Knowledge Representation and Learning
Final exam

Prof. Luciano Serafini

July 24, 2023

Exercise 1 (4 points) *List the pair of items from each one of the two lists (i.e, a list of (number),(letter) pairs) that corresponds to equivalent statements.*

(1) A is satisfiable	*(a) $\neg A$ is satisfiable*
(2) A is valid	*(b) $\neg A$ is valid*
(3) A is unsatisfiable	*(c) $\neg A$ is unsatisfiable*
(4) A is not valid	*(d) $\neg A$ is not valid*
(5) A is satisfiable and B is satisfiable	*(e) $A \wedge B$ is satisfiable*
(6) A is valid and B is valid	*(f) $A \wedge B$ is valid*
(7) A is unsatisfiable and B is unsatisfiable	*(g) $A \wedge B$ is unsatisfiable*
(8) A is not valid and B is not valid	*(h) $A \wedge B$ is not valid*
(9) A is satisfiable or B is satisfiable	*(i) $A \vee B$ is satisfiable*
(10) A is valid or B is valid	*(j) $A \vee B$ is valid*
(11) A is unsatisfiable or B is unsatisfiable	*(k) $A \vee B$ is unsatisfiable*
(12) A is not valid or B is not valid	*(l) $A \vee B$ is not valid*

Solution Let us recall the definiton of *satisfiable*, *valid*, *Unsatisfiabe*, and *not valid* formula.

A is satisfiable	if **there is** an interpretation $\mathcal{I}$ that $\mathcal{I} \models A$
A is valid	if **for every** interpretation $\mathcal{I}$, $\mathcal{I} \models A$
A is unsatisfiabe	if **for every** interpretation $\mathcal{I}$, $\mathcal{I} \not\models A$
A is not valid	if **there is** an interpretation $\mathcal{I}$ such that $\mathcal{I} \not\models A$

Where $\mathcal{I} \models A$ means that A is true in $\mathcal{I}$ and $\mathcal{I} \not\models A$ means that A is false in $\mathcal{I}$.

Using the above definition we can find the following connections between the items of the two lists.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)	(l)
(1)				$\Leftrightarrow$					$\Rightarrow$			
(2)			$\Leftrightarrow$	$\Rightarrow$					$\Rightarrow$	$\Rightarrow$		
(3)	$\Rightarrow$	$\Leftrightarrow$					$\Rightarrow$	$\Rightarrow$			$\Leftarrow$	
(4)	$\Leftrightarrow$	$\Leftarrow$						$\Rightarrow$				$\Leftarrow$
(5)				$\Rightarrow$	$\Leftarrow$	$\Leftarrow$			$\Rightarrow$			
(6)			$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Leftrightarrow$			$\Rightarrow$	$\Rightarrow$		
(7)	$\Rightarrow$	$\Rightarrow$				$\Rightarrow$	$\Rightarrow$				$\Leftrightarrow$	$\Leftarrow$
(8)	$\Rightarrow$							$\Rightarrow$			$\Leftarrow$	$\Leftrightarrow$
(9)			$\Leftarrow$	$\Leftarrow$	$\Leftarrow$	$\Leftarrow$			$\Leftrightarrow$	$\Leftarrow$		
(10)						$\Leftarrow$			$\Rightarrow$	$\Rightarrow$		
(11)		$\Leftarrow$					$\Rightarrow$	$\Rightarrow$				
(12)	$\Leftarrow$		$\Leftarrow$				$\Leftarrow$	$\Leftrightarrow$			$\Leftarrow$	$\Leftarrow$

From the above tabel we can extract the pairs of equivalent statements (shown in red).

$$(1, d) \qquad (2, c) \qquad (3, b) \qquad (4, a) \qquad (6, f) \qquad (7, k) \qquad (8, l) \qquad (9, i) \qquad (12, h)$$

$\square$

Exercise 2 (5 points) *Consider an undirected graph $G = (V, E)$ composed of a set of nodes $V = \{1, \dots, n\}$ and a set of undirected edges between them. Formulate the problem of finding a path that visits all the nodes starting from a node s and ending in a node e without passing twice the same edge.*

Solution Since the graph is undirected we have that $(i, j) \in E$ iff $(j, i) \in E$. Let $m = \frac{|E|}{2}$ the number or arcs in G. For every node $i \in V$ and for every $t \leq m$ we add the proposition p_i^t that means that i has been visited at time t Notice that $i \leq m$, since if we traverse more than m arcs it means that we have traversed one arc twice. We also add the “defined” proposition p_{ij}^t for every $(i, j) \in E$ that states that at time t we move from node i to node j. We use the propositional variable introduced above to formulate the requirements of the problem:

p_s^0		We start from s
p_e^m		We end at e
$\bigvee_{t=0}^{m} p_i$	$1 \leq i \leq n$	Every node is eventually visited
$\bigwedge_{1\leq i<j\leq n} \neg(p_i^t \wedge p_j^t)$	$1 \leq t \leq m$	At every time we can be in at most one node
$p_i^t \rightarrow p_i^t \vee \bigvee_{(i,j)\in E} p_j^{t+1}$	$1 \leq t < m$	At every time stamp we can either stay in the node or move to a connected node
$p_{ij}^t \leftrightarrow (p_i^t \wedge p_j^{t+1} \vee p_i^{t+1} \wedge p_j^t)$	$(i,j) \in E$	we define p_{ij}^t as the proposition stating that at time t we traversed the edge i, j in either direction
$\bigwedge_{1\leq t<t'<m} \neg(p_{ij}^t \wedge p_{ij}^{t'})$	$(i,j) \in E$	We cannot traverse the same arc twice

□

Exercise 3 (4 points) *Find all the cores of the following set of clauses.*

$$\{\neg a, b\}, \{\neg b, c\}, \{\neg c, d\}, \{\neg d, e\}$$

Solution The set of clauses is satisfiable, so every subset of them is satisfiable, therefore there are no core. □

Exercise 4 (4 points) *Compute the weighted model counting of the formula*

$$(A \rightarrow B) \wedge (B \rightarrow (C \vee D)) \tag{1}$$

via knowledge compilation with the following weight function:

lit	A	$\neg A$	B	$\neg B$	C	$\neg C$	D	$\neg D$
$w(lit)$	3	1	1	3	3	0.5	4	2

(2)

You can check your result by computing WMC using truth table (this is not strictly necessary for the exercize).

Solution We compute the WMC of the formula Φ by compiling it in the sd-DNNF form and then transform it in a computational circuit

$$
\begin{aligned}
(A \to B) \wedge (B \to (C \vee D)) & \qquad \text{to NNF} \\
(\neg A \vee B) \wedge (\neg B \vee C \vee D) & \qquad \text{to DNNF with shannon expansion on } B \\
(B \wedge (C \vee D)) \vee (\neg B \wedge \neg A) & \qquad \text{to d-DNNF} \\
(B \wedge (C \vee (\neg C \wedge D))) \vee (\neg B \wedge \neg A) & \qquad \text{to sd-DNNF} \\
(B \wedge (C \wedge (D \vee \neg D) \vee (\neg C \wedge D)) \wedge (A \vee \neg A)) \vee & \\
(\neg B \wedge \neg A \wedge (C \vee \neg C) \wedge (D \vee \neg D)) & \qquad \text{To circuit} \\
(1 \cdot (3 \cdot (4+2) + (0.5 \cdot 4)) \cdot (3+1)) + & \\
(3 \cdot 1 \cdot (3+0.5) \cdot (4+2)) = & \\
80 + 63 = 143 &
\end{aligned}
$$

□

Exercise 5 (4 points) *Formalize in first order logic the following english statements using the following predicates*

$B(x, y, z)$	*y is between x and z*
$S(x)$	*x is a square*
$T(x)$	*x is a triangle*
$C(x)$	*x is a circle*
$A(x, y)$	*x is above y*

(1) If a circle is between two squares then it is above some triangle

(2) two triangle cannot be one above the other unles one of them is between two squares;

(3) A circle must be always between two triangles;

(4) it is not possible that an object is between two distinct pairs of objects.

Solution

(1) If a circle is between two squares then it is above some triangle

$$\forall xyz(B(x, y, z) \wedge S(x) \wedge C(y) \wedge S(z) \to \exists w(A(y, w) \wedge T(w)))$$

(2) two triangle cannot be one above the other unless one of them is between two squares;

$$\forall xy(T(x) \wedge T(y) \wedge A(x, y) \to \exists zw(S(z) \wedge S(w) \wedge (B(z, x, w) \vee B(z, y, w))))$$

(3) A circle must be always between two triangles;

$$\forall x(C(x) \to \exists yz(T(y) \wedge T(z) \wedge B(y, x, z)))$$

(4) it is not possible that an object is between two distinct pairs of objects.

$$\forall xyz(B(x, y, z) \wedge B(x', y, z') \to x = x' \wedge z = z')$$

□

Exercise 6 (4 points) *Let Σ be the signature of a formula $\phi(x, y)$, Explain how a Σ-structure $\mathcal{I}$ that satisfies $\forall x \exists y \phi(x, y)$ can be extended in a Σ'-structure $\mathcal{I}'$ on the signature $\Sigma' = \Sigma \cup \{f/1\}$ obtained by extending Σ with a new unary function symbol f, such that $\mathcal{I}' \models \forall x \phi(x, f(x))$. How is it called this operation?*

Solution The transofrmation described in the exercize is called Skolemization. Skolemization allow to eliminate quantifiers by introducing new constant or function symbolis. It is proved that Skolemizaiton perserves satifiability. I.e., the original formula is satifiable if and only if the SKolemized formula does.

Let us now solve the exercize. Our objective is to define an interpretation of f, i.e., $\mathcal{I}'(f)$ such that $\mathcal{I}' \models \forall x \phi(x, f(x))$. This means that for every element $d \in \Delta^{\mathcal{I}}$ we have to find a d' such that $\mathcal{I}'(f)(d) = d'$ and such that $\mathcal{I} \models \phi(x, y)[x \leftarrow d, y \leftarrow d']$. Notice that for every element d of the interpretation domain $\Delta^{\mathcal{I}}$, the fact that $\mathcal{I} \models \forall x \exists y \phi(x, y)$. implies that $\mathcal{I} \models \exists y \phi(x, y)[x \leftarrow d]$. This implies that there is a d' such that $\imath \models \phi(x, y)[x \leftarrow d, y \leftarrow d']$. Let us define the interpretation of the function symbol $\mathcal{I}'(f) : d \mapsto d'$. Then we have that $\mathcal{I}' \models \phi(x, f(x))[x \leftarrow d]$. If we repreat this reasoning for all the d of the domain we have a complete definition of $\mathcal{I}'(f)$, such that $\mathcal{I} \models \phi(x, f(x))[x \leftarrow d]$ for all individuals $d \in \Delta^{\mathcal{I}}$. From this we can conclude that $\mathcal{I}' \models \forall x \phi(x, f(x))$. □

Exercise 7 (5 points) *Provide an explicit mathematical formula to compute the number of models of $\Phi \triangleq \forall x(A(x) \to B(f(x)))$ in the domain $\{1, 2, 3, \ldots, n\}$. To find this formula you first have to find (and describe in text) a procedure on how you can build a model for Φ.*

Solution For every i in A, we have to find an element j in B such that $f(i) = j$. if B contains b elements we have b possibilities. So if we have a elements in a we have b^a possibilities. For any other element j which is not in A we can fix $f(j)$ to be any element of the domain. so we have n possibilites. If A contains a elements we have n^{n-a} possibilites. Considering all the possible cardinalities of A and B, we therefore have

$$\text{FOMC}(\Phi, n) = \sum_{a=0}^{n} \binom{n}{a} \sum_{b=0}^{n} \binom{n}{b} b^a n^{n-a}$$

□