -71 + 21/4\ s_1 - 11/4\ s_2$$

- In order to minimize, it is convenient to increase s_2 (and keep $s_1 = 0$)
- Equalities have to be always satisfied...:

$$\begin{aligned} x_1 &= 2 + 1/2\ s_2 \\ x_2 &= 9/2 - 3/8\ s_2 \\ s_3 &= 3 - 3/4\ s_2 \end{aligned}$$

- while preserving non-negativity:

$$\begin{array}{lllllll} x_1 \geq 0 & \Rightarrow & 2 + 1/2 s_2 \geq 0 & \Rightarrow & s_2 & \geq & -4 \quad \textbf{always!} \\ x_2 \geq 0 & \Rightarrow & 9/2 - 3/8 s_2 \geq 0 & \Rightarrow & s_2 & \leq & 12 \\ s_3 \geq 0 & \Rightarrow & 3 - 3/4 s_2 \geq 0 & \Rightarrow & s_2 & \leq & 4 \end{array}$$

- New **feasible** and **better** solutions with $s_1 = 0$ and $0 \leq s_2 \leq 4$
- $s_2 = 4 \Rightarrow s_3 = 0$: new **basic**, **feasible** and **better** solution

Example

New basic solution! s_2 (now > 0) takes the place of s_3 (now $= 0$):

$$B = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$x_N = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad z_B = c^T x = c_B^T x_B + c_N^T x_N = -82$$

Same arguments as before: x_B and z **as a function of** x_N:

$$\begin{array}{rcrcrlcrl} x_1 & = & 4 & + & 1/3 & s_1 & - & 2/3 & s_3 \\ x_2 & = & 3 & - & 1/2 & s_1 & - & 1/2 & s_3 \\ s_2 & = & 4 & + & 5/3 & s_1 & - & 4/3 & s_3 \\ & & & & & & & & \\ z & = & -82 & + & 2/3 & s_1 & + & 11/3 & s_3 \end{array}$$

Optimal solution! Visited 2 out of $\binom{5}{3} = 10$ possible basis

LP in *canonical* form

PL $\min\{z = c^T x : Ax = b, x \geq 0\}$ is in **canonical form with respect to basis** B if all basic variables and the objective are explicitly written as functions of **non-basic variables only**:

$$
\begin{aligned}
z &= \bar{z}_B + \bar{c}_{N_1} x_{N_1} + \bar{c}_{N_2} x_{N_2} + \ldots + \bar{c}_{N_{(n-m)}} x_{N_{(n-m)}} \\
x_{B_i} &= \bar{b}_i - \bar{a}_{iN_1} x_{N_1} - \bar{a}_{iN_2} x_{N_2} - \ldots - \bar{a}_{iN_{(n-m)}} x_{N_{(n-m)}} \quad (i = 1 \ldots m)
\end{aligned}
$$

- $\bar{z}_B$ scalar (objective function value for the corresponding basic solution)
- $\bar{b}_i$ scalar (value of basic variable i)
- B_i index of the i-th basic variable $(i = 1 \ldots m)$
- N_j index of the j-th non-basic variable $(j = 1 \ldots n - m)$
- $\bar{c}_{N_j}$ coefficient of the j-th non-basic variable in the objective function (**reduced cost of the variable with respect to basis** B)
- $-\bar{a}_{iN_j}$ coefficient of the j-th non-basic variable in the constraints that makes explicit the i-th basic variable

Simplex method: optimality check

- **Reduced cost** of a variable: marginal unit increment of the objective function
- The reduced cost of a basis variable is $\bar{c}_{B_i} = 0$

Theorem: Sufficient optimality conditions

Given an LP and a feasible basis B, if all the reduced costs with respect to B are ≥ 0, then B is an optimal basis

$$\bar{c}_j \geq 0, \; \forall\, j = 1 \ldots n \quad \Rightarrow \quad B \text{ optimal}$$

- Notice: the inverse is not true! [there may be optimal basic solutions with negative reduced costs]

Simplex method: basis change

- From feasible basis B, obtain a $\tilde{B}$ **adjacent, feasible, improving**
- **One** column ($\approx$ variable) enters and one variable leaves the basis

- **Entering** variable (improvement): *any* $x_h : \bar{c}_h < 0$

$$z = \bar{z}_B + \bar{c}_h x_h = \bar{z}_{\tilde{B}} \leq \bar{z}_B$$

- **Leaving** variable (feasibility): [min ratio rule]

$$x_{B_i} \geq 0 \quad \Rightarrow \quad b_i - \bar{a}_{ih}\, x_h \geq 0,\ \forall\, i \quad \Rightarrow \quad x_h \leq \frac{\bar{b}_i}{\bar{a}_{ih}},\ \forall\, i : \bar{a}_{ih} > 0$$

$$t = \arg \min_{i=1\ldots m} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$$

$$x_h = \frac{\bar{b}_t}{\bar{a}_{th}} \geq 0 \quad \Rightarrow \quad x_{B_t} = 0 \text{ [} x_{B_t} \text{ leaves the basis!]}$$

Simplex method: check for unlimited LP

- Let x_h: $\bar{c}_h < 0$.

$$\begin{array}{rcccccc} z & = & \bar{z}_B & + & \bar{c}_h & x_h & \\ x_{B_i} & = & \bar{b}_i & - & \bar{a}_{ih} & x_h & (i = 1 \dots m) \end{array}$$

- If $a_{ih} \leq 0, \ \forall \ i = 1 \dots m$, feasible solution with $x_h \rightarrow +\infty$

Condition of unlimited LP

There exists a basis such that

$$\exists \, x_h : \ (\bar{c}_h \ < \ 0) \ \wedge \ (\bar{a}_{ih} \ \leq \ 0, \ \forall \ \ i = 1 \dots m)$$

Simplex method: summary

Init: PL in standard form $\min\{c^T x : Ax = b, x \geq 0\}$, and an initial feasible basis B

repeat

write the LP in canonical form with respect to B

$$z = \bar{z}_B + \bar{c}_{N_1} x_{N_1} + \bar{c}_{N_2} x_{N_2} + \ldots + \bar{c}_{N_{(n-m)}} x_{N_{(n-m)}}$$

$$x_{B_i} = \bar{b}_i - \bar{a}_{iN_1} x_{N_1} - \bar{a}_{iN_2} x_{N_2} - \ldots - \bar{a}_{iN_{(n-m)}} x_{N_{(n-m)}} \quad (i = 1 \ldots m)$$

if $(\bar{c}_j \geq 0, \forall j)$ **then** B is an optimal basis: **stop**

if $(\exists h : \bar{c}_h < 0$ and $\bar{a}_{ih} \leq 0, \forall i)$ **then** unlimited LP: **stop**

Entering variable: any $x_h : \bar{c}_h < 0$

Leaving variable: x_{B_t} with $t = \arg \min_{i=1 \ldots m} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$

$B \leftarrow B \oplus A_h \ominus A_{B_t}$ [basis change]

until (LP optimum found or unlimited)

Simplex tableau

- Represent the canonical form, can be used to operate Gauss-Jordan
- **Objective function as a constraint** (imposing the value of a new variable z):

  $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \rightsquigarrow \quad c_1x_1 + c_2x_2 + \ldots + c_nx_n - z = 0$

	x_{B_1}	$\ldots$	x_{B_m}	x_{N_1}	$\ldots$	$x_{N_{n-m}}$	z	$\bar{b}$
riga 0	$c_B^T 0$	$\ldots$	0	$c_N^T \square$	$\ldots$	$\square$	-1	$0\square$
riga 1	1		0	$\square$	$\ldots$	$\square$	0	$\square$
$\vdots$		$B \ddots$		$N\square$	$\ldots$	$\square$	$\vdots$	$b\square$
riga m	0		1	$\square$	$\ldots$	$\square$	0	$\square$

Tableau in canonical form

- Elementary row (z included) operations: up to reading x_B (and z) as functions of x_N

Tableau and canonical form

	x_{B_1}	$\dots$	x_{B_m}		x_{N_1}	$\dots$	$x_{N_{n-m}}$	
$-z$	0	$\dots$	0	$\square$	$\dots$	$\square\bar{c}_{N_1}$	$\dots$	$\bar{c}_{N_{n-}}$
x_{B_1}	1		0	$\square$	$\dots$	$\square\bar{a}_{1\,N_1}$	$\dots$	$\bar{a}_{1\,N_n}$
x_{B_i}		$\ddots$		$\square$	$\dots$	$\square\bar{a}_{i\,N_1}$	$\dots$	$\bar{a}_{i\,N_{n-}}$
x_{B_m}	0		1	$\square$	$\dots$	$\square\bar{a}_{m\,N_1}$	$\dots$	$\bar{a}_{m\,N}$

$$z = \bar{z}_B + \bar{c}_{N_1}\, x_{N_1} + \bar{c}_{N_2}\, x_{N_2} + \dots + \bar{c}_{N_{(n-m)}}\, x_{N_{(n-m)}}$$

$$x_{B_i} = \bar{b}_i - \bar{a}_{iN_1}\, x_{N_1} - \bar{a}_{iN_2}\, x_{N_2} - \dots - \bar{a}_{iN_{(n-m)}}\, x_{N_{(n-m)}} \quad (i = 1 \dots m)$$

Retrieving an intial feasible basis: **two-phases method**

- **Phase I**: solve an *artificial problem*

$$
\begin{aligned}
w^* = \quad \min w = \quad & 1^T y = y_1 + y_2 + \cdots + y_m \\
s.t. \quad & Ax + Iy = b \\
& x, y \geq 0
\end{aligned}
\qquad
y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m_+
$$

  If $w^* > 0$, the original problem is unfeasible, stop!

  If $w^* = 0$, then $y = 0$

  - if some y in the (degenarate) basis, change basis to put all y out, thus obtaining an x_B feasible for the original problem!

- **Phase II**: solve the problem starting from the provided basis B

Simplex algorithm with matrix operations (i)

$$\begin{aligned} \min z = \quad & c^T x \\ \text{s.t.} \quad Ax & = b \\ x & \geq 0 \end{aligned}$$

standard form

$$\begin{aligned} \min z = \quad & c_B^T x_B + c_N^T x_N \\ \text{s.t.} \quad B\, x_B + N\, x_N & = b \\ x_B, x_N & \geq 0 \end{aligned}$$

with (feasible) basis

$$\begin{aligned} x_B &= B^{-1} b - B^{-1} N\, x_N \\ z &= c_B^T x_B + c_N^T x_N = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N)\, x_N \end{aligned}$$

$$\begin{aligned} -z \quad & + \quad \bar{c}_N^T x_N = -z_B \\ I\, x_B \quad & + \quad \bar{N}\, x_N = \bar{b} \end{aligned}$$

canonical (or tableau) form

- $\bar{b} = B^{-1} b$
- $z_B = c_B^T B^{-1} b$
- $\bar{N} = B^{-1} N$
- $\bar{c}_N^T = c_N^T - c_B^T B^{-1} N$

Simplex algorithm with matrix operations (ii)

Map the coefficients of the canonical form...

$$
\begin{array}{ccccccccccccc}
z & = & \bar{z}_B & + & \bar{c}_{\nu_1} & x_{N_1} & + & \bar{c}_{\nu_2} & x_{\nu_2} & + & \dots & + & \bar{c}_{\nu_{(n-m)}} & x_{\nu_{(n-m)}} \\
x_{\beta_1} & = & \bar{b}_i & - & \bar{a}_{i\nu_1} & x_{\nu_1} & - & \bar{a}_{i\nu_2} & x_{\nu_2} & - & \dots & - & \bar{a}_{i\nu_{(n-m)}} & x_{\nu_{(n-m)}} \\
\dots \\
x_{\beta_m} & = & \bar{b}_i & - & \bar{a}_{i\nu_1} & x_{\nu_1} & - & \bar{a}_{i\nu_2} & x_{\nu_2} & - & \dots & - & \bar{a}_{i\nu_{(n-m)}} & x_{\nu_{(n-m)}}
\end{array}
$$

... into matrix operations

$$
\begin{array}{ccccccc}
-z & & & + & \bar{c}_N^T x_N & = & -z_B \\
& & I\, x_B & + & \bar{N}\, x_N & = & \bar{b}
\end{array}
$$

- $\bar{b} = B^{-1} b$
- $z_B = c_B^T B^{-1} b$
- $\bar{N} = B^{-1} N$
- $\bar{c}_N^T = c_N^T - c_B^T B^{-1} N$

- $\bar{b}_i = [B^{-1} b]_i$
- $u^T = c_B^T B^{-1}$ $\quad z_B = u^T b$
- $\bar{N}_{\nu_j} = B^{-1} N_{\nu_j}$
- $\bar{c}_{\nu_j} = [\bar{c}_N^T]_{\nu_j} = c_{\nu_j} - u^T N_{\nu_j}$

The (revised) simplex algorithm

1. Let $\beta[1], ..., \beta[m]$ be the column indexes of the **initial basis**
2. Let $B = \left[A_{\beta[1]}|...|A_{\beta[m]}\right]$ and compute B^{-1} e $u^T = c_B^T B^{-1}$
3. compute **reduced costs**: $\bar{c}_h = c_h - u^T A_h$ for non-basic variables x_h
4. If $\bar{c}_h \geq 0$ for all non-basic variables x_h, **STOP**: B is **optimal**
5. Choose any x_h having $\bar{c}_h < 0$
6. Compute $\bar{b} = B^{-1}b = \left[\ \bar{b}_i\ \right]_{i=1}^{m}$ e $\bar{A}_h = \bar{N}_h = B^{-1}A_h = \left[\ \bar{a}_{ih}\ \right]_{i=1}^{m}$
7. If $\bar{a}_{ih} \leq 0$, $\forall i = 1...m$, **STOP**: **unlimited**
8. Determine $t = \arg\min_{i=1...m} \left\{\bar{b}_i/\bar{a}_{ih}, \bar{a}_{ih} > 0\right\}$
9. Change basis: $\beta[t] \leftarrow h$.
10. Iterate from Step 2

Example

Solve:

$$
\begin{array}{lrcrcrcl}
\max & 3x_1 & + & x_2 & - & 3x_3 & & \\
\text{s.t.} & 2x_1 & + & x_2 & - & x_3 & \leq & 2 \\
 & x_1 & + & 2x_2 & - & 3x_3 & \leq & 5 \\
 & 2x_1 & + & 2x_2 & - & x_3 & \leq & 6 \\
 & x_1 \geq 0 & , & x_2 \geq 0 & , & x_3 \leq 0 & &
\end{array}
$$

Standard form

$$
\begin{array}{lrcrcrcrcrcrcl}
\min & -3x_1 & - & x_2 & - & 3\hat{x}_3 & & & & & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & \hat{x}_3 & + & x_4 & & & & & = & 2 \\
 & x_1 & + & 2x_2 & + & 3\hat{x}_3 & & & + & x_5 & & & = & 5 \\
 & 2x_1 & + & 2x_2 & + & \hat{x}_3 & & & & & + & x_5 & = & 6 \\
 & x_1 & , & x_2 & , & \hat{x}_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}
$$

Matrices and initial basis

$$
\begin{array}{lrcrcrcrcrcrcl}
\min & -3x_1 & - & x_2 & - & 3\hat{x}_3 & & & & & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & \hat{x}_3 & + & x_4 & & & & & = & 2 \\
 & x_1 & + & 2x_2 & + & 3\hat{x}_3 & & & + & x_5 & & & = & 5 \\
 & 2x_1 & + & 2x_2 & + & \hat{x}_3 & & & & & + & x_6 & = & 6 \\
 & x_1 & , & x_2 & , & \hat{x}_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}
$$

$$
A = \left[\; A_1 \mid A_2 \mid A_3 \mid A_4 \mid A_5 \mid A_6 \;\right] = \left[\begin{array}{c|c|c|c|c|c} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \qquad b = \left[\begin{array}{c} 2 \\ 5 \\ 6 \end{array}\right]
$$

$$
x^T = \left[\; x_1 \quad x_2 \quad \hat{x}_3 \quad x_4 \quad x_5 \quad x_6 \;\right] \qquad c^T = \left[\; -3 \quad -1 \quad -3 \quad 0 \quad 0 \quad 0 \;\right]
$$

Feasible initial basis (suppose given): $B = [A_4|A_5|A_6]$

$$
\beta[1] = 4 \qquad \beta[2] = 5 \qquad \beta[3] = 6
$$

Iteration 1: steps 2–5

$$x_B^T = \left[\begin{array}{ccc} x_4 & x_5 & x_6 \end{array}\right] \qquad c_B^T = \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right]$$

$$B = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad B^{-1} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$u^T = c_B^T B^{-1} = \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right]$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} 2 \\ 1 \\ 2 \end{array}\right] = -3 - 0 = -3$$

$$\bar{c}_2 = c_2 - u^T A_2 = -1 - \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} 1 \\ 2 \\ 2 \end{array}\right] = -1 - 0 = -1 \qquad h = 2 \ (x_2 \text{ enters})$$

$$\bar{c}_3 = c_3 - u^T A_3 = -3 - \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} 1 \\ 3 \\ 1 \end{array}\right] = -3 - 0 = -3$$

Iteration 1: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \quad \begin{matrix} x_4 \\ x_5 \\ x_6 \end{matrix}$$

$$\bar{A}_h = B^{-1}A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$t = \arg\min \left\{ \tfrac{2}{1} \quad \tfrac{5}{2} \quad \tfrac{6}{2} \right\} = \arg\left(\frac{2}{1}\right) = 1 \qquad \rightsquigarrow x_4 \text{ leaves}$$

$\beta[1] = 2$ (column 2 replaces $\beta[1]$ that was 4)

Iteration 2: steps 2–5

$$x_B^T = \begin{bmatrix} x_2 & x_5 & x_6 \end{bmatrix} \qquad c_B^T = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = -3 - (-2) = -1$$

$$\bar{c}_3 = c_3 - u^T A_3 = -3 - \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = -3 + 1 = -2 \qquad h = 3$$

($\hat{x}_3$ enters)

$$\bar{c}_4 = c_4 - u^T A_4 = \quad 0 - \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 - (-1) = 1$$

Iteration 2: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \begin{matrix} x_2 \\ x_5 \\ x_6 \end{matrix}$$

$$\bar{A}_h = B^{-1}A_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$t = \arg\min \left\{ \ \tfrac{2}{1} \quad \tfrac{1}{1} \quad \text{X} \ \right\} = \arg\left(\frac{1}{1}\right) = 2 \qquad \rightsquigarrow x_5 \text{ leaves}$$

$\beta[2] = 3$ (column 3 replaces column $\beta[2]$ that was 5)

Iteration 3: steps 2–5

$$x_B^T = \left[\begin{array}{ccc} x_2 & \hat{x}_3 & x_6 \end{array} \right] \qquad c_B^T = \left[\begin{array}{ccc} -1 & -3 & 0 \end{array} \right]$$

$$B = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 2 & 1 & 1 \end{array} \right] \qquad B^{-1} = \left[\begin{array}{ccc} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{array} \right]$$

$$u^T = c_B^T B^{-1} = \left[\begin{array}{ccc} -1 & -3 & 0 \end{array} \right] \left[\begin{array}{ccc} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{array} \right] = \left[\begin{array}{ccc} 3 & -2 & 0 \end{array} \right]$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - \left[\begin{array}{ccc} 3 & -2 & 0 \end{array} \right] \left[\begin{array}{c} 2 \\ 1 \\ 2 \end{array} \right] = -3 - (4) = -7 \qquad h = 1$$

(x_1 enters)

It is not necessary to compute all reduced costs, stop as soon **one of them** is negative!

Iteration 3: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \quad \begin{matrix} x_2 \\ \hat{x}_3 \\ x_6 \end{matrix}$$

$$\bar{A}_h = B^{-1}A_1 = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$$

$$t = \arg\min \left\{ \ \tfrac{1}{5} \quad \mathsf{X} \quad \mathsf{X} \ \right\} = \arg\left(\frac{1}{5}\right) = 1 \qquad \rightsquigarrow x_2 \text{ leaves}$$

$\beta[1] = 1$ (column 1 replaces column $\beta[1]$ that was 2)

Iteration 4

$$x_B^T = \begin{bmatrix} x_1 & \hat{x}_3 & x_6 \end{bmatrix} \qquad c_B^T = \begin{bmatrix} -3 & -3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = \begin{bmatrix} -3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6/5 & -3/5 & 0 \end{bmatrix}$$

$$\bar{c}_2 = c_2 - u^T A_2 = -1 - \begin{bmatrix} -6/5 & -3/5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -1 - (12/5) = 7/5$$

$$\bar{c}_4 = c_4 - u^T A_4 = 0 - \begin{bmatrix} -6/5 & -3/5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 - (6/5) = 6/5$$

$$\bar{c}_5 = c_5 - u^T A_5 = 0 - \begin{bmatrix} -6/5 & -3/5 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 - (3/5) = 3/5$$

Optimal solution

Standard form (the one we solved by simplex method):

- $x_B^* \begin{bmatrix} x_1 \\ \hat{x}_3 \\ x_6 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 8/5 \\ 4 \end{bmatrix}$
- $x_1^* = 1/5$; $x_2^* = 0$; $\hat{x}_3^* = 8/5$; $x_4^* = 0$; $x_5^* = 0$; $x_6^* = 4$
- $z_{MIN}^* = c^T x^* = c_B^T x_B^T = \begin{bmatrix} -3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1/5 \\ 8/5 \\ 4 \end{bmatrix} = -27/5$

Optimal solution for the initial problem:

- $x_1^* = 1/5$
- $x_2^* = 0$
- $x_3^* = -\hat{x}_3^* = -8/5$
- first constraint satisfied with equality (since $x_4^* = 0$)
- second constraint satisfied with equality (since $x_5^* = 0$)
- third constraint satisfied with a slack of 4 (since $x_6^* = 4$)
- $z_{MAX}^* = -z_{MIN}^* = 27/5$.