Knowledge Representation and Learning
Final exam

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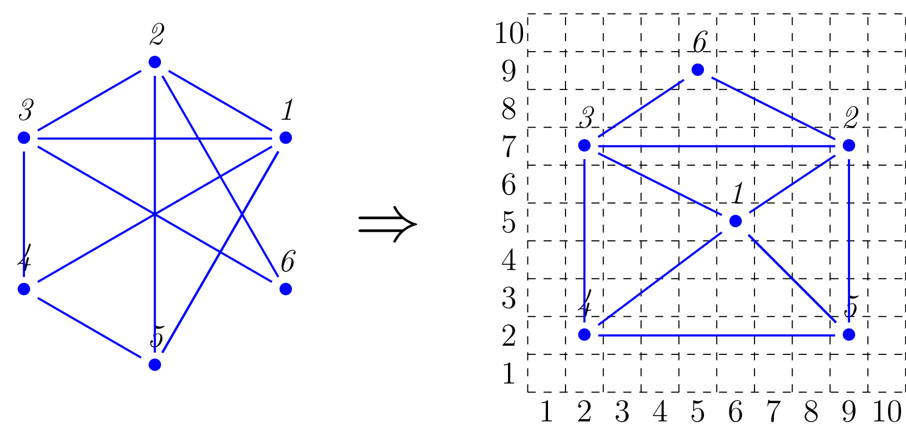
Exercise 1 (6 points) *Suppose that you have an undirected graph* $G = (V, E)$ *of* n *vertexes* V *and a set of edges* E*. Suppose that you have a grid of* $m \times m$ *places in which you can position the nodes. Formalize the problem of positioning the* n *nodes on the* $m \times m$ *grid so that the edges in* E *do not intersect. An example of a graph of 6 nodes displaced on a grid of* 10×10 *squares is given below:*



PS: Vertices can be placed only in the center of each square. You can suppose that the coordinates of the center of the ij*-square are* (i, j)*. So the solution of be above examle place node 1 at (6,5), node 2 at (9,7), etc.*

Solution The problem asks to place each vertex $v \in V$ is a place of the $n \times n$ grid. The decisions that we have to take to solve the problem is if the vertex v is placed in position x. Therefore, we introduce a propositional variable p_{vx} for every $v \in V$ and every $x \in [n] \times [n]$ (where $[n]$ denotes the set $\{1, 2, \dots, n\}$). the fact that the vertex v is placed in position x. We then add the following formulas that formulate the constraints on the problem:

$$\bigwedge_{v \in V} \sum_{x \in [n] \times [n]}^{n} p_{vx} = 1 \quad \text{Every vertex } v \text{ must be placed in one and only one position}$$

The second constraint is that if we have two edges (v_1, w_1) and (v_2, w_2) in E then the vertexes v_i and w_i with $i = 1, 2$ cannot be places in position x_i and y_y respectively if the segment from x_1 to y_1 intersect the seqment from x_2 to y_2. Let intersect(x_1, y_1, x_2, y_2) be an atom that is true if and only if the two segments $\overline{x_1 y_1}$ and $\overline{x_2 y_2}$ intersect. Then we add the following formula:

$$\bigwedge_{\substack{(v_1,w_1),(v_2,w_2)\in E \\ \text{intersect}((x_1,y_1),(x_2,y_2))}} \neg(p_{v_1 x_1} \wedge p_{w_1 y_1} \wedge p_{v_2 x_2} \wedge p_{w_2 y_2})$$

To detect if intersect$((x_1, y_1), (x_2, y_2))$ is true we can use any algorithm. □

Exercise 2 (4 points) *Show how the connectives $\wedge$, $\vee$, $\neg$ and $\leftrightarrow$ can be rewritten in terms of the connectives $\rightarrow$ and $\bot$ ($\bot$ is the formula that is always false).*

Solution We start by rewriting $\neg$, then we use the rewriting of $\neg$ to rewrite $\vee$, then we use the rewriting of $\neg$ and $\vee$ to rewrite $\wedge$, and finally we use the rewriting of $\wedge$ to rewrite $\leftrightarrow$.

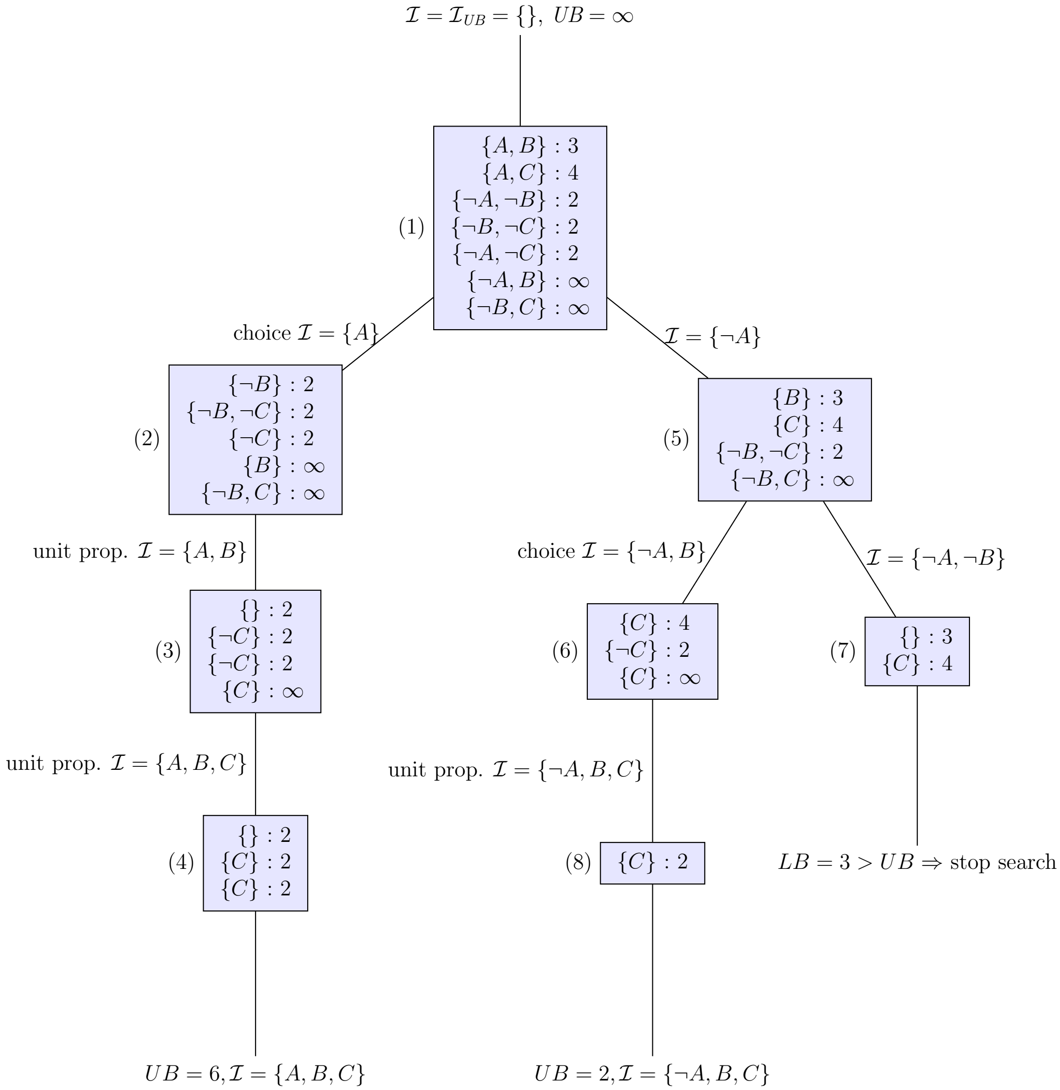
$$\begin{aligned}
\neg A &\equiv A \rightarrow \bot \\
A \vee B &\equiv \neg A \rightarrow B \equiv (A \rightarrow \bot) \rightarrow B \\
A \wedge B &\equiv \neg(\neg A \vee \neg B) \equiv \neg(A \rightarrow \neg B) \equiv (A \rightarrow (B \rightarrow \bot)) \rightarrow \bot \\
A \leftrightarrow B &\equiv A \rightarrow B \wedge B \rightarrow A \equiv ((A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow \bot)) \rightarrow \bot
\end{aligned}$$

□

Exercise 3 (5 points) *Solve the following mas-sat problem using either B&B or Fu & Malik algorithm*

$$\begin{array}{rrr}
3 : \{A, B\} & 4 : \{A, , C\} & 2 : \{\neg A, \neg B\} \\
2 : \{\neg B, \neg C\} & 2 : \{\neg A, \neg C\} & \\
\infty : \{\neg A, B\} & \infty : \{\neg B, C\} &
\end{array}$$

Solution



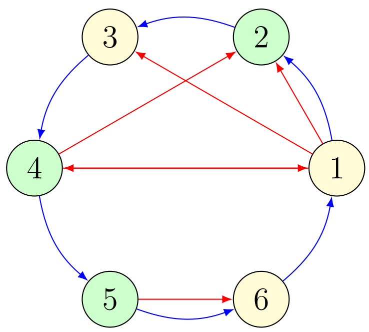
The algorithm terminates returning $\mathcal{I} = \{\neg A, B, C\}$ with a cost of 2 $\square$

Exercise 4 (4 points) *Find the weights of the literals: A, $\neg A$, B, $\neg B$, C and $\neg C$ that produce the following weight function*

A	B	C	$w(\mathcal{I})$
1	1	1	-1
1	1	0	1
1	0	1	1
1	0	0	-1
0	1	1	1
0	1	0	-1
0	0	1	-1
0	0	0	1

Solution One can notice that when there is an even (including 0) number of propositions which are true the weight of the model is +1 and when there is an odd number of propositions which are false the weight is −1. This behaviour can be obtained by associating the weight −1 to each positive and not associating any weight to the negative literals (or equivalently associating the weight +1). □

Exercise 5 (4 points) *Consider the following first order interpretation on the domain* $\{1, \ldots, 6\}$ *with two unary predicates* Y *and* G *(yellow and green), a unary function symbol* f *(shown in blue) and a binary relation* R *(shown in red).*



Check if the following formulas are true or false. If they are false explain why

1. $\forall x \exists y (R(x, y) \lor Y(f(x)))$
2. $\exists x (G(x) \land R(f(x), x) \land R(x, f(f(f(x)))))$
3. $\forall x \forall y (G(x) \land R(x, y) \rightarrow Y(y) \lor Y(f(y)))$
4. $\exists x (\forall y (Y(y) \rightarrow R(x, y)))$

Solution

1. $\forall x \exists y (R(x, y) \lor Y(f(x)))$ is false, indeed if $x \leftarrow 3$ there is no y such that $R(3, y)$ is true, and $Y(f(3)) = Y(4)$ is false.
2. $\exists x (G(x) \land R(f(x), x) \land R(x, f(f(f(x)))))$ is false since for all the three green items $R(f(x), x)$ is always false;
3. $\forall x \forall y (G(x) \land R(x, y) \rightarrow Y(y) \lor Y(f(y)))$ is true since for all the pair x, y where x is green and $R(x, y)$ is true (i.e., for $((4, 1), (4, 2), (5, 6))$, where have that either y is yellow ($Y(1)$ and $Y(6)$) or $f(y)$ is yellow ($Y(f(2)) = Y(3)$).
4. $\exists x (\forall y (Y(y) \rightarrow R(x, y)))$ is false. The states that there is a x that is connected via R with all the yellow items. One can see from the picture that the model does not contains such an item.

□

Exercise 6 (4 points) *Use resolution to prove that*

$$\forall x \forall y \forall z (P(x, y) \wedge P(z, y) \rightarrow P(x, z)) \tag{1}$$

is a logical consequence of

$$\forall x \forall y \forall z (P(x, y) \wedge P(y, z) \rightarrow P(x, z))$$
$$\forall x \forall y (P(x, y) \rightarrow P(y, x))$$

Solution We first have to negate the consequence obtaining

$$\exists x \exists y \exists z (P(x, y) \wedge P(z, y) \wedge \neg P(x, z))$$

By Skolemization we obtain:

$$P(a, b) \wedge P(c, b) \wedge \neg P(c, a)$$

Therefore we have the following set of clauses

$$\{P(a, b)\}$$
$$\{P(c, b)\}$$
$$\{\neg P(a, c)\}$$
$$\{\neg P(x, y), \neg P(y, z), P(x, z)\}$$
$$\{\neg P(x, y), P(y, x)\}$$

We can now apply resolution and unification

$\{\neg P(x, y), \neg P(y, z).P(x, z)\}$ $\{P(a, b)\}$

$\{\neg P(b, z), P(a, z)\}$ $\{P(b, c)\}$

$\{P(a, c)\}$ $\{\neg P(x, y), P(y, x)\}$

$P(c, a)\}$ $\{\neg P(c, a)\}$

$\{\}$

□

Exercise 7 (4 points) *Provide a direct mathematical expression for counting the models of*

$$\forall x A(x) \vee \exists y B(y)$$

on a domain of n elements.

Solution We consider the two separate cases.

1. If $\forall x A(x)$ is true, we have that A must be interpreted in the entire domain, therefore there is only one possibility, and B can be interpreted in any subset of the domain, which amounts to 2^n possibilities.

2. If $\forall x A(x)$ is false, then A can be interpreted in any strict subset of $[n]$ and therefore there are $2^n - 1$ possibilities, while B must be interpreted in a non empty subset of $[n]$ and therefore there are $2^n - 1$ possibilities. The total possibilities are therefore $(2^n - 1)^2$.

Notice that the two cases are disjoint and cover all the possibilities. Therefore we can add the two results obtaining

$$(2^n - 1)^2 + 2^n = 2^{2n} - 2^n + 1$$

□