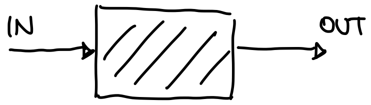
COMPUTABILITY (09/10/2023)

effective procedure (algorithm) ⇝ computable function } existence of non-computable functions

* **Effective procedure**

sequence of elementary steps

input data ⇝ output data



deterministic ⇝ function (partial) $f: \{\text{inputs}\} \to \{\text{outputs}\}$

Def.: A function f is computable if there exists an algorithm such that the induced function is f.

* $GCD(x,y)$ = greatest common divisor (Euclid's)

* $f(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$

* $g(n) = n^{th}$ prime number

* $h(n) = n^{th}$ digit in π

* $f(n) = \begin{cases} 1 & \text{if in } \pi \text{ there are exactly } n \text{ consecutive } 5\text{'s} \\ 0 & \text{otherwise} \end{cases}$

if $\pi = 3,14 \dots 7\,5\,5\,5\,5\,2 \dots$

$f(4) = 1$

idea : - compute all digits of π

- check if there are n digits 5 in a row

NOT AN ALGORITHM!

So is f computable?

* $g(n) = \begin{cases} 1 & \text{if } \pi \text{ includes at least } n \text{ digits 5 in a row} \\ 0 & \text{otherwise} \end{cases}$

if $\pi = 3{,}14 \ldots 7\,5\,5\,5\,5\,2 \ldots$

$g(4) = 1$
$g(3) = 1$
$g(2) = 1$
$g(1) = 1$
$g(0) = 0$

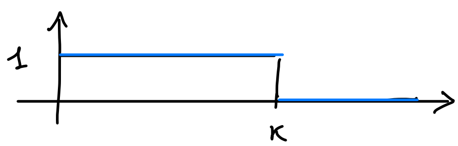
if $g(n) = 1$ $\quad \forall m \leq n \quad g(m) = 1$

take $\quad k = \sup \{ n \mid \text{there } n \text{ digits 5 in a row in } \pi \}$

↓

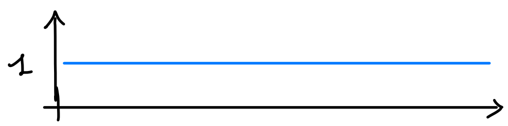
two possibilities

* k finite $\qquad g(n) = \begin{cases} 1 & \text{if } n \leq k \\ 0 & \text{otherwise} \end{cases}$




```
function g(n):
    if n ≤ k:
        return 1
    else
        return 0
```

* k is infinite $\qquad g(n) = 1 \quad \forall n$




```
function g(n):
    return 1
```

Can we use the same argument for f? NO

$$f(n) = \begin{cases} 1 & \text{if in } \pi \text{ there are exactly } n \text{ consecutive 5's} \\ 0 & \text{otherwise} \end{cases}$$

Let $A = \{n \mid \text{there are exactly } n \text{ digits 5 in a row in } \pi\}$

and take

```
function f(n):
    if n ∈ A:
        return 1
    else
        return 0
```

* Existence of non-computable functions

* Characteristics of a "good" algorithm
  - → finite length
  - → there exists a computing agent which executes the algorithm
    - → memory (unbounded)
    - → discrete steps, deterministic, not probabilistic
    - → finite limit to number and power of instructions
    - → the computation can
      - → terminate after a finite number of steps ⇝ output
      - → diverge (never terminate) → no output

* **Math notation**

* $\mathbb{N} = \{0, 1, 2, \dots\}$

* A, B sets $\qquad A \times B = \{(a,b) \mid a \in A, b \in B\}$

  A set $\qquad A^n = \underbrace{A \times A \times \dots \times A}_{n \text{ times}}$

* relations $\qquad r \subseteq A \times B$

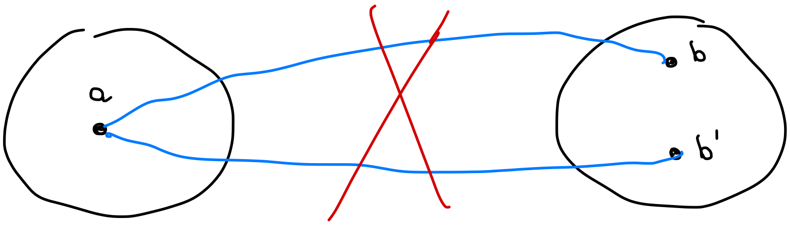
  m divides n $\qquad \text{divides} = \{(n, n*k) \mid n, k \in \mathbb{N}\}$

  $\text{divides} \subseteq \mathbb{N} \times \mathbb{N}$

* functions (partial) $\qquad f : A \to B$

  special relation $f \subseteq A \times B$

  $\forall a \in A, \forall b, b' \in B \quad (a,b), (a,b') \in f \Rightarrow b = b'$



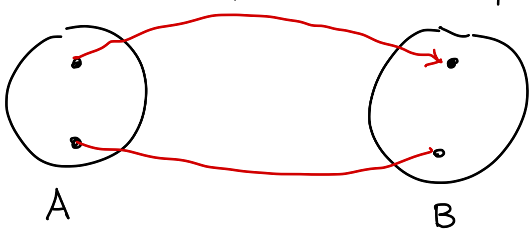
$\text{dom}(f) = \{a \in A \mid \exists b \in B . (a,b) \in f\}$

we write $f(a) = b$ instead of $(a,b) \in f$

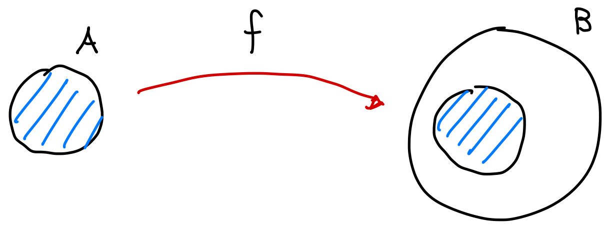
if $a \in \text{dom}(f)$ $\qquad f(a)\downarrow$
if $a \in \text{dom}(f)$ $\qquad f(a)\uparrow$

* **Cardinality**

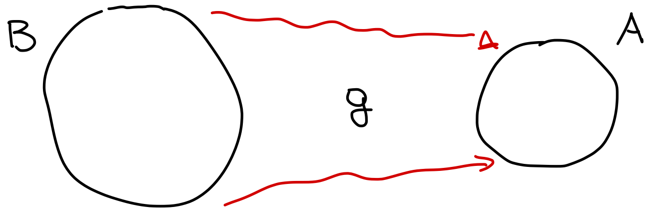
  A, B sets $\qquad |A| = |B|$ if there is $f : A \to B$ bijective



$|A| \leq |B|$ if there is $f: A \to B$ injective



"equivalently" if there is a surjection $g: B \to A$



* A countable (denumerable)

$$|A| \leq |\mathbb{N}|$$

surjective function $f: \mathbb{N} \to A$

$$\underbrace{f(0) \quad f(1) \quad f(2) \quad \dots}_{\text{list of all elements of } A}$$

* A, B countable $\Rightarrow$ $A \times B$ countable

Idea: A a_0 a_1 a_2 ...

B b_0 b_1 b_2 ....

	a_0	a_1	a_2	
b_0	(a_0, b_0)	(a_1, b_0)	(a_2, b_0)	...
b_1	(a_0, b_1)	$(a_1, b_0$	(a_1, b_1)	...
b_2	(a_0, b_2)	(a_1, b_2)	(a_2, b_2)	...
	⋮	⋮	⋮	

dove tail

* $A_0, A_1, A_2, \dots$ countable sets $\Rightarrow$ $\bigcup_{i \in \mathbb{N}} A_i$ countable

* Existence of non computable functions

we restrict to unary functions over the naturals $f : \mathbb{N} \to \mathbb{N}$
(partial)

$$\mathcal{F} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}$$ set of all functions

Fix a model of computation $\rightarrow$ set of algorithms $\mathcal{A}$

and given $A \in \mathcal{A}$ $\rightsquigarrow$ $f_A : \mathbb{N} \to \mathbb{N}$

Functions computable in $\mathcal{A}$

$$\mathcal{F}_{\mathcal{A}} = \{ f \mid \text{there exists } A \in \mathcal{A} \text{ s.t. } f_A = f \}$$
$$= \{ f_A \mid A \in \mathcal{A} \}$$

Clearly $\mathcal{F}_{\mathcal{A}} \underset{\text{yes}}{\overset{\subsetneq}{?}} \mathcal{F}$

* An algorithm is a sequence of instructions from a set I ↑ finite

$$\mathcal{A} = I \cup I \times I \cup I \times I \times I \cup \dots$$
$$= \bigcup_{i \geq 1} I^n$$

↑ countable union of countable sets $\Rightarrow$ countable

$$|\mathcal{A}| \leq |\mathbb{N}|$$

Now

$$\mathcal{A} \to \mathcal{F}_{\mathcal{A}}$$
$$A \mapsto f_A$$ surjective

$$\boxed{|\mathcal{F}_{\mathcal{A}}| \leq |\mathcal{A}| \leq |\mathbb{N}|}$$

* The set of all functions $\mathcal{F}$ is not countable

Why? Assume that it is so

$$|\mathcal{F}| \leq |\mathbb{N}|$$

i.e. we can list elements of $\mathcal{F}$

$$\begin{array}{ccccc} & f_0 & f_1 & f_2 & f_3 \; \cdots \\ 0 & f_0(0) & f_1(0) & f_2(0) & \cdots \\ 1 & f_0(1) & f_1(1) & f_2(1) & \cdots \\ 2 & f_0(2) & f_1(2) & f_2(2) & \cdots \\ 3 & \vdots & \vdots & \vdots & \\ \vdots & & & & \end{array}$$

change it systematically

if $\downarrow$ ⇝ $\uparrow$

if $\uparrow$ ⇝ 0

define $d : \mathbb{N} \to \mathbb{N}$

$$d(n) = \begin{cases} 0 & f_n(n)\uparrow \\ \uparrow & f_n(n)\downarrow \end{cases}$$

d is a function in $\mathcal{F}$ so there is $n \in \mathbb{N}$ s.t. $d = f_n$

→ if $f_n(n)\downarrow \Rightarrow d(n)\uparrow \neq f_n(n)$ contradiction

→ if $f_n(n)\uparrow \Rightarrow d(n) = 0 \neq f_n(n)$ 〃

⇒ absurd

↳ $\mathcal{F}$ not countable $|\mathcal{F}| > |\mathbb{N}|$ $\square$

* Putting things together

$$\mathcal{F}_A \subseteq \mathcal{F}$$

$$|\mathcal{F}_A| \leq |\mathbb{N}| < |\mathcal{F}| \Rightarrow \boxed{\mathcal{F}_A \subsetneq \mathcal{F}}$$

* How many non-computable functions?

$$|\mathcal{F} \setminus \mathcal{F}_A| > |\mathbb{N}|$$