Calculus 1
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Exercise 1 (punti 8) Consider the function

$$f(x) = x\, e^{\frac{1}{x}}$$

(a) find its (maximal) domain, determine its sign and study the possibility that f is odd or even:

Solution:

Domain: the domain is clearly equal to $D := \mathbb{R}\backslash\{0\}$

The function is neither even nor odd.

Sign: since $e^{\frac{1}{x}} > 0$ for every $x \neq 0$ one has

$$f(x) \geq 0 \iff x \geq 0$$

(b) compute limits and possible asymptotes ;

Limits at 0**:**

$$\lim_{x\to 0-} f(x) = 0 \cdot 0 = 0$$

$$\lim_{x\to 0+} f(x) = 0 \cdot \infty$$

indeterminate form ... Let us try the change of variable $y = \frac{1}{x}$

$$\lim_{x\to 0+} f(x) = \lim_{y\to +\infty} \frac{e^y}{y} = +\infty$$

So $x = 0$ is a (right) vertical asymptote at $x = 0$.

Limits at $\pm\infty$

$$\lim_{x\to +\infty} x\, e^{\frac{1}{x}} = +\infty \times 1 = +\infty$$

$$\lim_{x\to -\infty} x\, e^{\frac{1}{x}} = -\infty \times 1 = -\infty$$

Asymptotes:

$$\lim_{x\to +\infty} \frac{x\, e^{\frac{1}{x}}}{x} = \lim_{x\to +\infty} e^{\frac{1}{x}} = 1$$

$$\lim_{x\to +\infty} xe^{\frac{1}{x}} - x \overset{y=\frac{1}{x}}{=} \lim_{y\to 0+} \frac{1}{y}(e^y - 1) = \lim_{y\to 0+} \frac{1}{y}(y + o(y)) = 1 \implies \text{there is an asymptote at } +\infty : \quad y = x + 1$$

$$\lim_{x\to -\infty} \frac{x\, e^{\frac{1}{x}}}{x} = \lim_{x\to -\infty} e^{\frac{1}{x}} = 1$$

$$\lim_{x\to -\infty} xe^{\frac{1}{x}} - x \overset{y=\frac{1}{x}}{=} \lim_{y\to 0-} \frac{1}{y}(e^y - 1) = \lim_{y\to 0-} \frac{1}{y}(y + o(y)) = 1 \implies \text{there is an asymptote at } -\infty : \quad y = x + 1$$

(c) study the differentiability of f and find the derivative where possible (if necessary study the limits of the derivative); discuss the monotonicity of f and determine its supremum and infimum; if existing determine relative (=local) and absolutely ()=global) minima and maxima of f;

$$\forall x \neq 0,\ \ f'(x) > 0 \iff e^{\frac{1}{x}} - xe^{\frac{1}{x}}\frac{1}{x^2} = e^{\frac{1}{x}} - e^{\frac{1}{x}}\frac{1}{x} > 0 \iff 1 - \frac{1}{x} > 0 \iff x > 1 \text{ or } x < 0$$

and

$$f'(x) = 0 \iff x = 1$$

f is increasing on $]1, +\infty[$ and on $]-\infty, 0[$, and it is deacreasing on $]0, 1[$. Hence it has a local minimum at $x = 1$. The function is both upper unbounded and lower unbounded.

Finally

$$\lim_{x \to 0-} f'(x) = 0$$

so $y = 0$ is a left tangent at $x = 0$

(d) determine the second derivative and study the convexity of the function ;

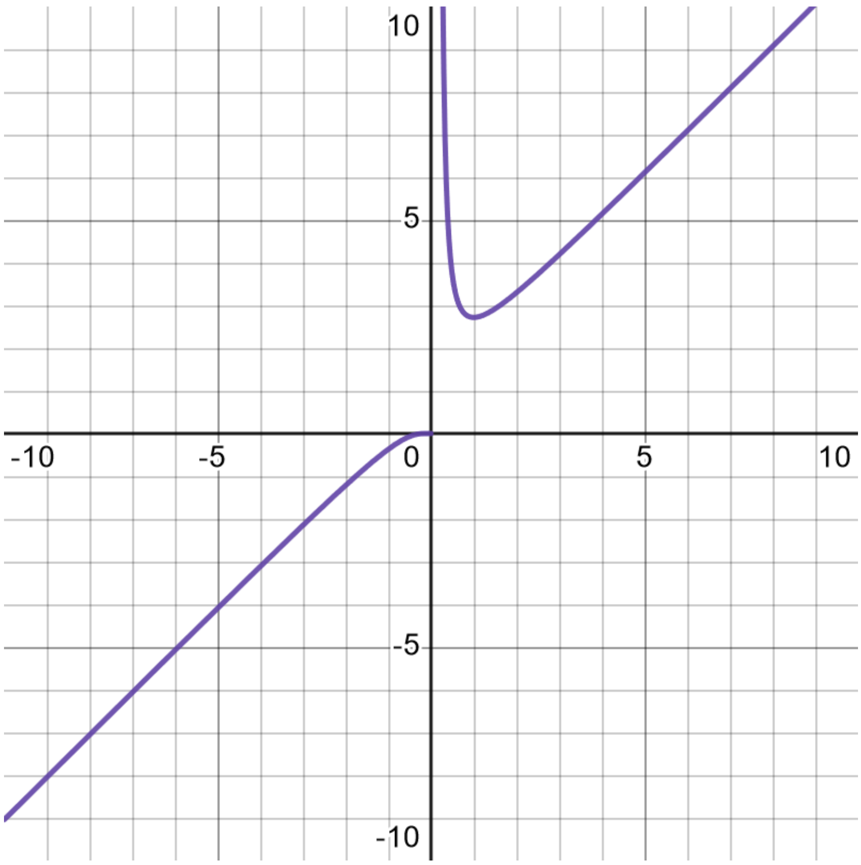
$$f''(x) = \frac{d}{dx}\left(e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right)\right) = -e^{\frac{1}{x}}\frac{1}{x^2}\left(1 - \frac{1}{x}\right) + e^{\frac{1}{x}}\frac{1}{x^2} = e^{\frac{1}{x}}\frac{1}{x^3}$$

so that

$$f''(x) > 0 \iff x > 0, \qquad f''(x) < 0 \iff x < 0.$$

Hence the function is convex on the interval $]0, +\infty[$ and concave on the interval $]-\infty, 0[$

(e) draw a qualitative graph of f.



Exercise 2 (punti 8) Consider the equation on complex numbers

$$z^6 + 2iz^3 - 1 = 0.$$

Find the solutions with their multiplicity, and draw the on the complex plane

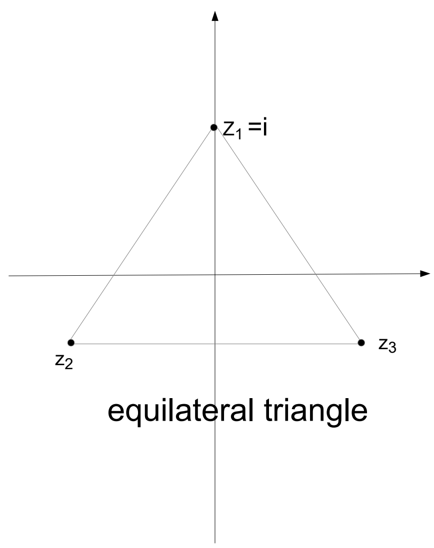
Solution Consider the substitution $w = z^3$, so that the equation becomes the second order equation

$$w^2 + 2iw - 1 = 0.$$

whose left-hand side is easily seen to be the square of the first order polynomial $(w+1)$, namely we obtain

$$(w + i)^2 = 0.$$

Therefore we have the only solution $w = -i$ with multiplicity equal to 2. If we write $z^3 = w = -i = e^{3\pi i/2}$ we obtain the three solutions $z_1 = e^{\pi i/2}$, $z_2 = e^{7\pi i/6}$ $z_3 = e^{11\pi i/6}$, each one with multiplicity 2.



Exercise 3 (punti 8) Study the behaviour of the following series for the values of the parameter $\alpha > 0$

$$\sum_{n=1}^{+\infty} n^\alpha \left(1 - \sqrt{\frac{n^2}{n^2+1}}\right)^{\alpha-1}.$$

Let us investigate the order of the sequence (this is not the only method):

$$n^\alpha \left(1 - \sqrt{\frac{n^2}{n^2+1}}\right)^{\alpha-1} = n^\alpha \left(1 - \sqrt{\frac{n^2}{n^2+1}}\right)^{\alpha-1} \frac{\left(1 + \sqrt{\frac{n^2}{n^2+1}}\right)^{\alpha-1}}{\left(1 + \sqrt{\frac{n^2}{n^2+1}}\right)^{\alpha-1}} =$$

$$n^\alpha \frac{\left(1 - \frac{n^2}{n^2+1}\right)^{\alpha-1}}{\left(1 + \sqrt{\frac{n^2}{n^2+1}}\right)^{\alpha-1}} = n^\alpha \frac{\left(\frac{1}{n^2+1}\right)^{\alpha-1}}{\left(1 + \sqrt{\frac{n^2}{n^2+1}}\right)^{\alpha-1}} \sim n^{\alpha-2\alpha+2} = n^{-\alpha+2}$$

Therefore, the series (which has positive terms) converges if and only if $-\alpha + 2 < -1$, i.e. if and only if $\alpha > 3$.

Exercise 4 (punti 8)
(a) Use De L'Hôpital Theorem to show that

$$\lim_{x\to\infty} \frac{\arctan(x+1) - \arctan(x)}{\frac{1}{x^2}} = 1;$$

Solution The limit is a form 0/0 so that we can apply De L'Hôpital Theorem provided the limit of the ratio of the derivatives does exist. Denoting with f and g the numerator and the denominator we have

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{(1+(x+1)^2)^{-1}}{(1+(x)^2)^{-1}} = 1$$

(b) Use the result stated in the previous point to discuss the behaviour of the generalized integral

$$\int_1^{\infty} \frac{1}{x^\alpha \left[\arctan(x+1) - \arctan(x)\right]}\, dx$$

for all values of the parameter $\alpha \in \mathbb{R}$.

From

$$\lim_{x\to\infty} \frac{\arctan(x+1) - \arctan(x)}{\frac{1}{x^2}} = 1$$

we get that the integrand

$$\frac{1}{x^\alpha \left[\arctan(x+1) - \arctan(x)\right]}$$

is of the same order as

$$\frac{1}{x^{\alpha-2}}$$

so that the integral converges if and only if $\alpha - 2 > 1$, i.e. if and only if $\alpha > 3$.