Università degli Studi di Padova

Network Science

A.Y. 23/24

ICT for Internet & multimedia, Data science, Physics of data

Course overview

Network science 23/24

Lecturer

tomaso.erseghe@unipd.it
room 217, DEI/A

lectures: mon 8:30-10:00 & fri 10:30-12:00
www.dei.unipd.it

office hours: contact me by email

Università degli Studi di Padova

In this course you’ll also meet

Prerequisites

Basic requirements (that you already satisfy)

Calculus and linear algebra
Familiarity with a programming language
(Python, R, MatLab, C, Java, etc.)
Probability theory / Statistics

Other useful knowledge

Networking processes in economics, telecommunications, semantics, etc …
Otpimization, machine learning, deep learning, etc …

Which programming language?

- Python

  very good at scraping data (e.g., via Twitter APIs), polishing, plotting graphs, implementing algorithms

- R

  very good for memory storage, plotting graphs, implementing algorithms

- MatLab

  An alternative for algorithms and graph plotting

  University license available

  https://www.ict.unipd.it/servizi/servizi-utenti-istituzionali/contratti-software-e-licenze/matlab

Università degli Studi di Padova

What about you?

Why did you pick the course?

Which is your background?
Who knows about deep learning?

Do you know Python?
and CoLab?

What do you expect
from this course?

Do you have a laptop?

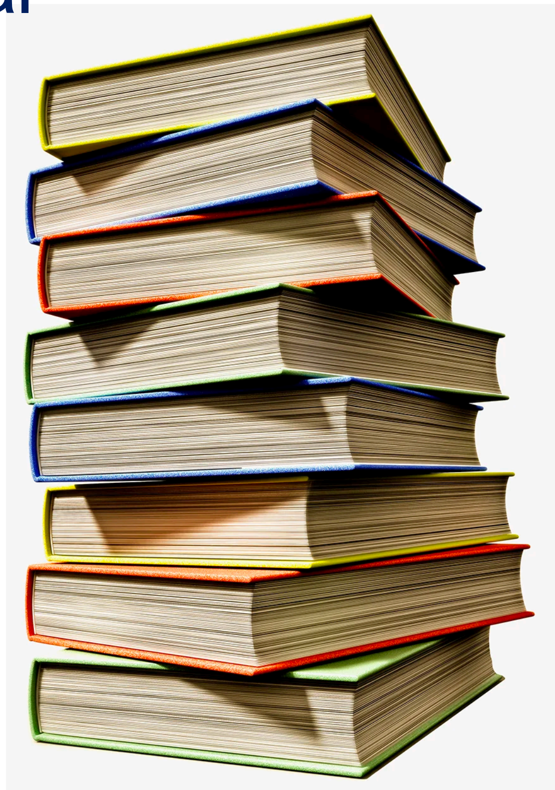
Are you interested in an
interdisciplinary work?

Textbooks

No textbook! ☺

Slides/videos & additional material available

@ stem.elearning.unipd.it



A few useful books

but we will not cover them completely

- A.L. Barabási, «Network science»

  http://barabasi.com/networksciencebook

  (these slides = Ch.1 "Introduction")

- J. Lescovec, «Machine learning with graphs»

  http://web.stanford.edu/class/cs224w

- M. Newman, «Networks: an introduction»

  Oxford University Press, 2010

- R. van der Hofstad, «Random graphs and complex networks»

  http://www.win.tue.nl/~rhofstad/NotesRGCN.html

Project based exam

- **Written exam**
  - multiple choice questions (30 min)
  - 2 open questions (30+30 min)
- **Project**
  - extract network analytics using your preferred programming language(s)
  - oral presentation: slides + code
  - 10 min presentation (slides)
  - 5 min for questions

Final grade: 50% written exam, 50% project

Exam sessions

Written exam:

- Jan 15, 2024 (Mon) - 8:30, Me
- Feb 2, 2024 (Fri) - 9:00, Le
- Feb 20, 2024 (Tue) - 9:00, Le
- July 3, 2024 (Wed) - 9:00, Le
- Sep 11, 2024 (Wed) - 9:00, Le

Oral sessions to be organised in the days that follow, plus:

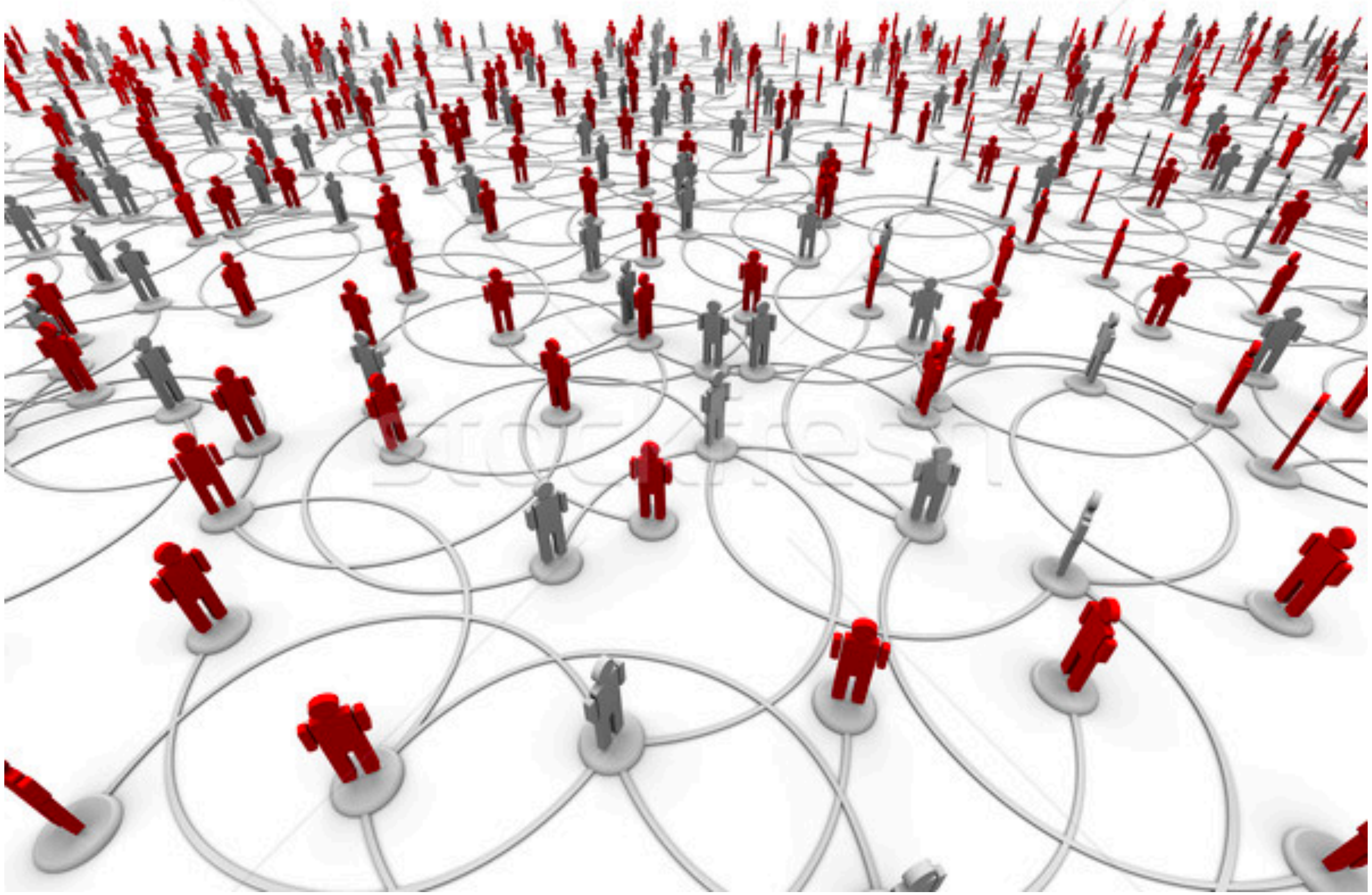
- IP day Feb 8, 2024 (Thu) - 9:00, Aula Magna

PS: You will be asked to enrol in
www.uniweb.unipd.it

Contents

a brief overview

This course is about networks

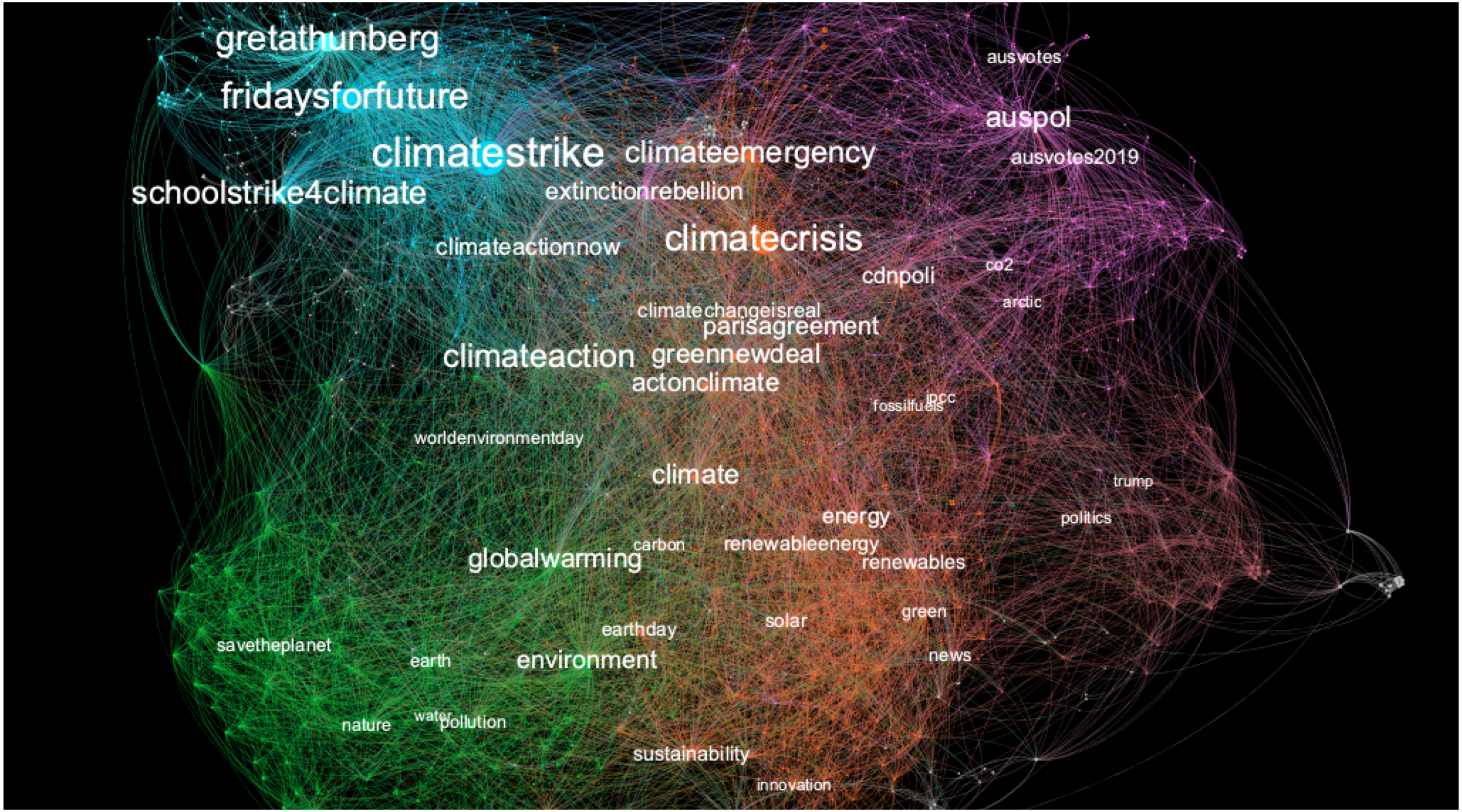


Network = anything that interconnects

e.g., people sharing friendship in a social network platform

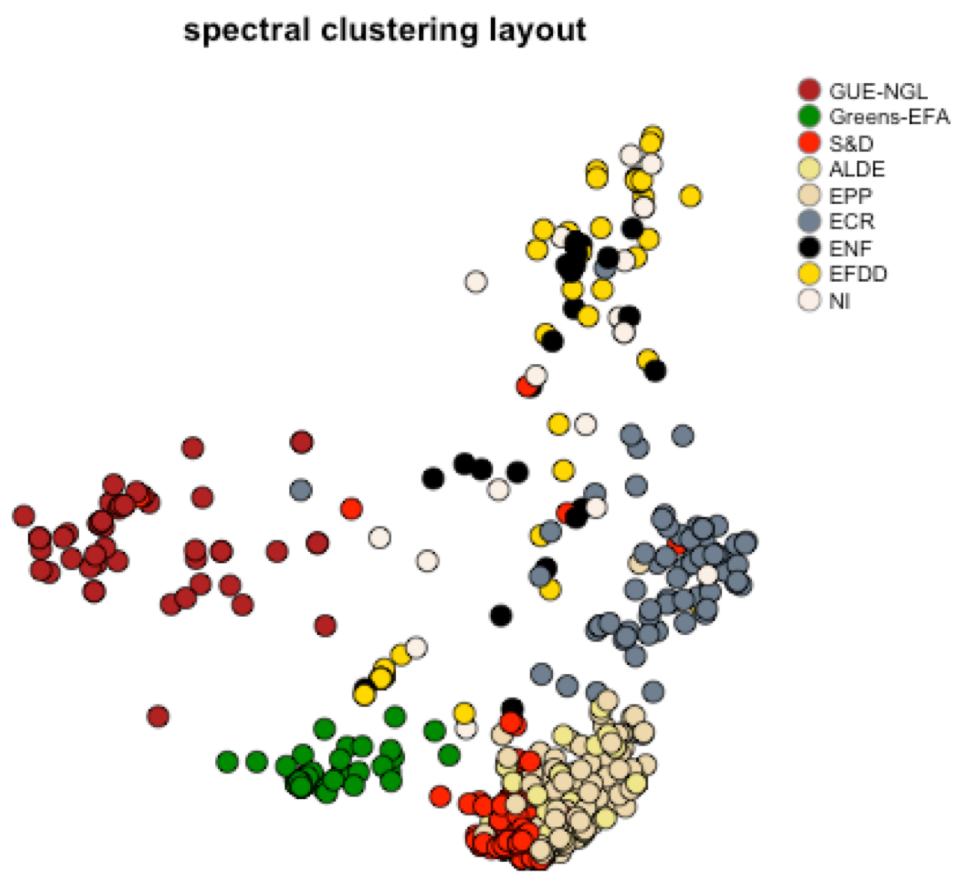
Network example

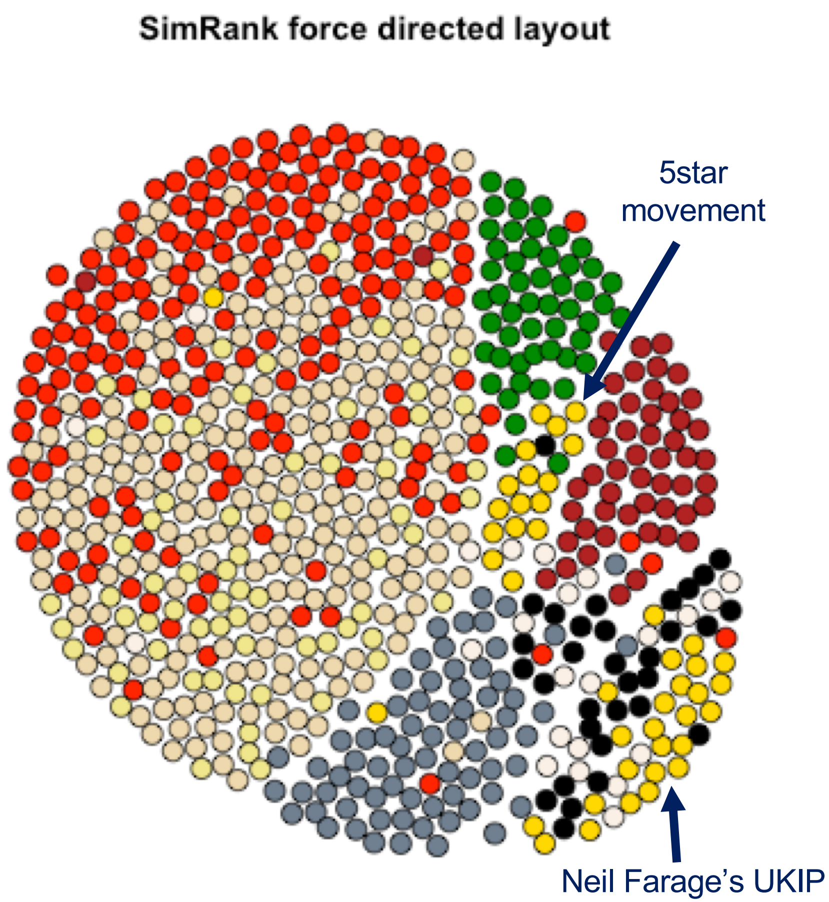
2019 hashtag network related to #climatechange from Twitter, after #gretathunberg



Network examples (cont'd)

April-May 2016 political network (votes at the EU parliament)

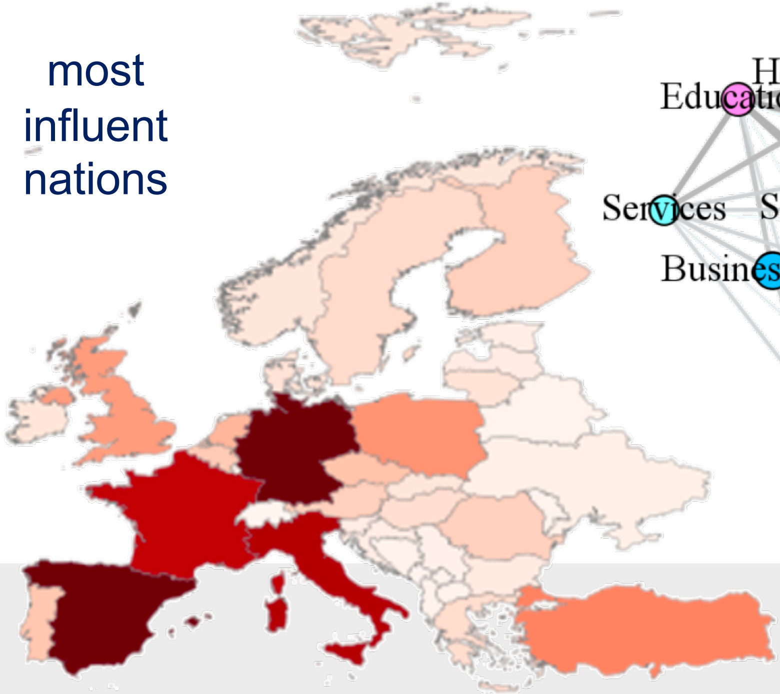




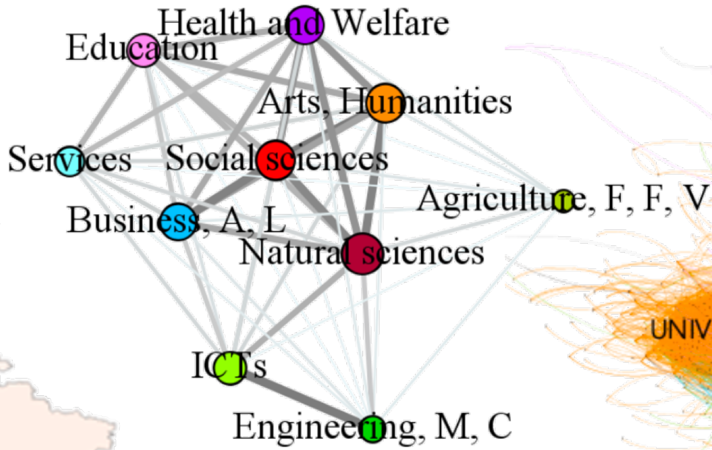
Network examples (cont'd)

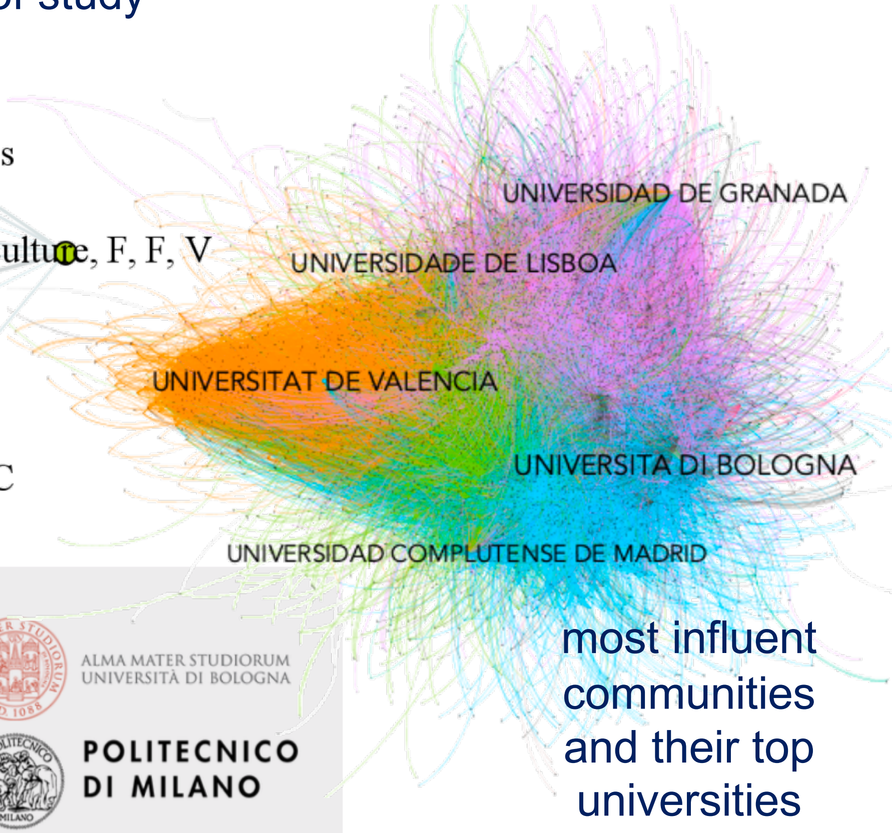
Erasmus exchanges network 2019

most influent nations



fields of study





most influent communities and their top universities

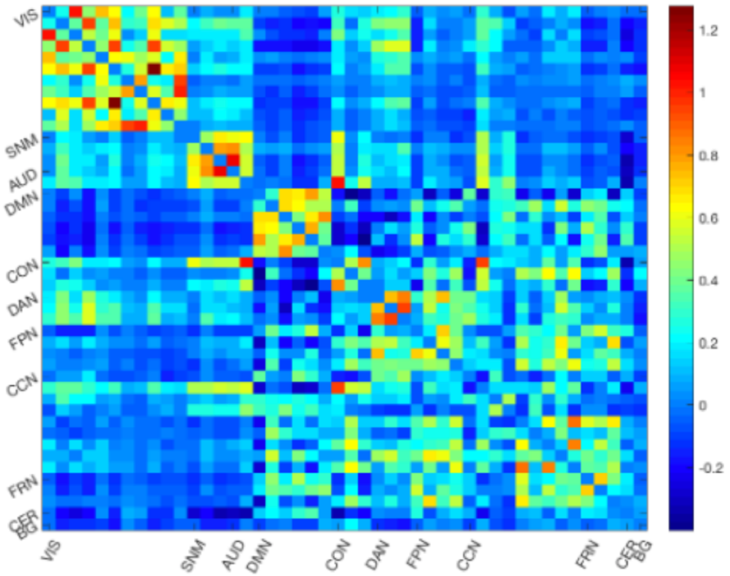
Which are the more connected Italian institutions?

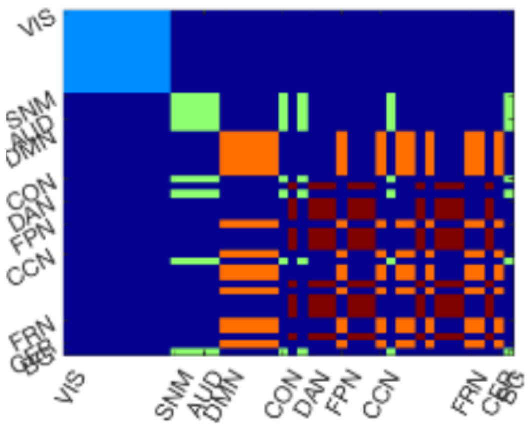
01

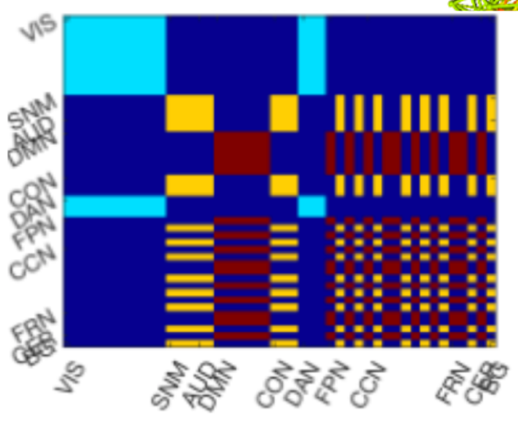
- Alma Mater Studiorum Università di Bologna
- Politecnico di Milano
- Università degli Studi di Padova
- Sapienza Università di Roma
- Università degli Studi Firenze

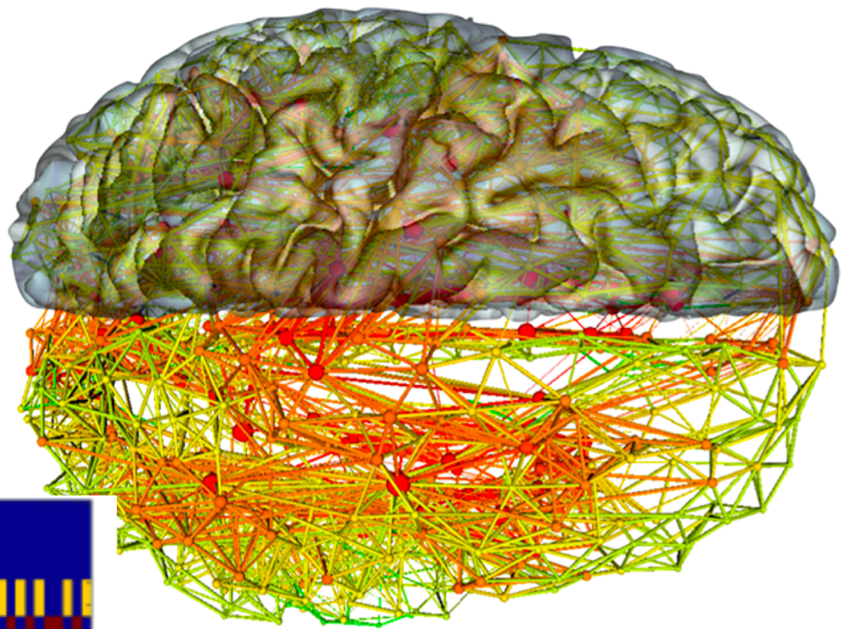
Network examples (cont'd)

the brain network – functional connectivity









What is then network science?

Network science

From Wikipedia, the free encyclopedia

For other uses, see Network (disambiguation).

Network science is an academic field which studies complex networks such as telecommunication networks, computer networks, biological networks, cognitive and semantic networks, and social networks, considering distinct elements or actors represented by *nodes* (or *vertices*) and the connections between the elements or actors as *links* (or *edges*). The field draws on theories and methods including graph theory from mathematics, statistical mechanics from physics, data mining and information visualization from computer science, inferential modeling from statistics, and social structure from sociology. The United States National Research Council defines network science as "the study of network representations of physical, biological, and social phenomena leading to predictive models of these phenomena."[1]

… what about social network analysis?

Social network analysis

From Wikipedia, the free encyclopedia

Social network analysis (**SNA**) is the process of investigating social structures through the use of networks and graph theory.[1] It characterizes networked structures in terms of *nodes* (individual actors, people, or things within the network) and the *ties*, *edges*, or *links* (relationships or interactions) that connect them. Examples of social structures commonly visualized through social network analysis include social media networks,[2][3] memes spread,[4] information circulation,[5] friendship and acquaintance networks, business networks, knowledge networks,[6][7] difficult working relationships,[8] social networks,

Social network analysis has emerged as a key technique in modern sociology. It has also gained a significant following in anthropology, biology,[12] demography, communication studies,[3][13] economics, geography, history, information science, organizational studies,[6][8] political science, public health,[14][7] social psychology, development studies, sociolinguistics, and computer science[15] and is now commonly available as a consumer tool (see the list of SNA software).[16][17][18][19]

And how do we study networks?

With a holistic character

(the whole is greater than the sum of its parts)

With mathematical rigour

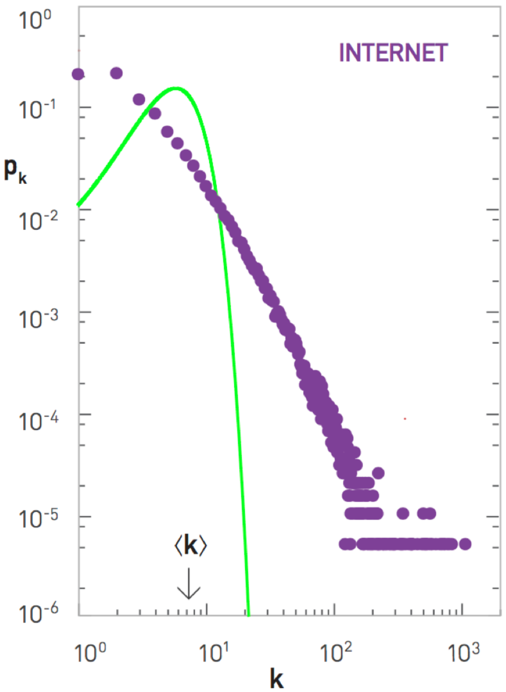
The approach is

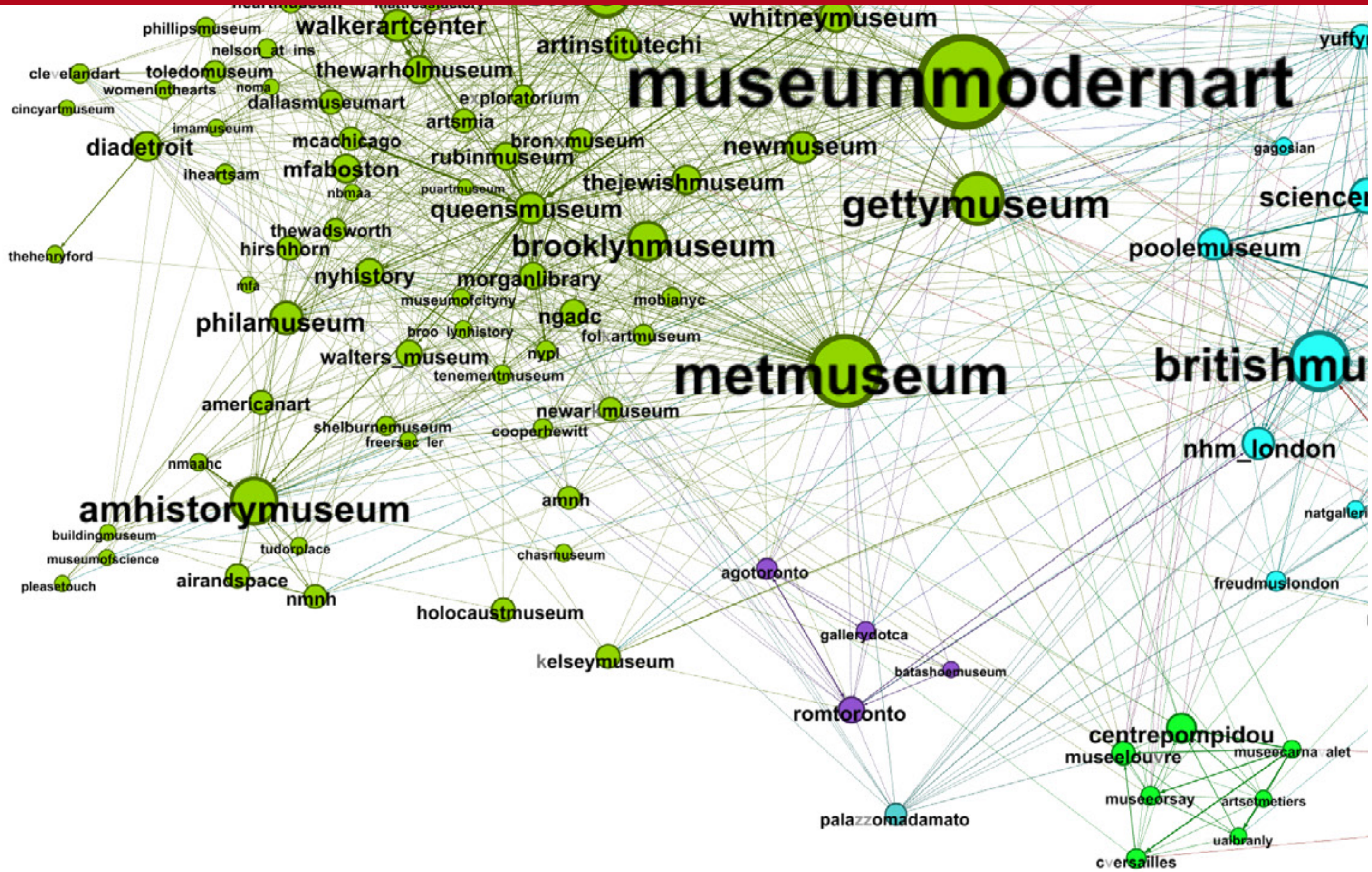
empirical (driven by concrete data),
precise (requires a proper formalism),
interdisciplinary (can be applied to several fields), and
challenging (in data size and in objectives)

UNIVERSITÀ DEGLI STUDI DI PADOVA

And what do we study?

the power law



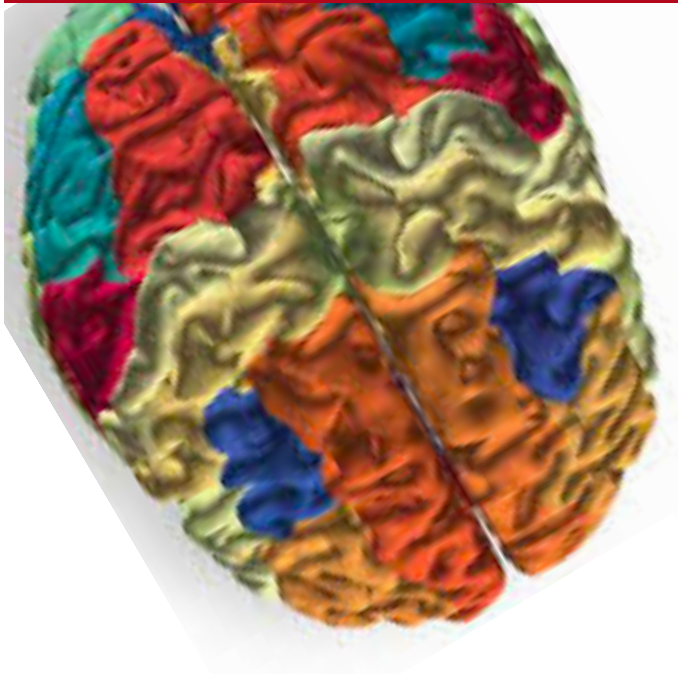


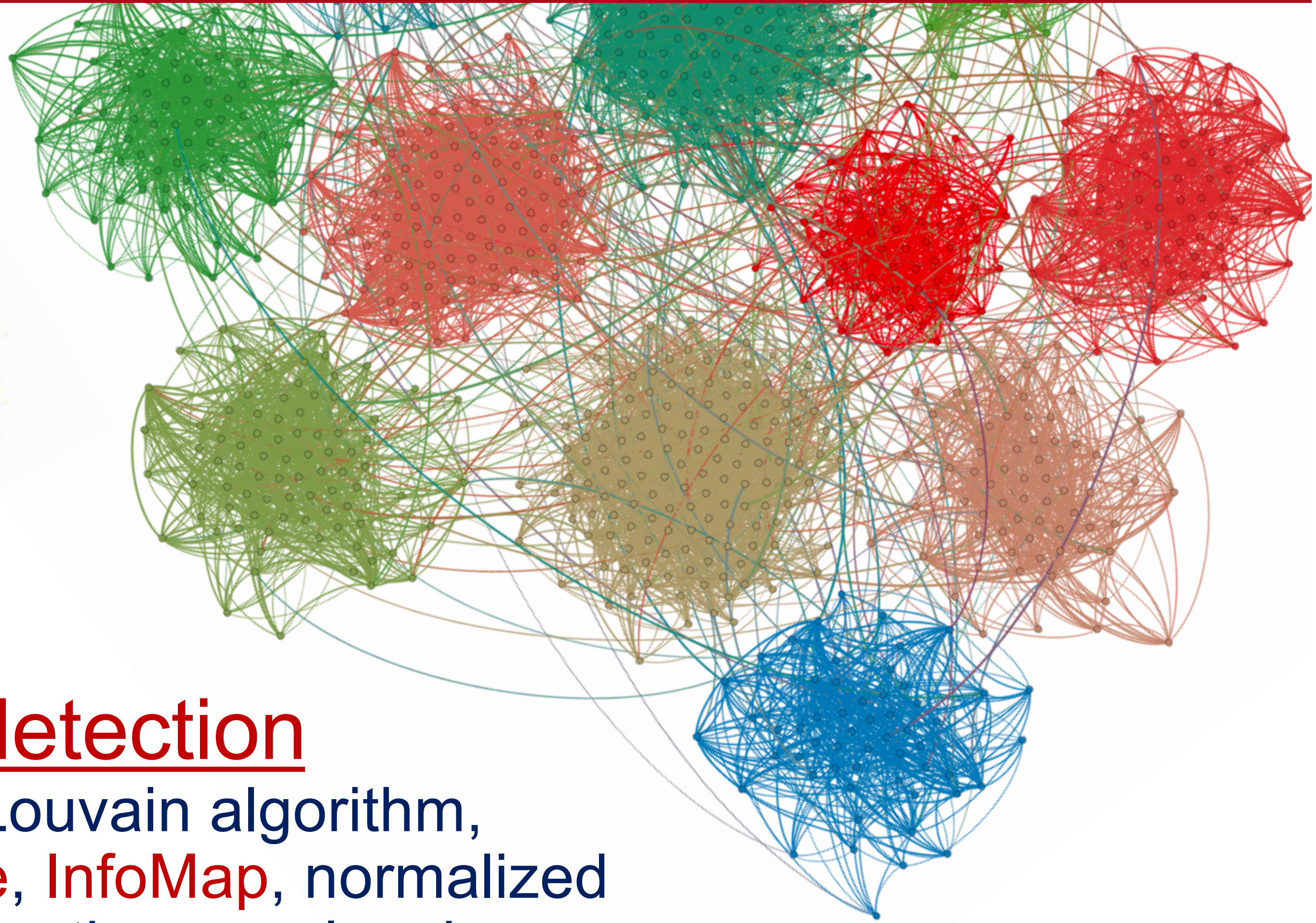
many network analytics, e.g., centrality

- degree, PageRank, HITS, betweenness, etc.

Google

And what do we study? (cont'd)





community detection

- modularity, Louvain algorithm, conductance, InfoMap, normalized mutual information, overlapping communities, BigCLAM, stochastic block models

And what do we study? (cont'd)

semantic networks

- network layout, data collection, sentiment analysis, BERTAgent, topic detection, latent Dirichlet allocation, variational autoencoders, BERTopic and the Transformer architecture

Project

a brief overview

Università degli Studi di Padova

What about the project?

create your own group (1 to 3 people)

choose your dataset (possibly create your own dataset)

apply the ideas learned during the course

show that you can do clever things

try extracting meaningful measures/analytics that describe an interesting aspect of your network

write good code

each contributor to the group should focus on a different aspect (no everything together)

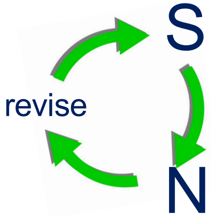
present the project in a clear and convincing way, using clear and convincing plots

UNIVERSITÀ DEGLI STUDI DI PADOVA

What about interdisciplinary projects?

mainly related to semantic networks

in collaboration with the twin course of Social Network Analysis @ Communication Strategies



SNA students suggest research questions

NS students conceive appropriate algorithmic solutions

in brainstorming sessions
the instructor will help/give feedback ☺

Your SNA colleagues

Interdisciplinary projects 23/24

MIME Master's degree ICT Internet Multimedia Engineering

Dipartimento di Ingegneria dell'Informazione

DiSLL Dipartimento di Studi Linguistici e Letterari

Dipartimento di Psicologia dello Sviluppo e della Socializzazione

UNIVERSITÀ DEGLI STUDI DI PADOVA

IP examples from past years

on **Twitter**

- 2019 – Evolution of Climate Change Perception on Twitter - Focusing on Greta Thunberg Impact
- 2019 – UN Women Twitter profile's reaction to the MeToo movement
- 2020 – NBA and Premier League players around #blacklivesmatter and the racial issue on Twitter
- 2020 – Republicans vs Democracts on Twitter
- 2020 – Haters gonna (make you) hate - Semantic analysis of hate during 2019 European elections
- 2021 – Sports brands and eco-sustainability
- 2022 – Sexism in Politics
- 2022 – What is the perception around the world in terms of Menstruation Stigma in 2021?
- 2022 – Cancel culture on social media - Social network analysis on famous cases of cancellation

on **TikTok**

- 2022 – PoliTok: How do Italian politicians use TikTok as tool to promote their political ideas and influence the young generation during the 2022 elections?

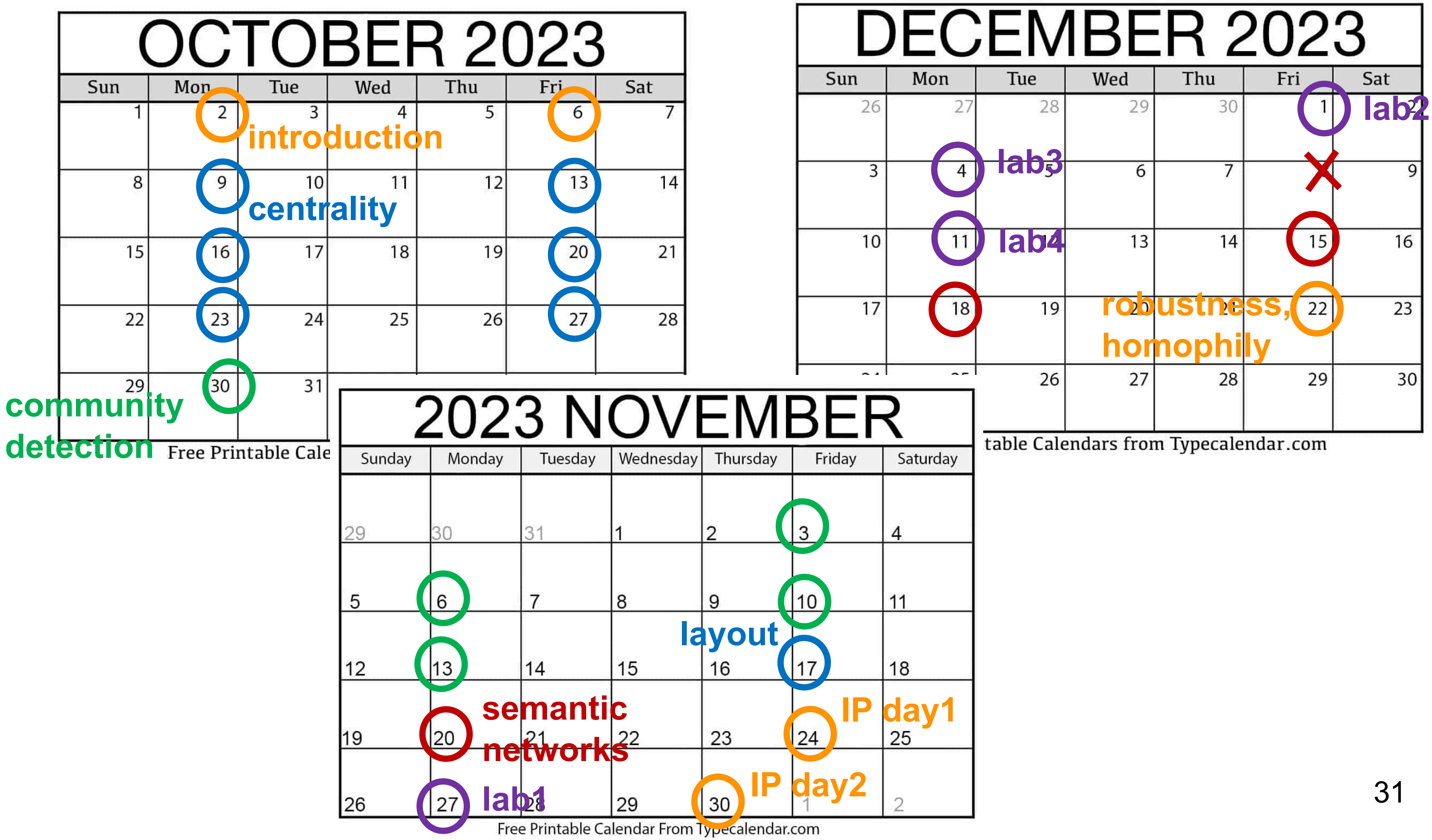
other

- 2019 – Noodles and Spaghetti - How people make pasta in eastern countries
- 2021 – Erasmus+ Programme: a social network analysis study of the 2014-2019 exchanges
- 2021 – Nationality vs. movie prestige: from the Oscars to International Film Festivals

Calendar

tentative

Calendar


OCTOBER 2023
introduction
centrality
community detection
Free Printable Cale
DECEMBER 2023
lab2
lab3
lab4
robustness, homophily
table Calendars from Typecalendar.com
2023 NOVEMBER
layout
semantic networks
IP day1
IP day2
lab1
Free Printable Calendar From Typecalendar.com

Contents

- Misc (4 lectures)
  introduction; graphs; graph layout: ForceAtlas2, Gephi, UMAP; robustness; homophily
- Centrality (6 lectures)
  degree centrality, power law, preferential attachment, fitness, Bianconi-Barabasi model, scale-free regime, PageRank, convergence properties, Local PageRank, Approximate and signed PageRank, Row-normalized PageRank, HITS, closeness, betweenness, clustering coefficient
- Community detection (5 lectures)
  modularity, Louvain algorithm, consensus clustering, Modularity for directed and signed networks and overlapping communities, Minimum cut criterion, spectral clustering, InfoMap, Normalized mutual information, F1 score, Dice correlation, BigCLAM, stochastic block models, Dendrograms, Girvan-Newman, HDBSCAN
- Semantic networks (3 lectures)
  Reddit, cleaning steps: spaCy, LIWC, BERTAgent, semantic networks, TF-IDF, modularity, latent Dirichlet analysis, variational autoencoders, BERTopic, performance comparison
- Python labs (4 lectures)
- IP projects (2 lectures)

To do list

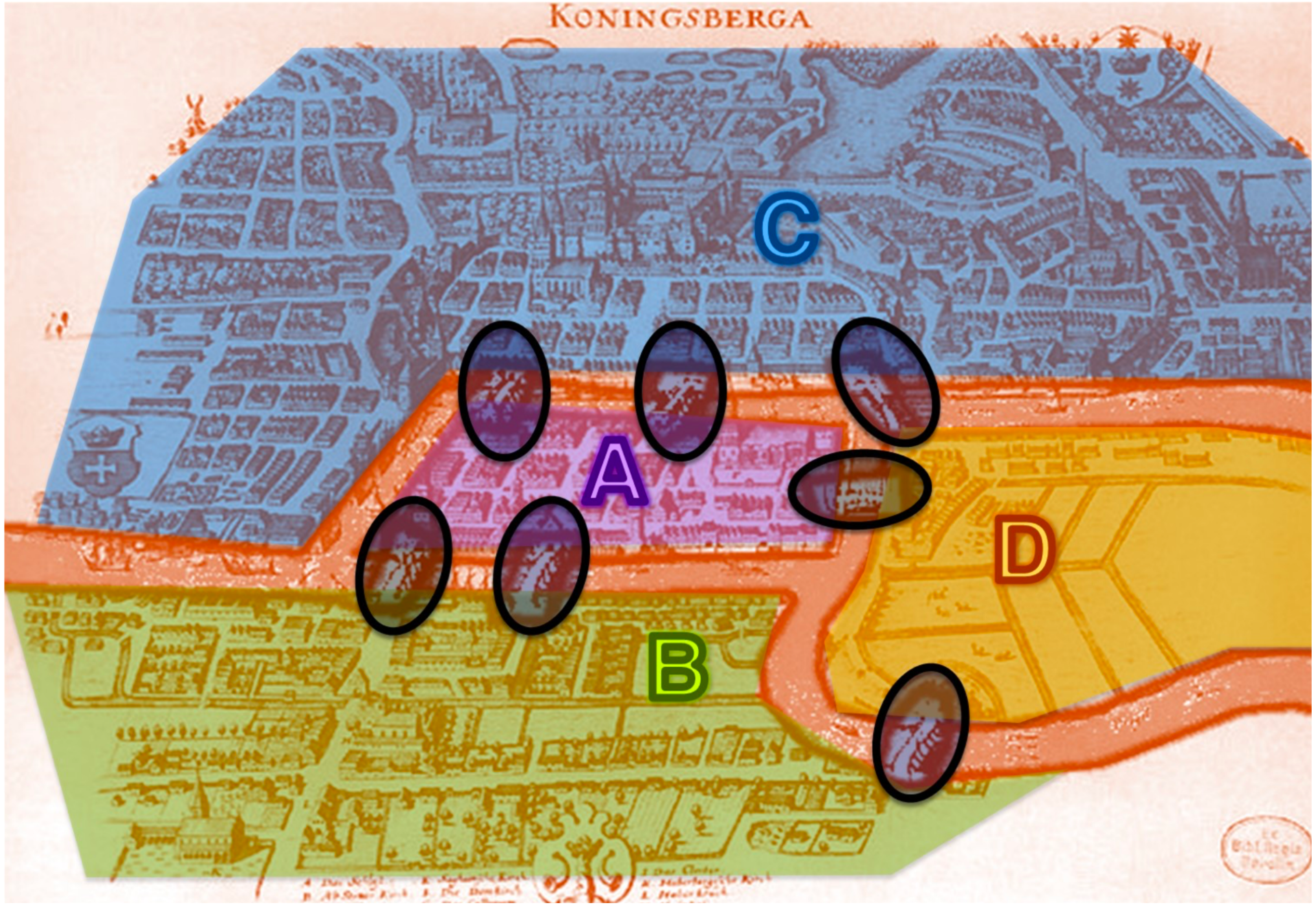
- ❑ Enrol @ stem.elearning.unipd.it ☺
- ❑ Have a laptop available
- ❑ Ensure you know Python's basics
- ❑ Activate a Google account (with the @unipd.it email)
  - → Google Drive
  - → Google CoLab
- ❑ Activate a Reddit account (using Google's account)
  - → Reddit apps https://www.reddit.com/prefs/apps
- ❑ Install Gephi on your laptop https://gephi.org/
- ❑ Review everything you know about deep learning and/or optimization
- ❑ Organize yourselves into working groups (max 3 people)

Graphs

an introduction

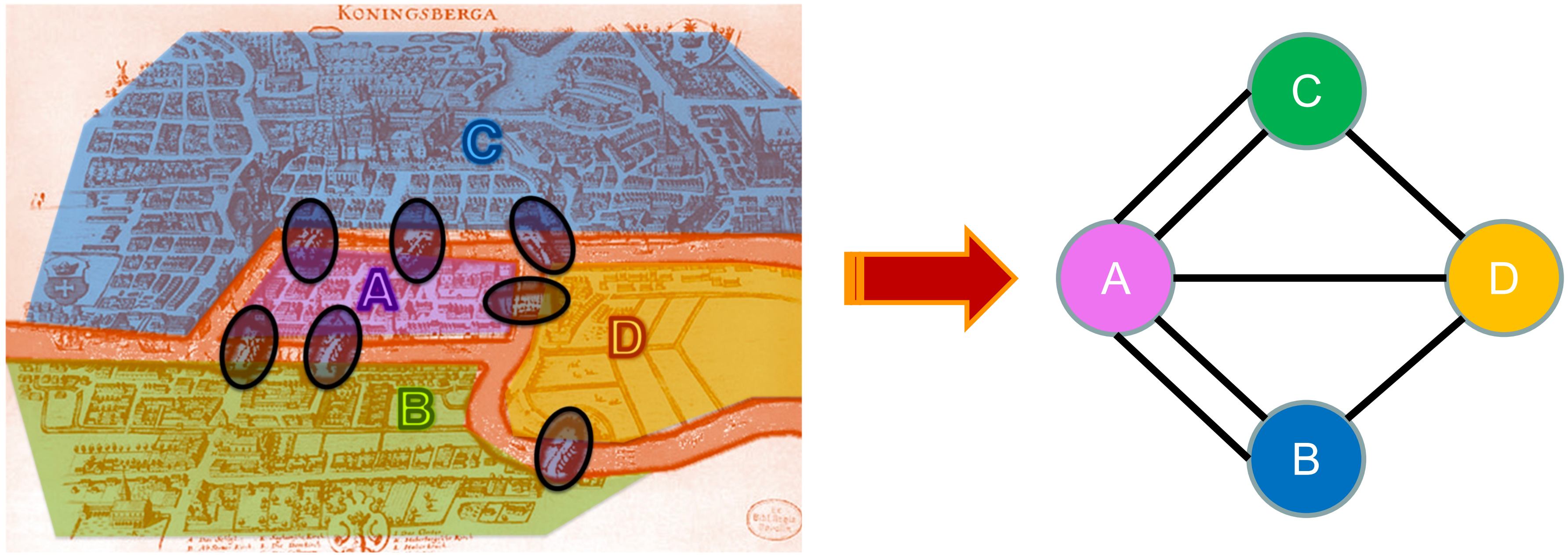
Euler and the 7 bridges of Könisberg

(Prussia, 1736) today Kaliningrad



How to walk through the city by crossing each bridge only once?

Networks as graphs



Graph $\mathcal{G}$ ($\mathcal{V}$,$\mathcal{E}$): network

- Vertices (set $\mathcal{V}$) : nodes, people, concepts
- Edges (set $\mathcal{E}$): links, relations, associations

Edges ↑ *mathematics*; links ↑ *technology*; relations ↑ *social psychology*; associations ↑ *social cognition*

Directed versus undirected

- A connection relationship can have a privileged direction or can be mutual
  - Either a directed or an undirected link

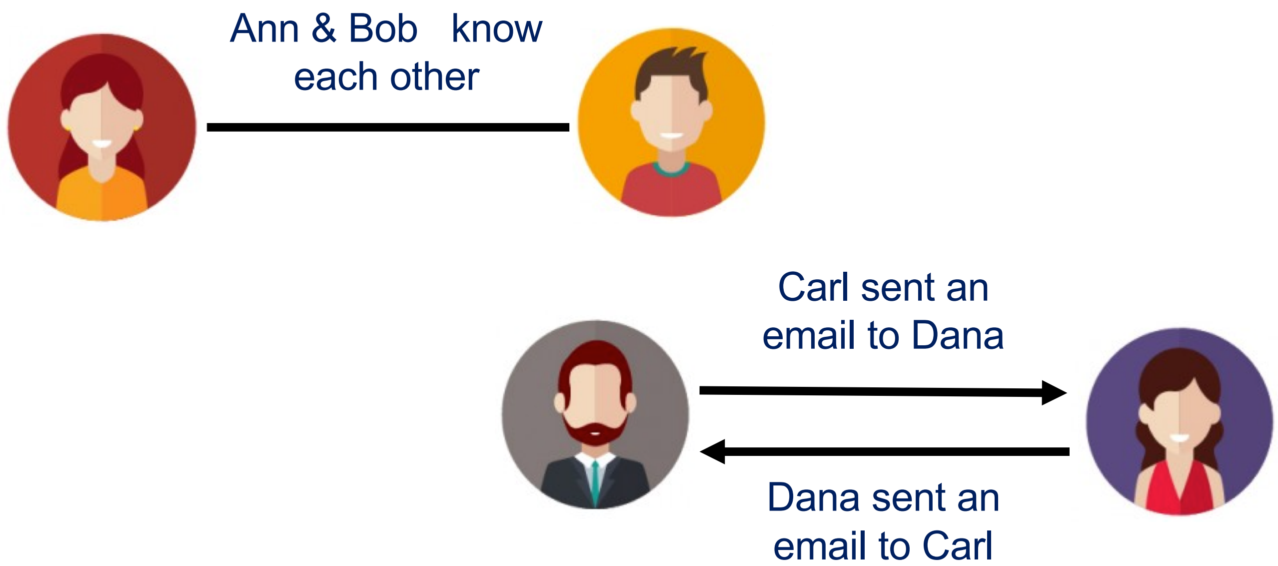
- If the network has only (un)directed links, it is also called itself (un)directed network
  - Certain networks can have both types

Some examples

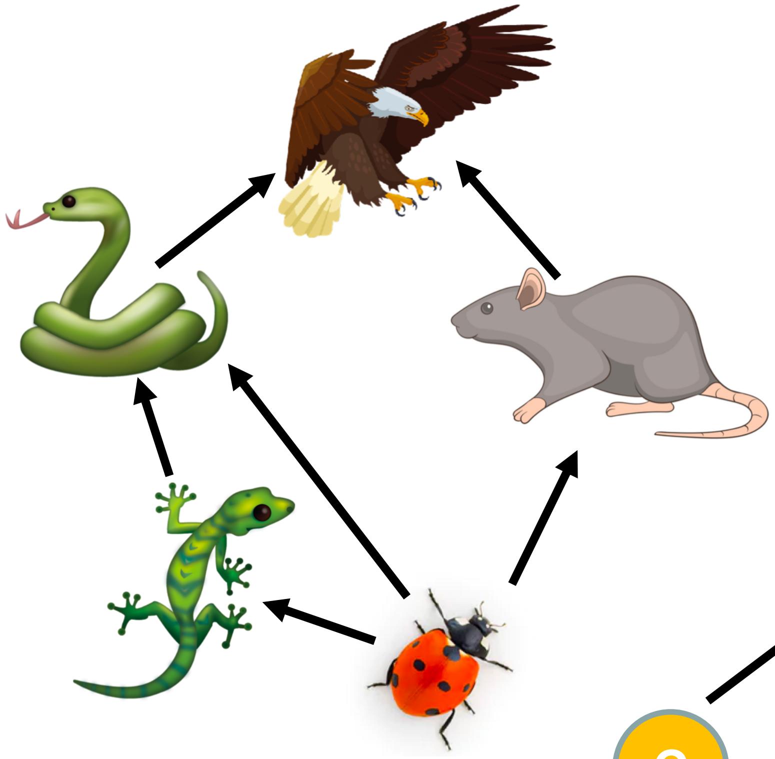
network	nodes and links	type
Facebook	Profiles and friendship	undirected
Instagram	Accounts and followers	directed
the www	Webpages and links	directed
citation network	Papers and references	directed
social network	People and friends/acquaintances	undirected
movie network	Actors and co-starring	undirected
genealogy	People and parenthood	directed

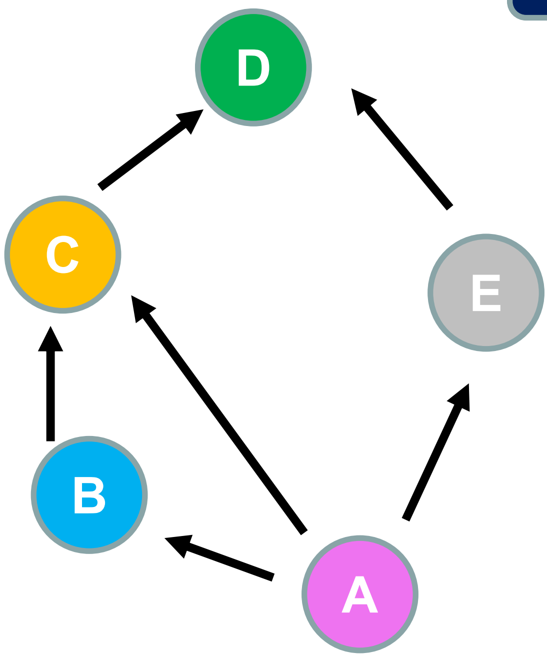
Directed versus undirected

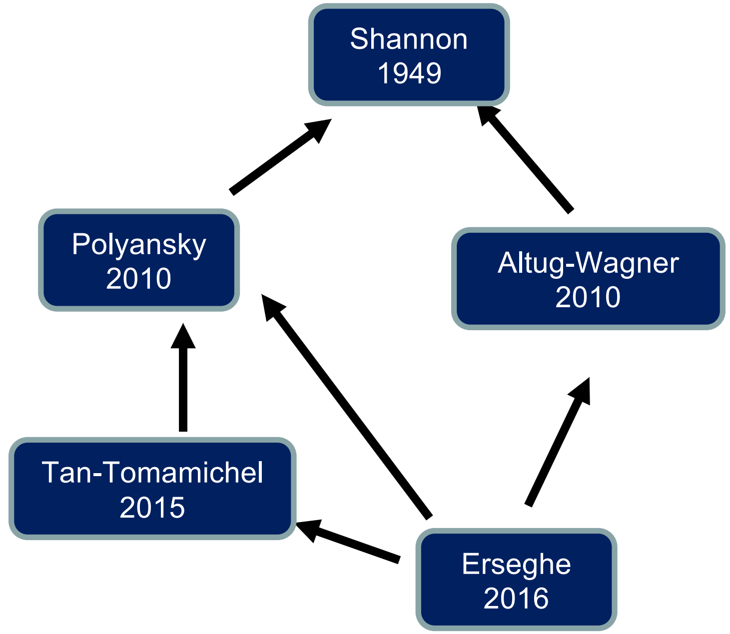
- At first glance undirected → directed by duplicating links, but not necessarily quite the same though



Generality of representation





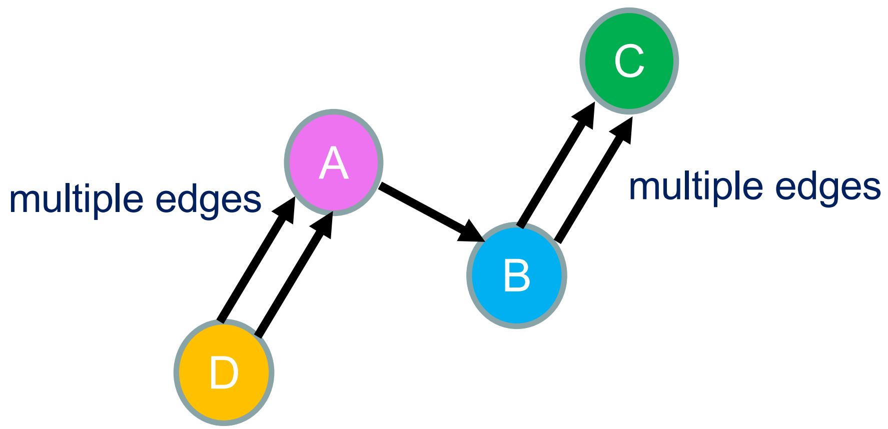


Weighted graps

and adjacency matrix

Multi-graphs

- Multi-graphs (or pseudo-graphs)
  Some network representations require multiple links (e.g., number of citations from one author to another)

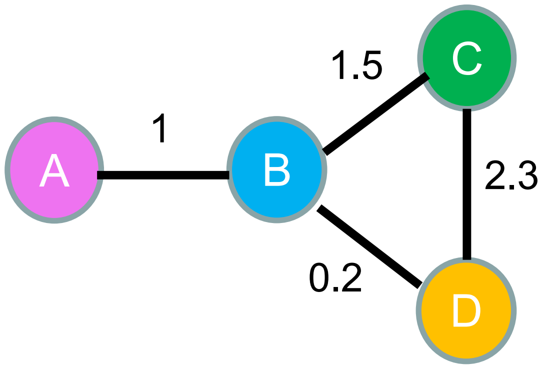


Weighted graphs

Weighted graph

Usually a **weight** w_{ij} is associated to a link $(i,j) \in \mathcal{E}$, e.g., to underline that the links are not identical (strong/weak relationships)

Can be seen as a generalization of multi-graphs (weight = # of links)

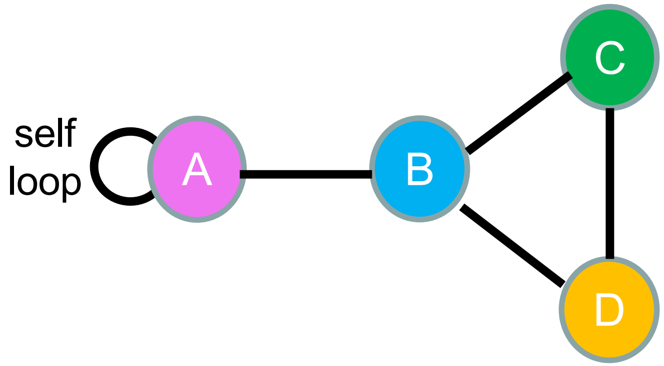


Self interactions

- In many networks nodes do not interact with themselves

  if $i \in \mathcal{V}$ then $(i,i) \notin \mathcal{E}$

- To account for self-interactions, we add loops to represent them



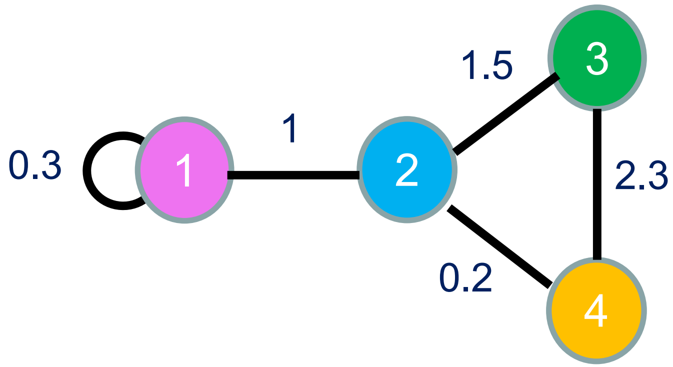
Adjacency matrix

- An adjacency matrix $A = [a_{ij}]$ associated to graph $\mathcal{G}\ (\mathcal{V},\mathcal{E})$ has

  entries $a_{ij} = 0$ for $(i,j) \notin \mathcal{E}$ (not a connection)

  if nodes i and j are connected then $a_{ij} \neq 0$

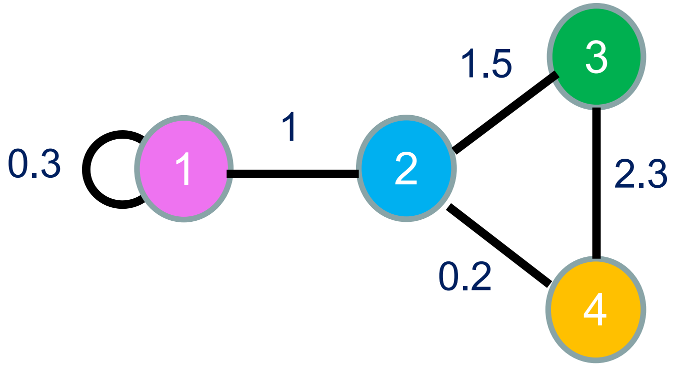
  in plain graphs $a_{ij} = 1$ for $(i,j) \in \mathcal{E}$



$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0.2 \\ 0 & 1.5 & 0 & 2.3 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$

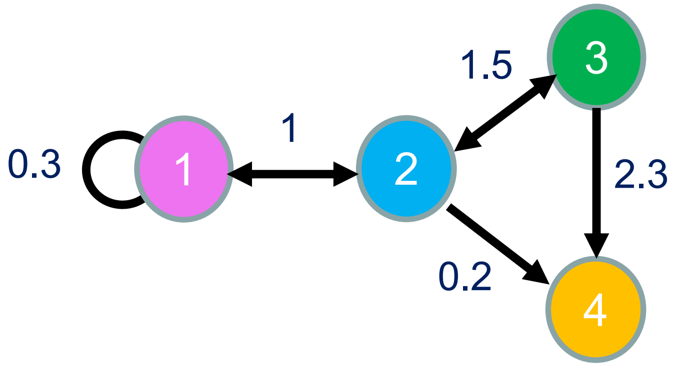
Symmetries

- Undirected graph = symmetric matrix



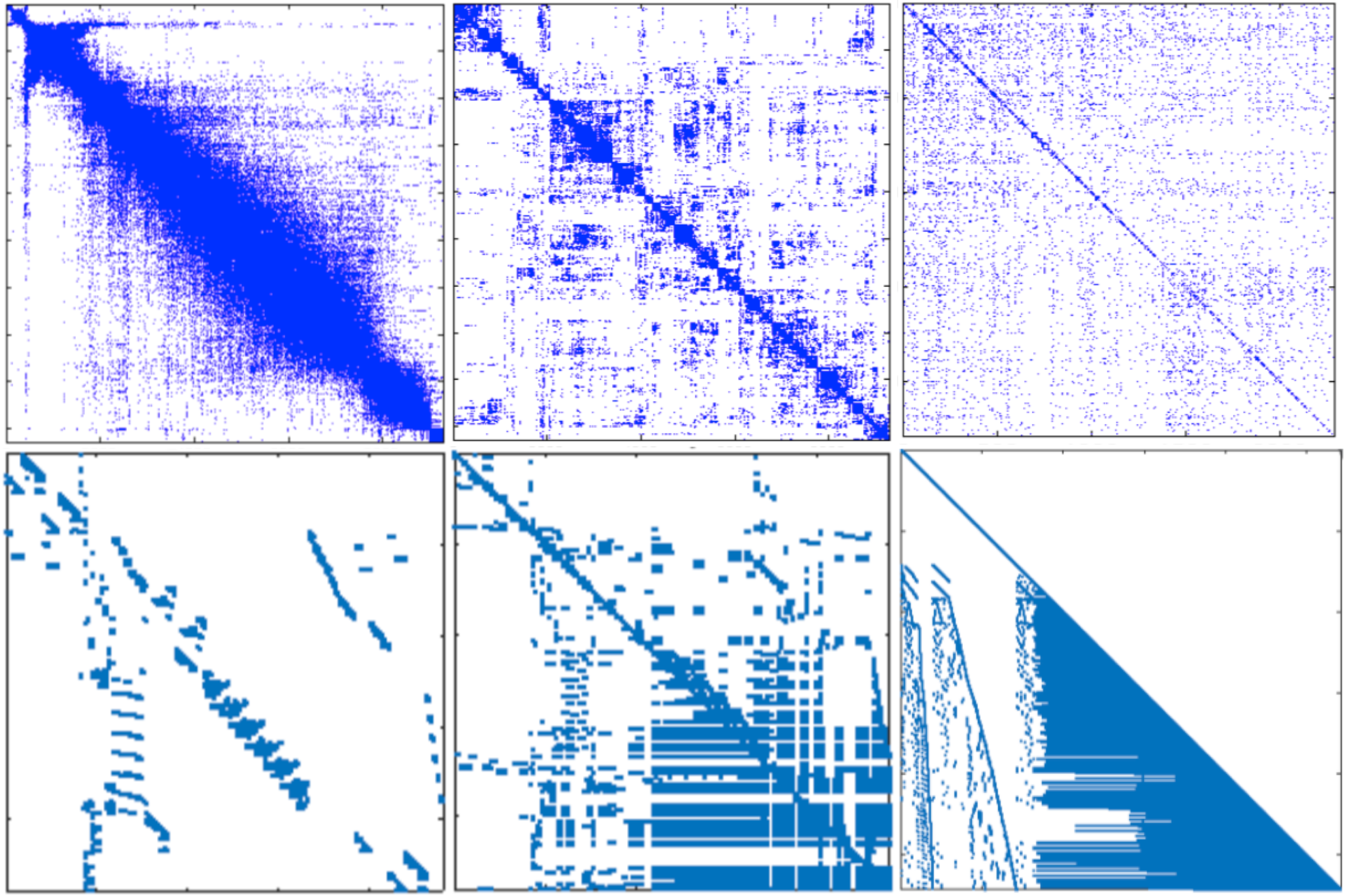
$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0.2 \\ 0 & 1.5 & 0 & 2.3 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$

- Directed graph = asymmetric matrix



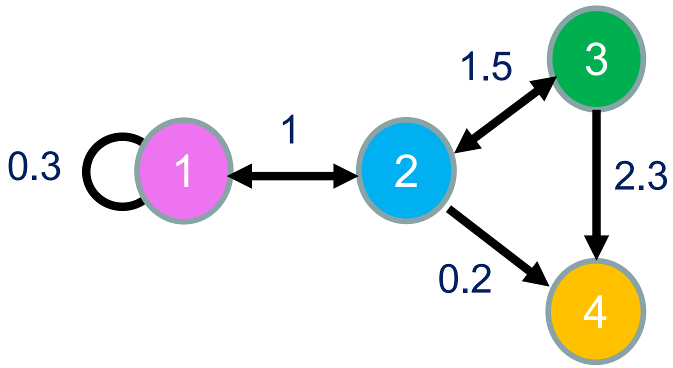
$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$

Symmetries



Convention

- The weight a_{ij} is associated to
  i th row
  j th column
  directed edge *j* → *i* starting from node *j* and leading to node *i*



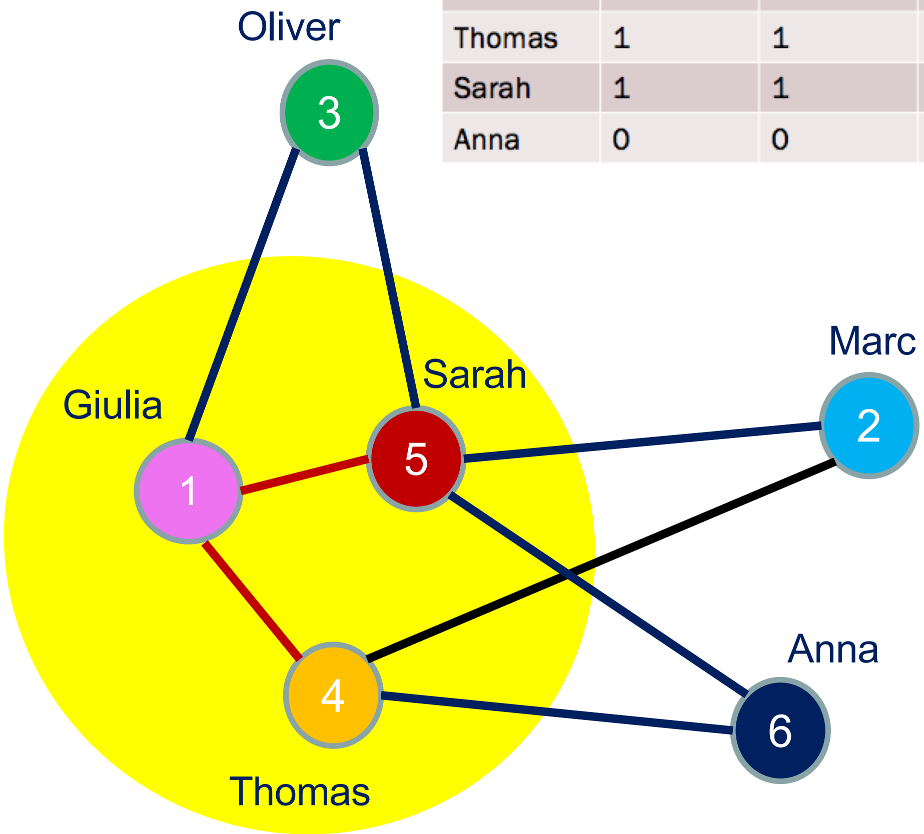
$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$

a_{24}, a_{34}, a_{42}, a_{43}

Question

which of these representations do you like best?

	Giulia	Marc	Oliver	Thomas	Sarah	Anna
Giulia	X					
Marc	0	X				
Oliver	1	0	X			
Thomas	1	1	0	X		
Sarah	1	1	1	0	X	
Anna	0	0	0	1	1	x



$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

edge list

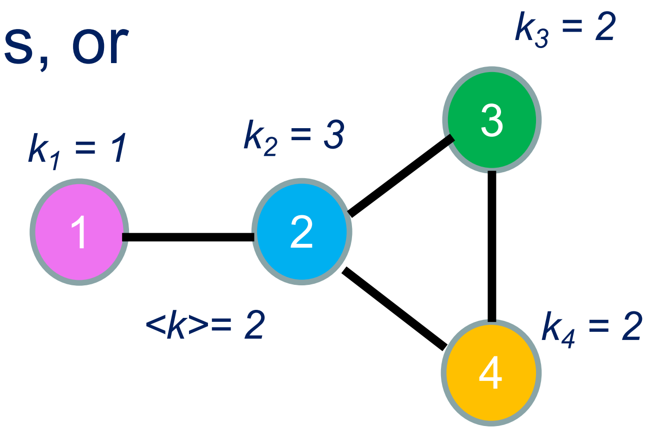
1 → 3
1 → 4
1 → 5
2 → 4
2 → 5
3 → 5
4 → 6
5 → 6

Node degree

in directed and undirected networks

Node degree
undirected networks

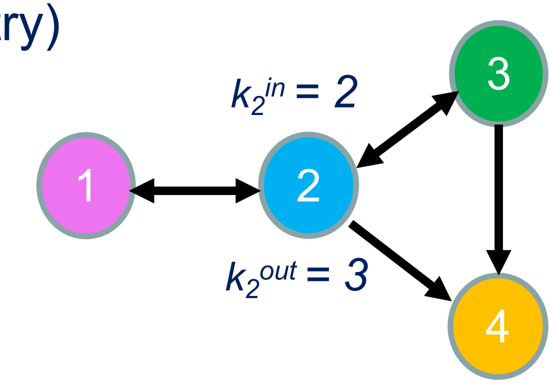
- The degree k_i of node i in an undirected networks is
  - the # of links i has to other nodes, or
  - the # of nodes i is linked to



- The # of nodes is $N = |\mathcal{V}|$
- The # of edges is $L = |\mathcal{E}| = \frac{1}{2} \sum_i k_i$
- The average degree is $<k> = \sum_i k_i / N = 2L / N$

Node degree
directed networks

- For directed networks we distinguish between
  - in-degree k_i^{in} = # of entering links
  - out-degree k_i^{out}= # of exiting links
  - (undirected: $k_i^{in} = k_i^{out}$ due to the symmetry)

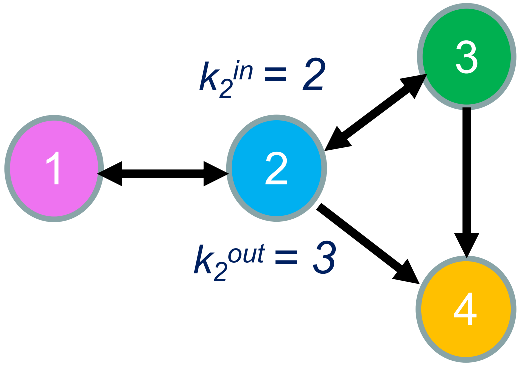


- The # of links is $L = \sum_i k_i^{in} = \sum_i k_i^{out}$
  (no need for factor ½)

  The average # of links is $\langle k \rangle = L / N$

Adjacency matrix and degree

- The in (out) degree can be obtained by summing the adjacency matrix over rows (columns)



no self-loops in this case!!!

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$k_2^{in} = 2$

$k_2^{out} = 3$

- A few useful linear algebra expressions

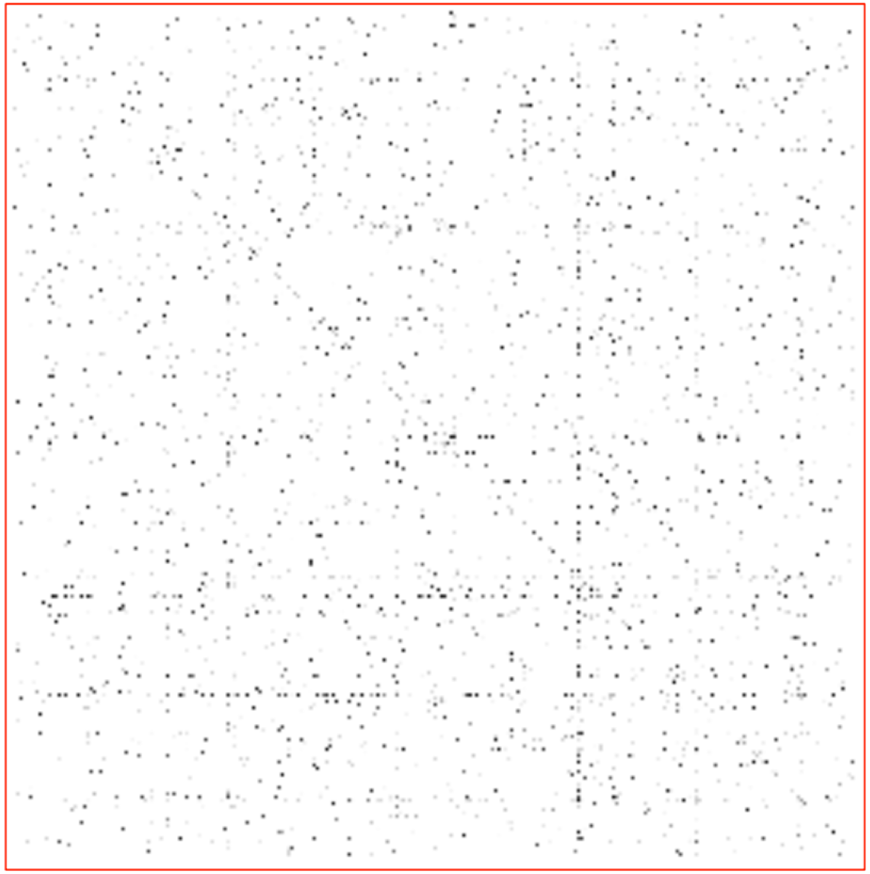
$$\boldsymbol{k}^{in} = \boldsymbol{A} \cdot \boldsymbol{1} \qquad \boldsymbol{k}^{out} = \boldsymbol{A}^T \cdot \boldsymbol{1} = (\boldsymbol{1}^T \cdot \boldsymbol{A})^T$$

valid also for weighted graphs

Real networks are sparse

- The adjacency matrix is typically sparse

  good for tractability !

$A =$ 

protein interaction network

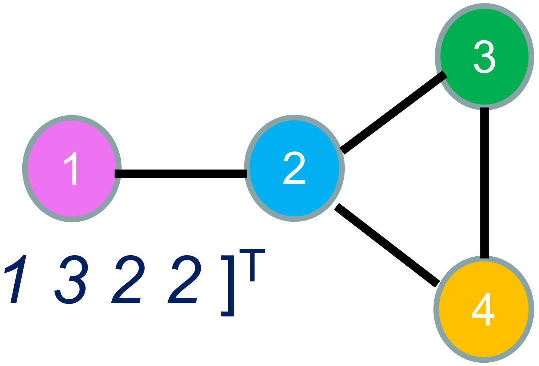
Real networks are sparse

- The maximum degree is $<k>_{max} = N - 1$
- In real networks $<k> \ll N - 1$

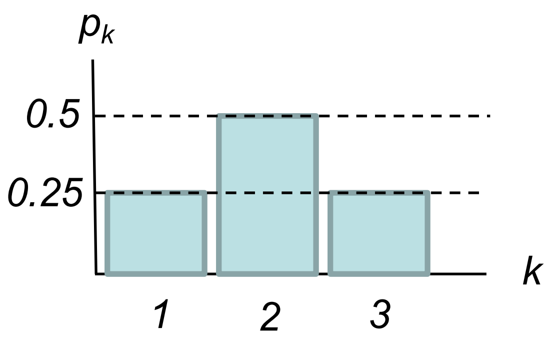
network	type	N	L	<k>
www	directed	3.2×10^5	1.5×10^6	4.60
Protein	directed	1870	4470	2.39
Co-authorships	undirected	23133	93439	8.08
Movie actors	undirected	7×10^5	29×10^6	83.7

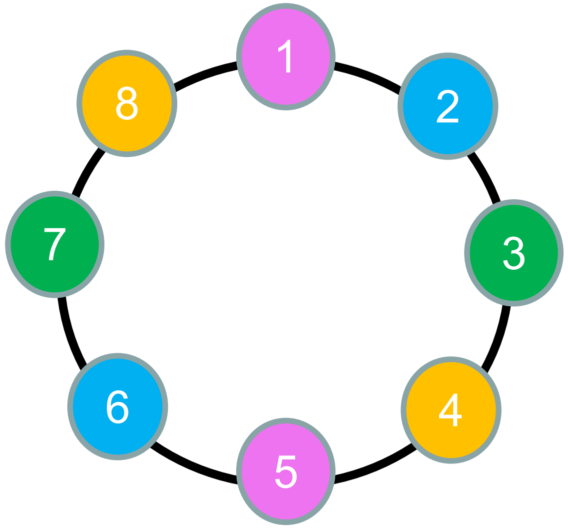
- Degree distribution p_k , a probability distr.

  p_k is the fraction of nodes that have degree exactly equal to k (i.e., # of nodes with that degree / N)

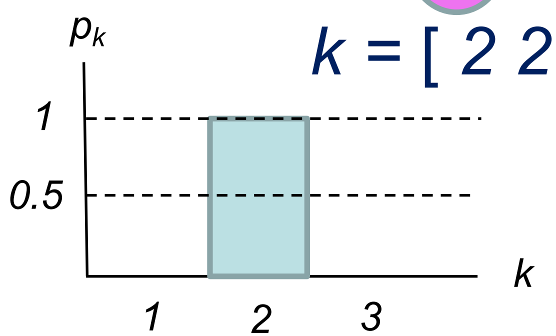


$k = [\ 1\ 3\ 2\ 2\ ]^T$





$k = [\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ ]^T$

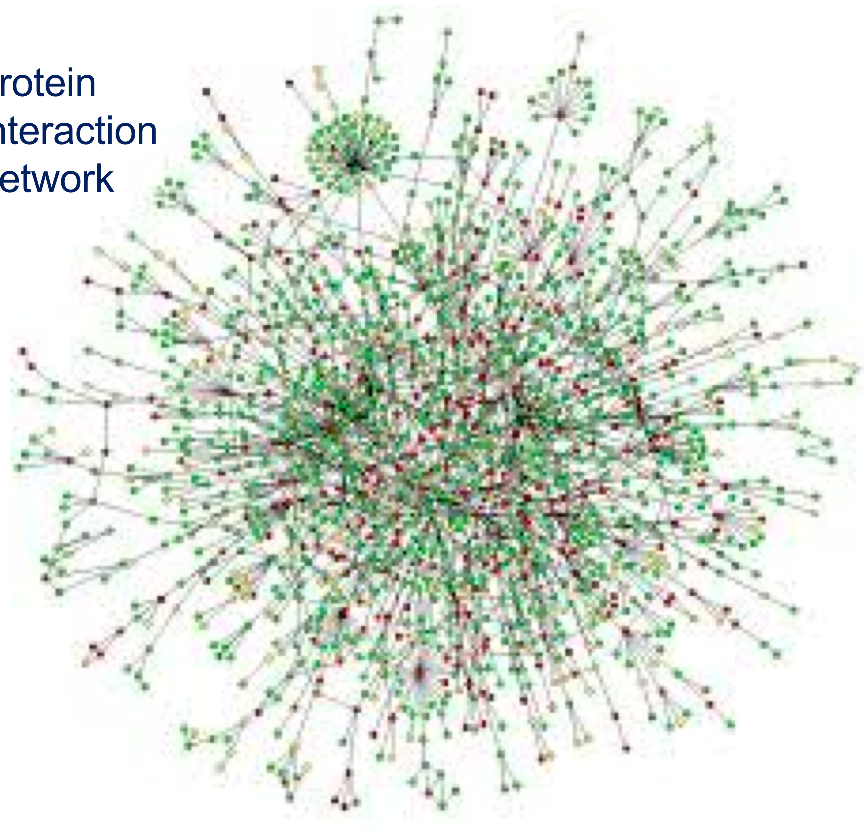


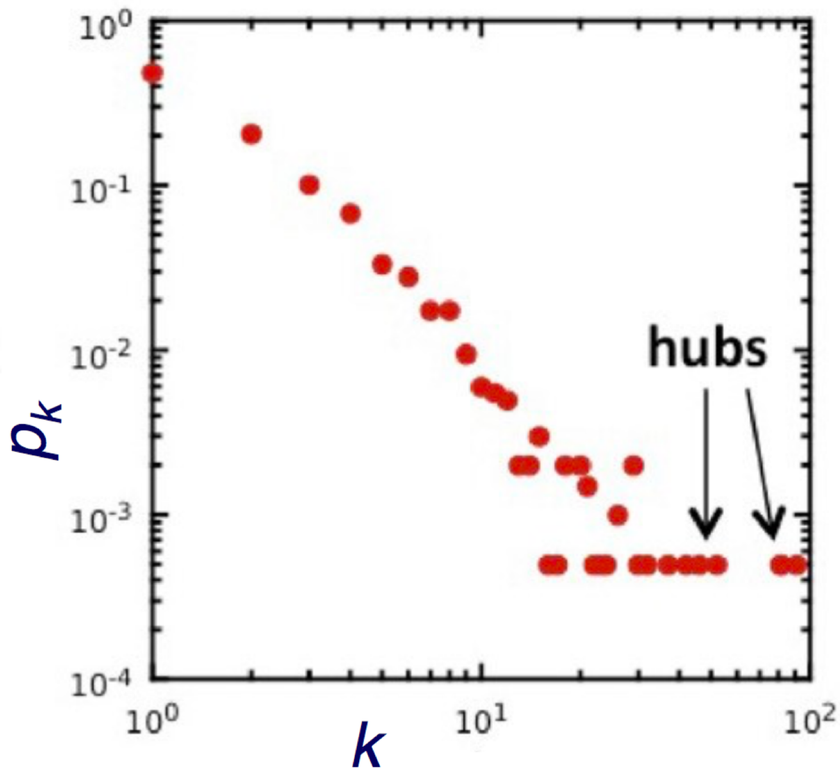
Degree distribution

- In real world (large) networks, degree distribution is typically heavy-tailed

nodes with high degree = hubs

protein interaction network



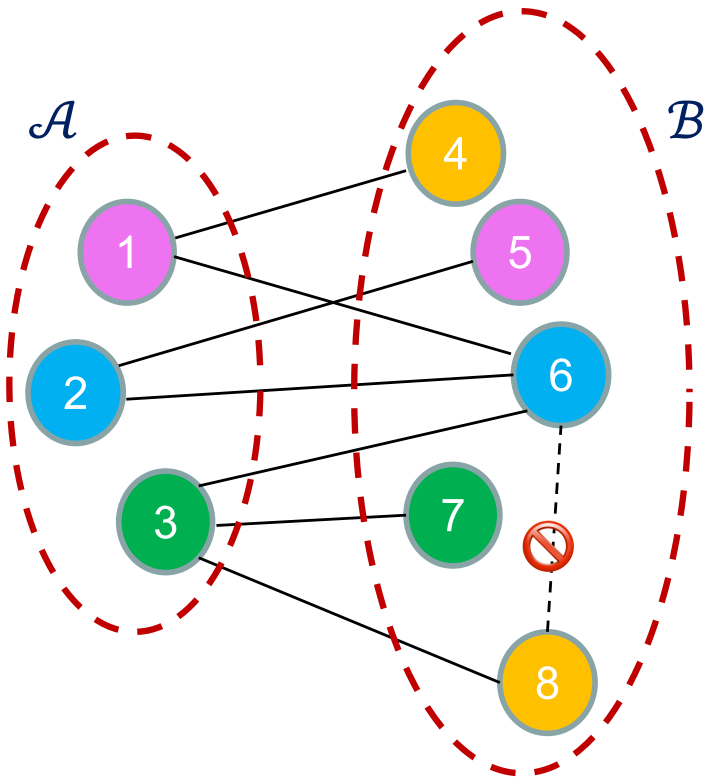


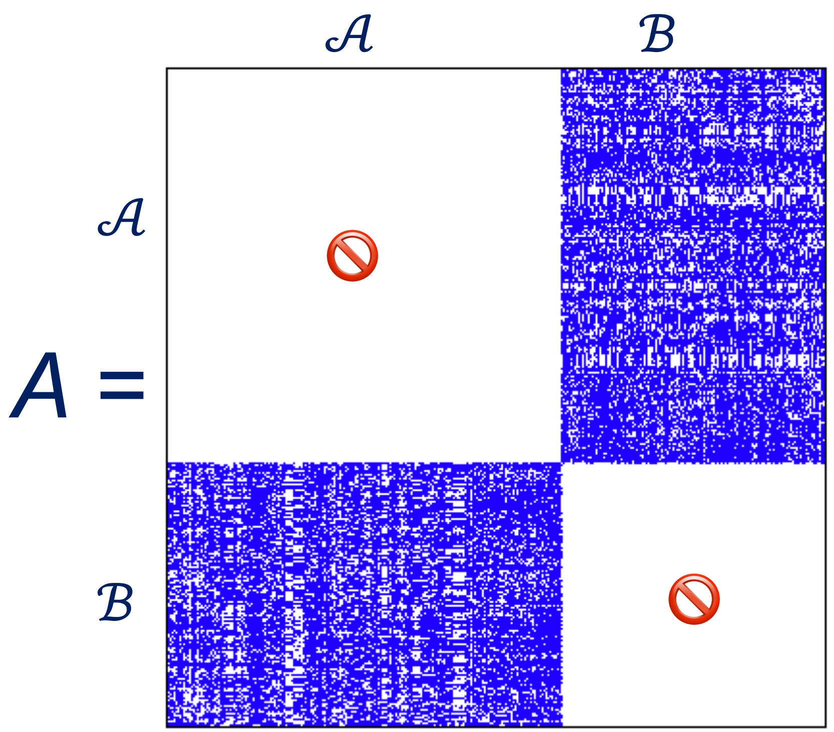
Other graph types

of interest to us

Bipartite graphs

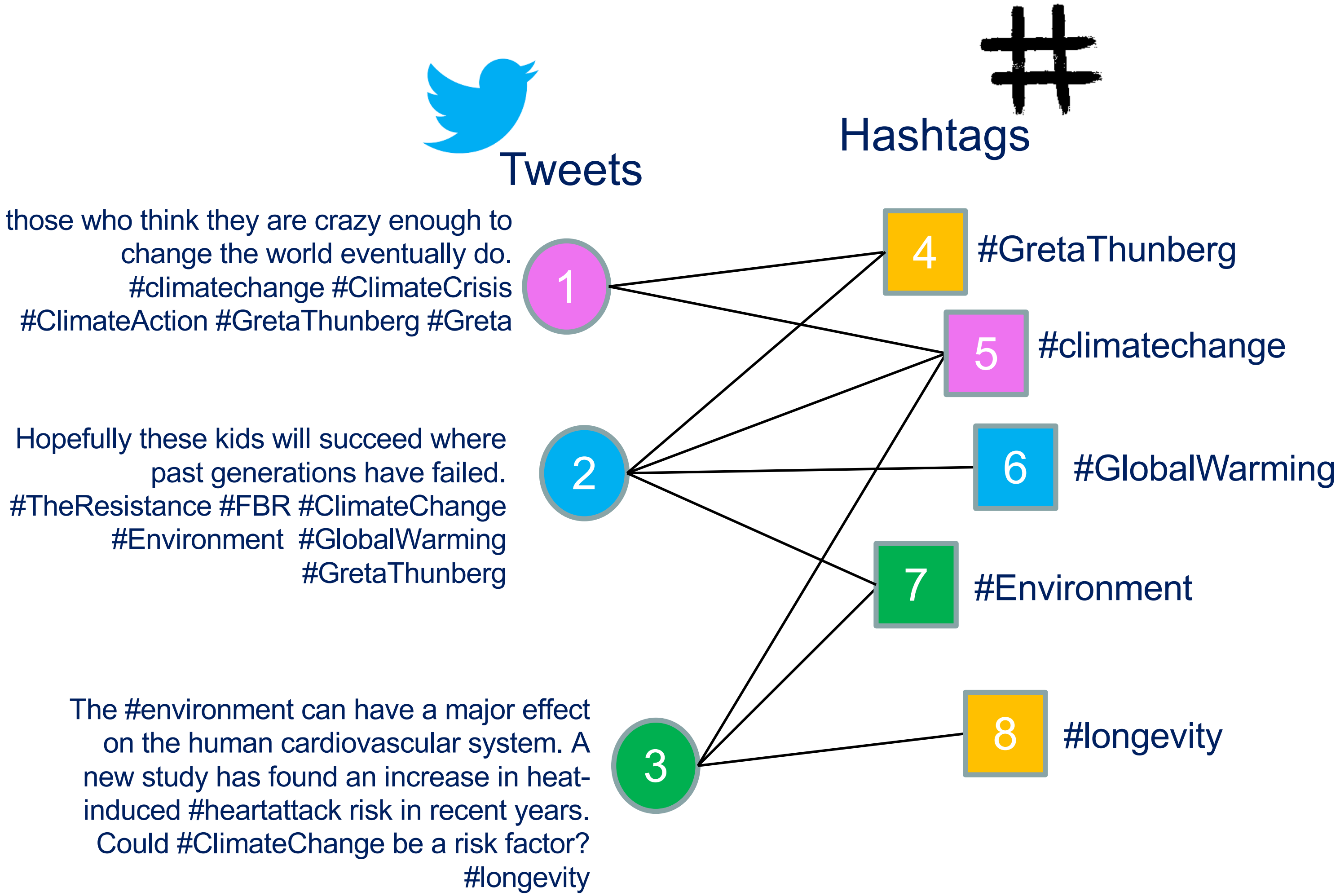
- Connections are available only between the groups $\mathcal{A}$ and $\mathcal{B}$





UNIVERSITÀ DEGLI STUDI DI PADOVA

Bipartite graph example



Meaning

- Bipartite graphs represent memberships/relationships, e.g., groups ($\mathcal{A}$) to which people ($\mathcal{B}$) belong

  examples: movies/actors, classes/students, conferences/authors

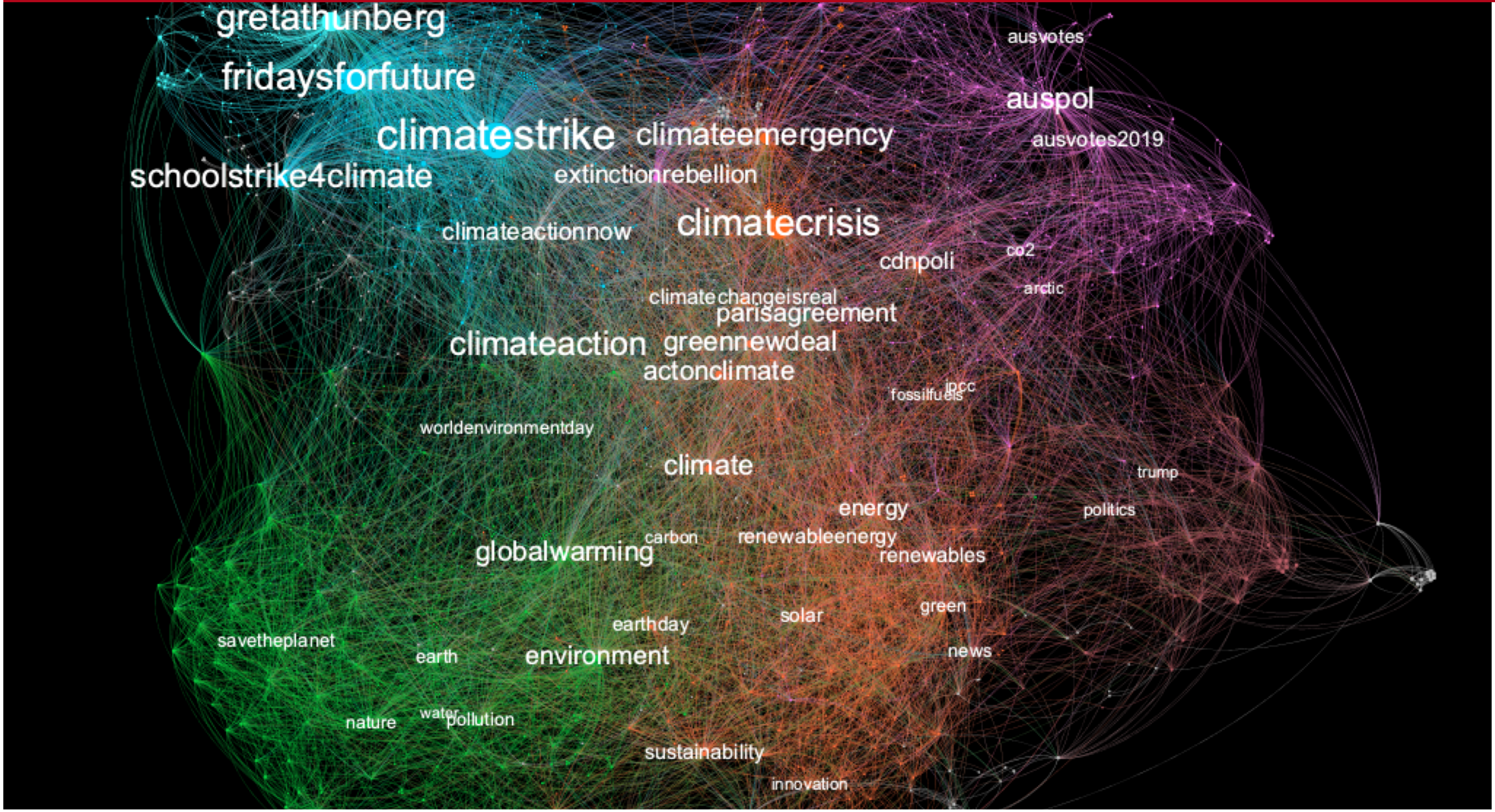
- We can build separate networks (projections) for $\mathcal{A}$ and $\mathcal{B}$ (sometimes this is useful)

  in the movies/actors example being linked can be interpreted in two ways: "actors in the same movie" (projection on $\mathcal{B}$), or "movies sharing the same actor" (projection on $\mathcal{A}$)

Università degli Studi di Padova

Projection on a semantic network

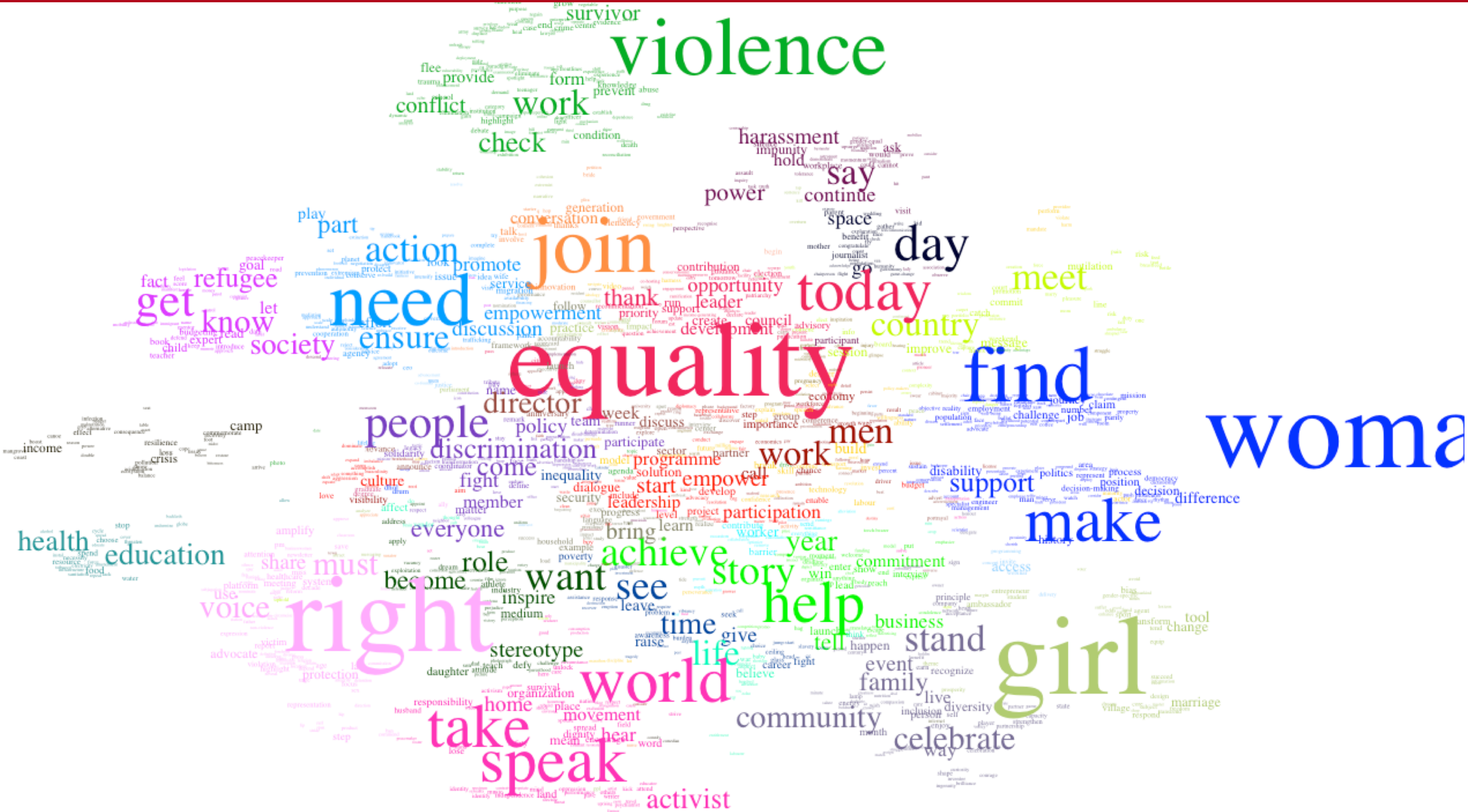
#hashtags that appear in the same tweet are linked



#climateaction tweets <u>after</u> Greta Thunberg

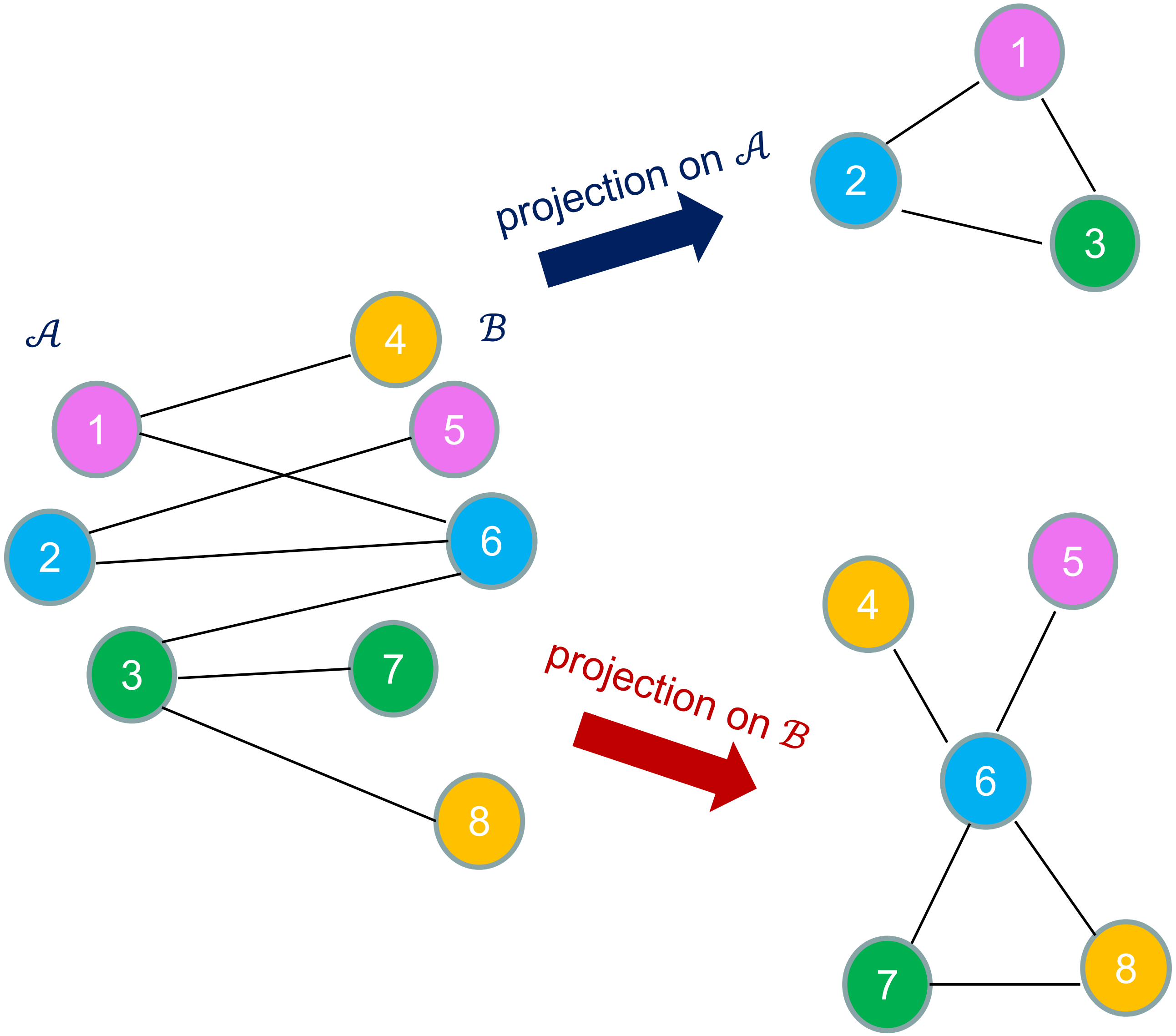
Projection on a semantic network

words that appear in the same tweet are linked



#metoo tweets

Abstract example



Nodes are linked if they have a common neighbour in $\mathcal{B}$

PS: we say that nodes *i* and *j* have a common neighbour *k* if both *i* and *j* are connected to *k*

Nodes are linked if they have a common neighbour in $\mathcal{A}$

Projections

- The two projections on $\mathcal{A}$ and $\mathcal{B}$ can be obtained by inspecting the squared adjacency matrix A^2

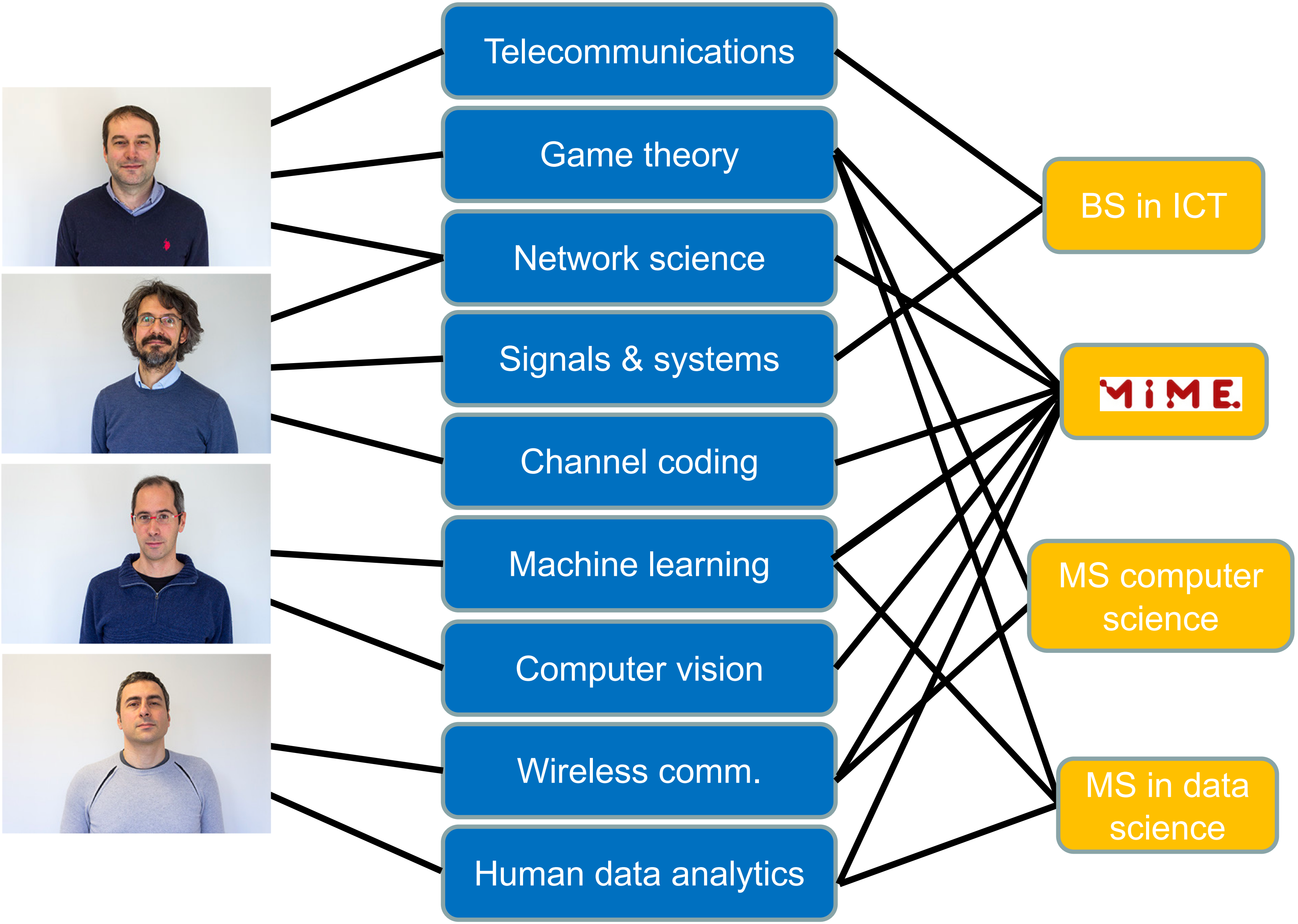
$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

A_1 (top-left 3×3 block), A_2 (bottom-right 5×5 block)

# of common neighbors of *i*=6 and *j*=5

# of neighbors of *i*=6

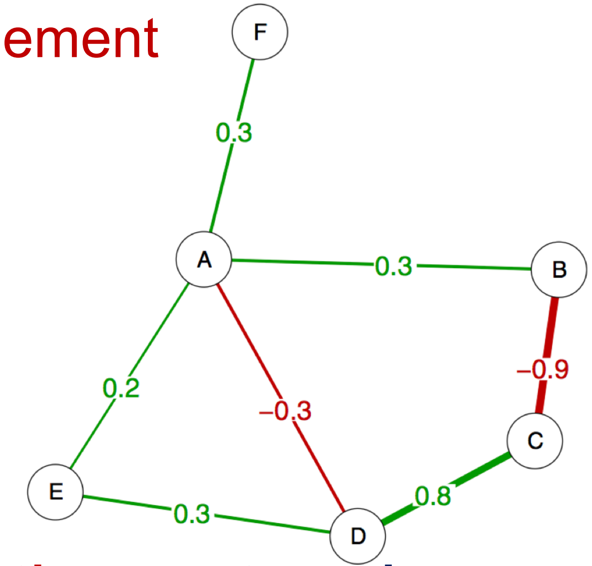
Tri-partite graphs



Signed graphs

- Edges can have signed values

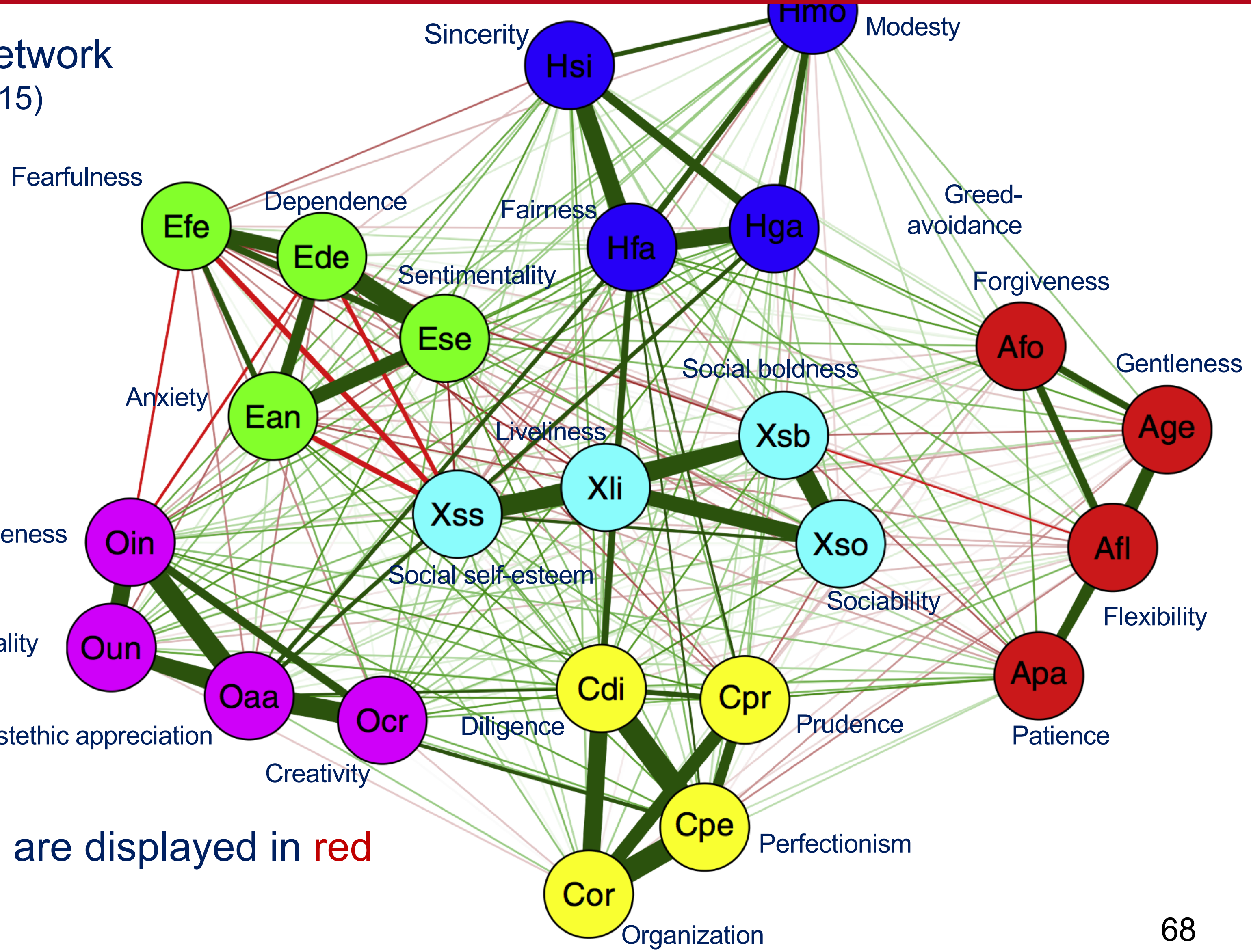
  positive if there is an agreement between nodes
  negative if there's a disagreement



- This is typical of correlation networks

Signed graph example

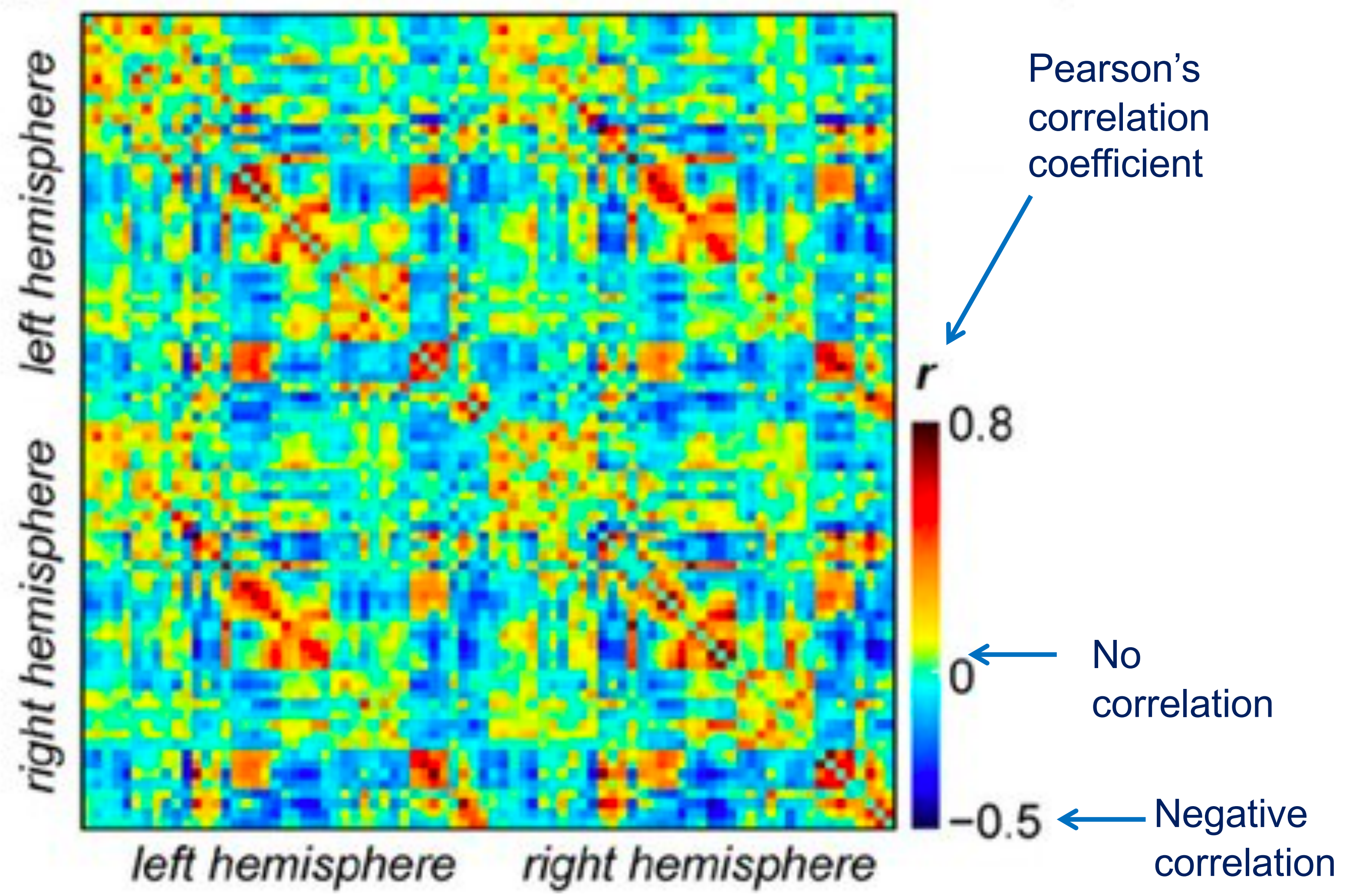
A personality network
(Costantini et al, 2015)



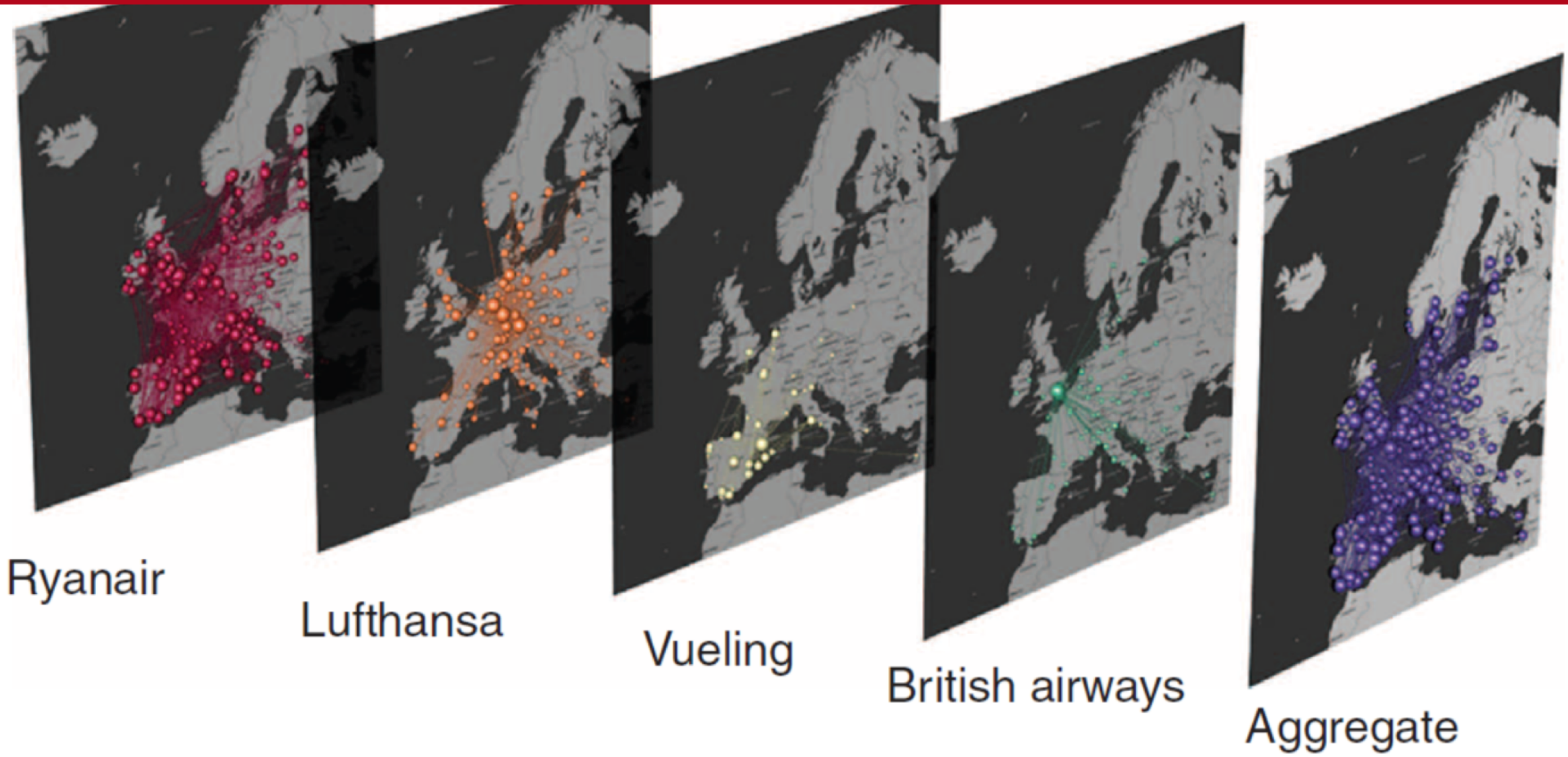
Negative links are displayed in red

Signed graph example

An fMRI adjacency matrix (fMRI = functional magnetic resonance imaging)



Multi-layer networks

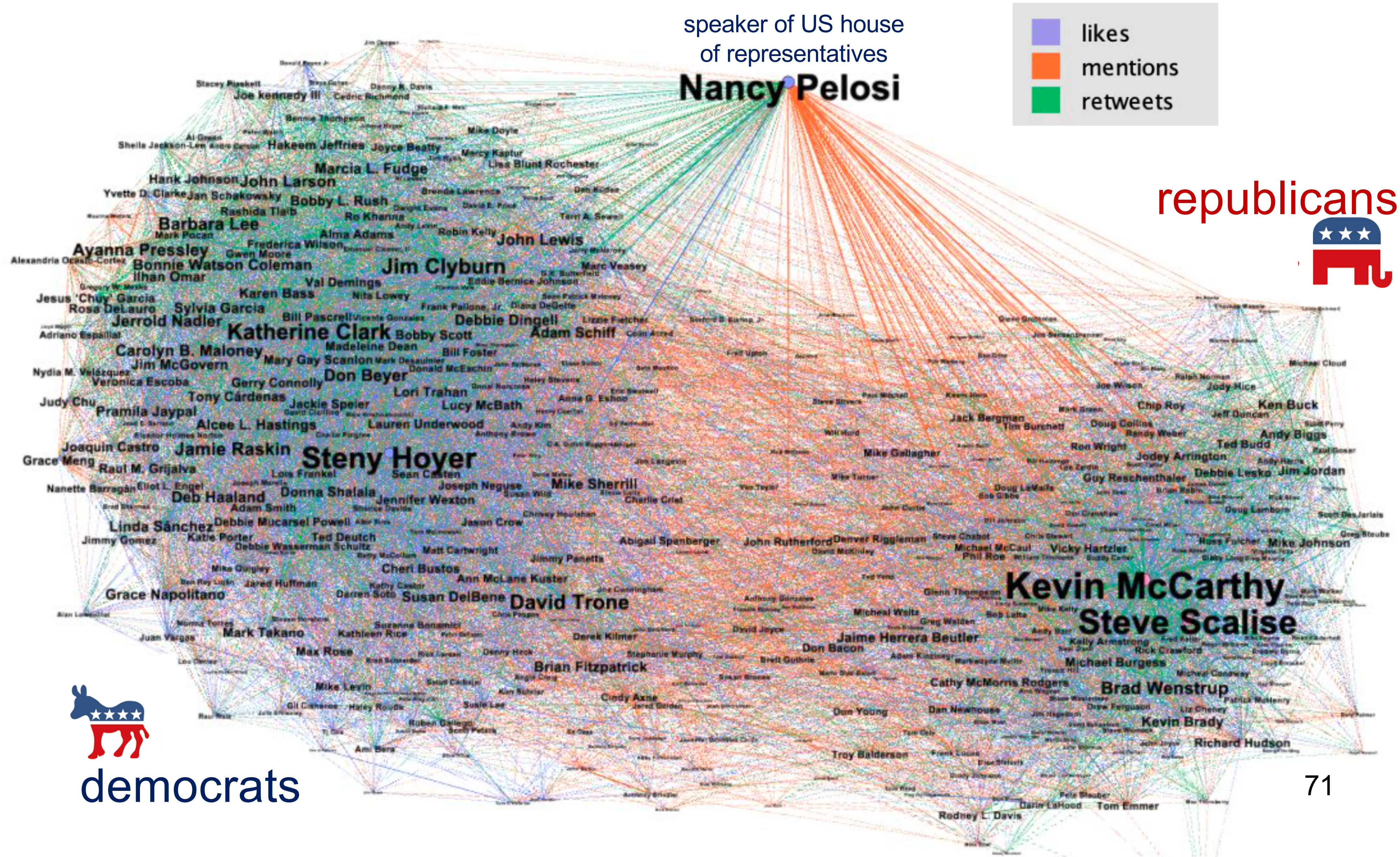


described by a set of adjacency matrices $\boldsymbol{A}_\ell$

average connection $\boldsymbol{A} = \sum_\ell \boldsymbol{A}_\ell$

An example

US republicans and democrats interactions on Twitter (2020)



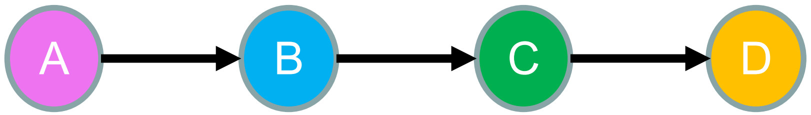
Paths and connectivity

in graphs

Paths

Path

a sequence of interconnected nodes (meaning each pair of nodes adjacent in the sequence are connected by a link)

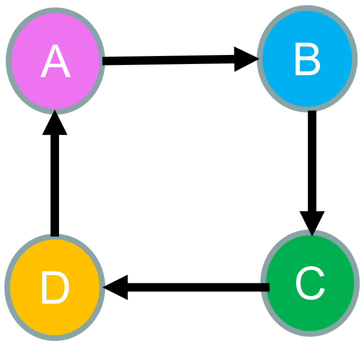


Path length

of links involved in the path (if the path involves *n* nodes then the path link is *n-1*)

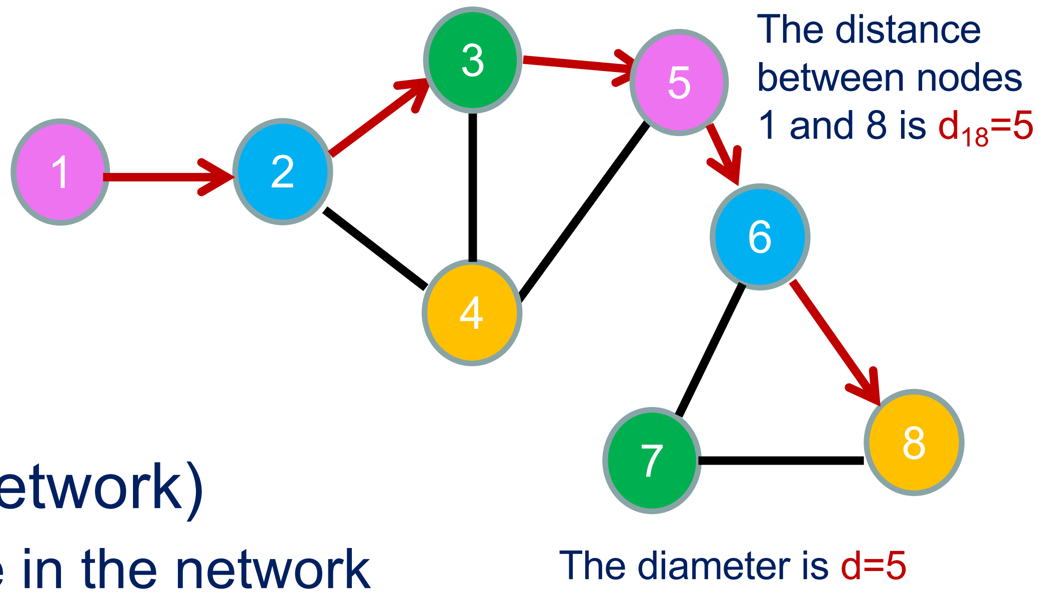
Cycle

path where starting and ending nodes coincide



Distances

- **Shortest path** (between any two nodes)
  - the path with the minimum length, which is called the distance
  - it is not unique!

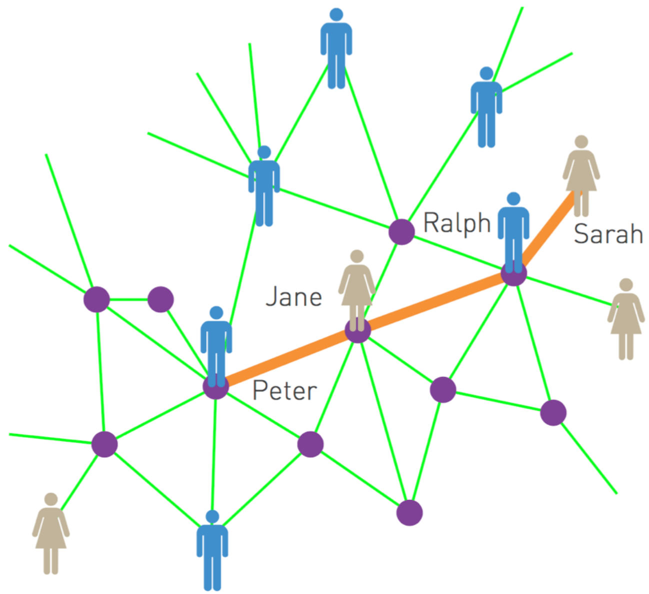


- **Diameter** (of the network)
  - the highest distance in the network
- **Algorithms**
  - available to compute distances: Dijkstra, Bellman-Ford, BFS

Average path length

average distance between all nodes pairs (apply an algorithm to all node couples, and take the average)

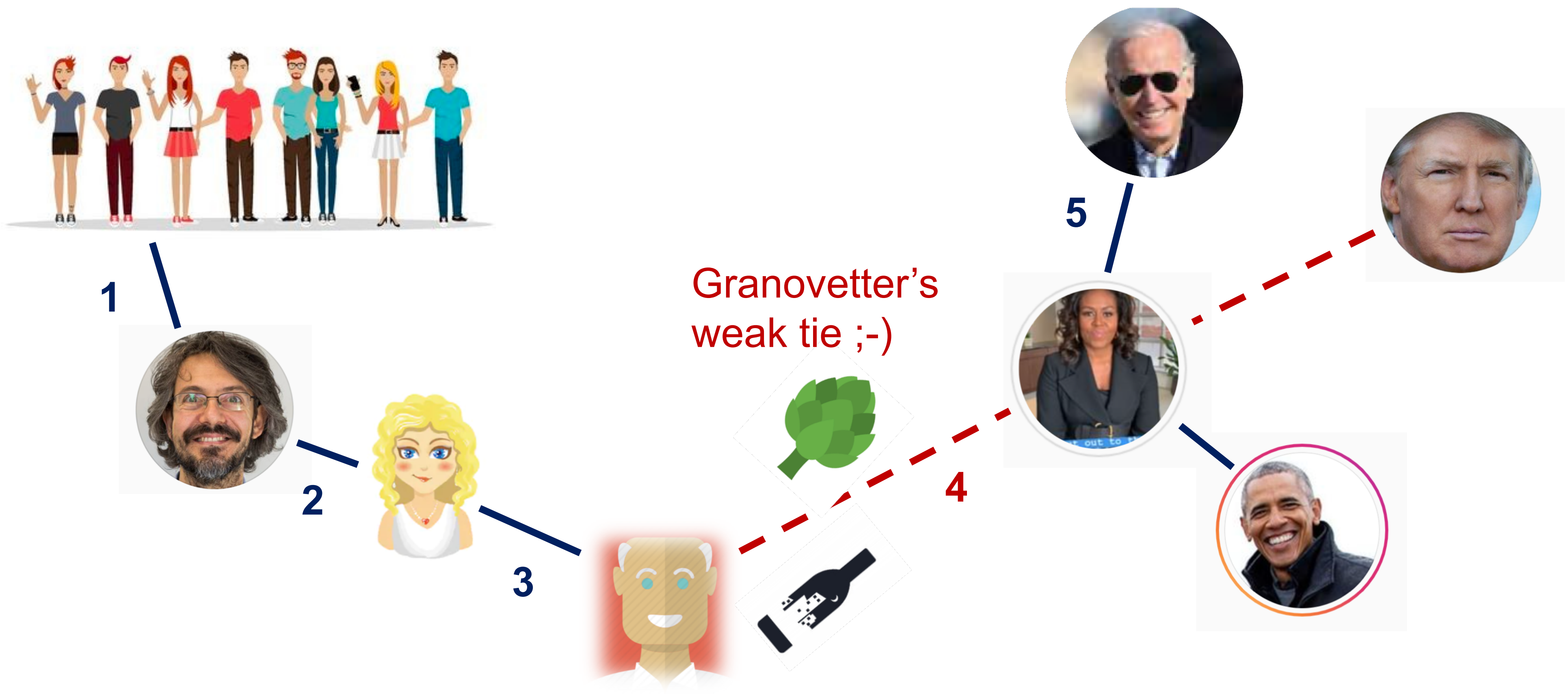
- In real networks distance between two randomly chosen nodes is generally short
- Milgram [1967]: *6 degrees of separation*



- What does this mean?
We are more connected than we think

Small world

we and the US presidents



Connectivity in undirected networks

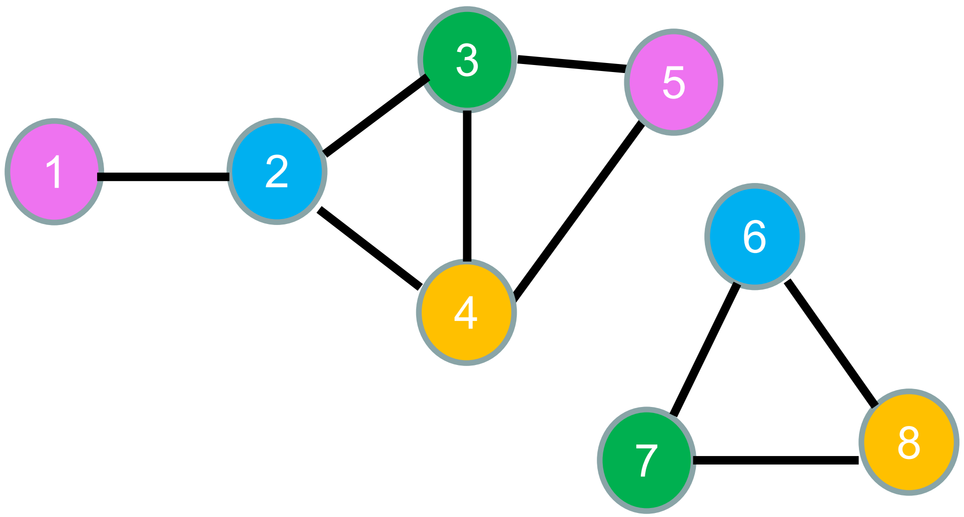
- **Connected graph (undirected)**

  for all couples (*i*,*j*) there exists a path connecting them

  if disconnected, we count the # of connected components (e.g., use BFS and iterate)

- **Giant component (the biggest one)**

- **Isolates (the other ones)**

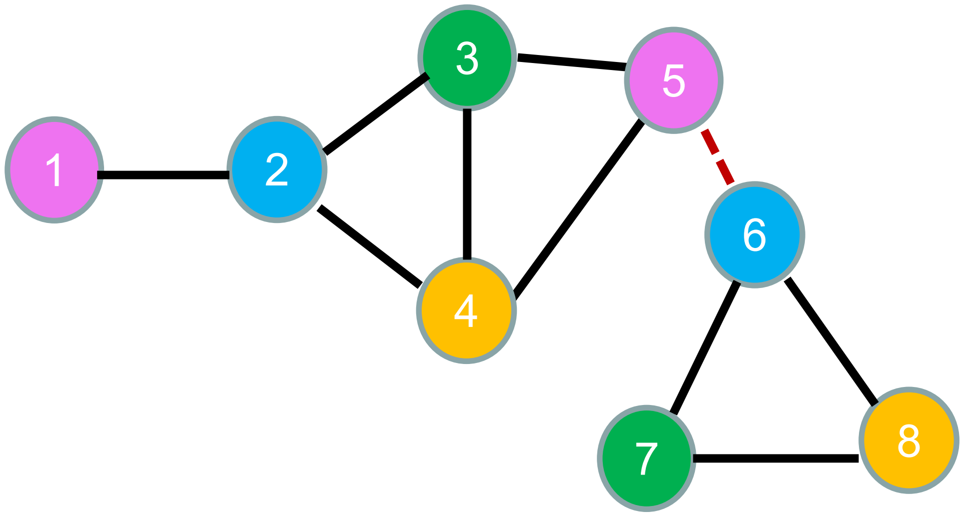


$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

block-diagonal matrix

- A **bridge** is a link between two connected components

its removal would make the network disconnected



$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Connectivity in directed networks

For directed networks we distinguish between

- **Strongly** connected components

  where $i \rightarrow j$ and $j \rightarrow i$ for all choices of (i,j) in the component

- **Weakly** connected components

  connected in the undirected sense (i.e., disregard link directions)

Condensation graph

- Strong connectivity induces a partition in disjoint strongly connected sets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_K$
- By reinterpreting the sets as nodes we obtain a condensation graph $\mathcal{G}^*$ where $i \rightarrow j$ is an edge if a connection exists between sets $\mathcal{V}_i \rightarrow \mathcal{V}_j$