Knowledge Representation and Learning
Final exam

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Exercise 1 (4 points) *Find a formula ϕ that has the following truth table, and explain the method you have followed to find it.*

p	q	r	ϕ
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

Solution A possible way to proceed is based on three main steps (1) building is by associating a conjunction of literals that fully describes each interpretation that satisfies ϕ; (2) put them in a disjunction and (3) simplify the resulting disjunction as much as possible.

1. For every interpretation $\mathcal{I}$ on the set of proposition $\mathcal{P}$ we can define the conjunction of the literals that are satisfied by $\mathcal{I}$

$$\psi_{\mathcal{I}} \triangleq \bigwedge_{\substack{l \in Lit \\ \mathcal{I} \models l}} l$$

   where Lit is the set of literals on the propositional variables in $\mathcal{P}$. Notice that $\psi_{\mathcal{I}}$ is satisfied by $\mathcal{I}$ and $\mathcal{I}$ is the only model that satisfies $\psi_{\mathcal{I}}$.

2. Let us put in disjunction all the $\psi_{\mathcal{I}}$ such that $\mathcal{I} \models \phi$.

$$(p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \tag{1}$$

   Notice taht the above formula has exactly the same models than ϕ since is it the disjunction of the formulas that are true in each model of ϕ.

3. We can then simplify it. First notice that r is true in all the disjunction, so r must be true, We therefore obtain:

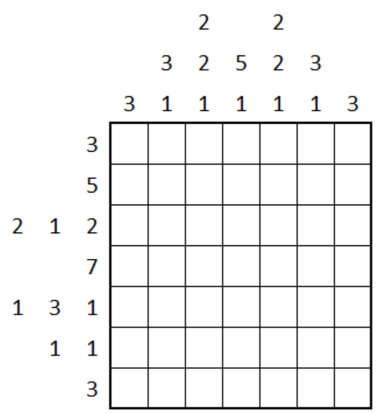
$$((p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)) \wedge r \tag{2}$$

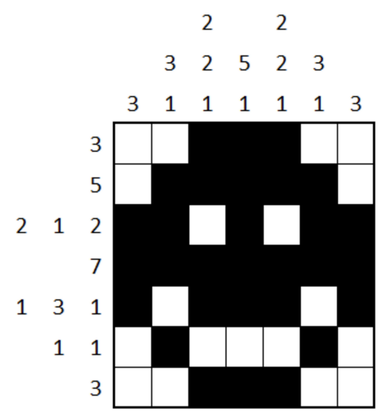
Furthermore notice that $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$ is equialent to $\neg(p \wedge \neg q)$ which is equivalent to $p \rightarrow q$

$$(p \rightarrow q) \wedge r$$

$\square$

Exercise 2 (5 points) *A nonogram is a grid with a series of numbers on the left of each row and above each column of the grid. Each of these numbers represents a consecutive run of shaded spaces in the corresponding row or column. Each consecutive run is separated from other runs by at least one empty space. The puzzle is complete when all of the numbers have been satisfied. See for instance the example belwo and the corresponding solution*





Describe a possible encoding of a generic configuration of a nanogram in propositional logic so that the solution can be obtained by running a sat solver. Show how you encode the clue contained in the second column of the above example.

Solution Let m, n be the size of the grid of a nanogram. We consider $n \times m$ propositional variables p_{rc} for $1 \leq r \leq n$ and $1 \leq c \leq m$. p_{rc} represents the proposition that the cell at row r and column c is black. A clue $n_1, \dots, n_k$ associated to the r-th row can be modelled by adding $k \cdot m$ propositional variables $rclue(r, j)_c$ for $j = 1, \dots, k$ and $c = 1, \dots, m$. $rclue(r, j)_c$ is true when the run associated to the j-th element of the clue of r starts at column c. Using this set of proposition, we can model the constraitns that must be satisfied by a solution of a nanogram. For every row r

if $n_1, \dots, n_k$ is the associated clue, we add the following axioms:

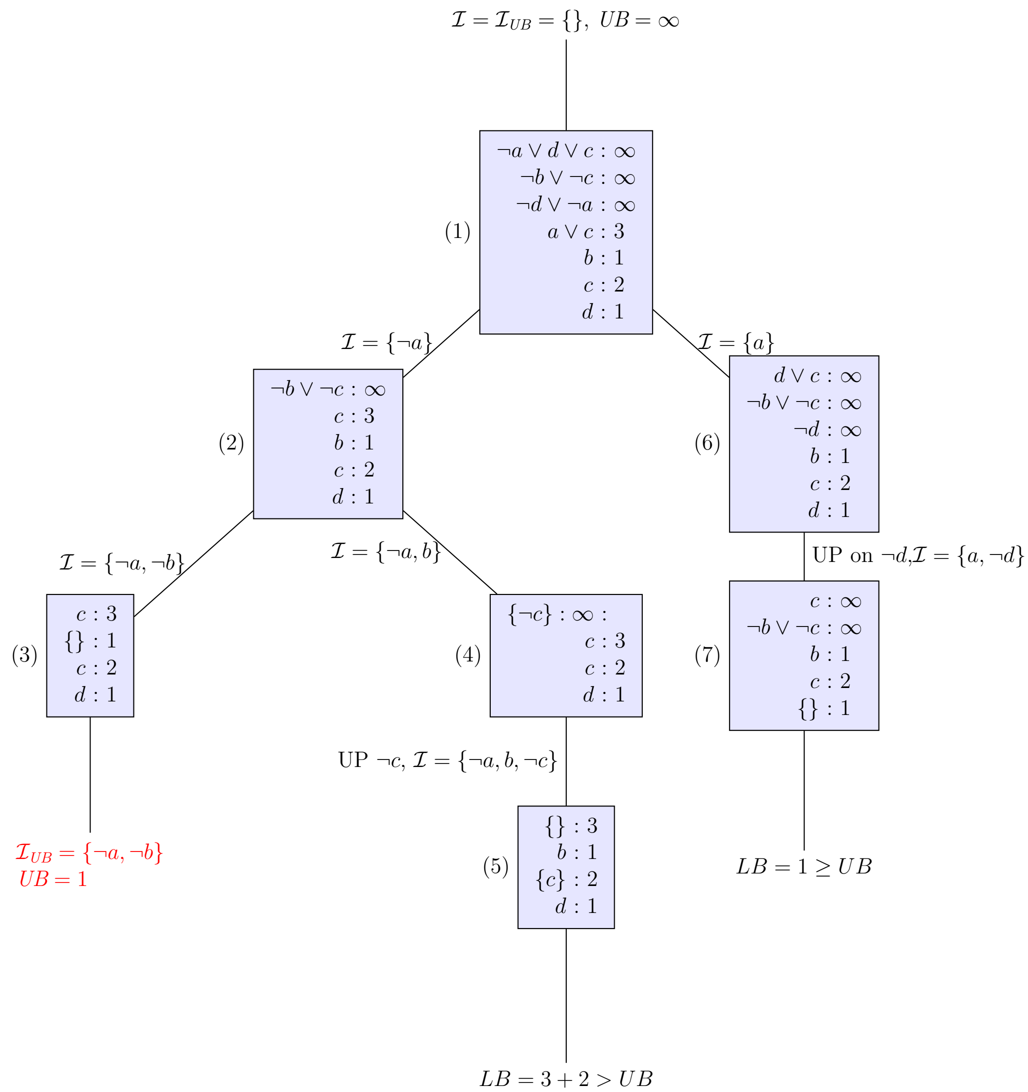
$\sum_{c=1}^{m-n_j} rclue(r,j)_c = 1$	The j-th run of the clue of the r-th row starts at exaclty one column between 1 and n_j
$\bigwedge_{c+n_i \geq c'} rclue^r_{ic} \to \neg rclue^r_{(i+1)c'}$	The $j+1$-run at least one step after the end of the i-th run
$rclue(r,j)_c \to \bigwedge_{c'=c}^{c+n_j-1} p_{rc'}$	the next n_j cells after the start of the j-th run must be black
$rclue(r,j)_c \wedge rclue(r,j+1)_{c'} \to \bigwedge_{c''=c+n_i}^{c'-1} \neg p_{rc''}$	The cells between the end of the j-th run and the beginning of the $j+1$-th run are not black
$rclue^r_{1c} \to \bigwedge_{c'=1}^{c-1} \neg p_{rc'}$	The cells before the initial run are not black
$rclue^r_{kc} \to \bigwedge_{c'=c+n_k}^{m} \neg p_{rc'}$	The cells after the last run are not black

The clues for the columns can be formalized in an analogours using the propositional variables $cclue(c,j)_r$. □

Exercise 3 (4 points) *Use B&B algorithm to solve the following maxSat problem*

$$
\begin{array}{ccc}
(\neg a \vee d \vee c : \infty) & (a \vee c : 3) & (c : 2) \\
(\neg b \vee \neg c : \infty) & (b : 1) & (d : 1) \\
(\neg d \vee \neg a : \infty) & &
\end{array}
$$

Solution



$\square$

Exercise 4 (5 points) *Suppose you have three coins: the faces of the first coin are black and white, the faces of the second coin are yellow and green, and the faces of the third coin are red and green. In an experiment you toss the first coin; if you obtain a black you toss the second coin otherwise you toss the third coin.*

1. *Model this experiment in propositional logic and*
2. *use model counting to determine what are the number of possible outcomes?*
3. *Let p, q and r be the probability of obtaining a black, yellow, and red faces when tossing the first, second and third coin respectively. Compute the probability of obtaining an outcome which is either red or green.*

Solution We can use the language B, W, R, G, Y to state that in the outcome there is a coin with a black, white, red, green, and yellow faces respectively. Notice that this is possible since there is no possibility to have outcomes with two coins with the same color face. We can now formalize the constraints of the game in terms of the following formulas:

$B + W = 1$	The toss of the first coin can have only one result among black and white
$R + Y + G = 1$	The toss of the second or third coin can have only one result since only one coin among the two is tossed
$B \rightarrow Y \vee G$	If you have a black then you can have only one among yellow and green since you toss the second coin
$W \rightarrow R \vee G$	If you have a white then you can have only one among red and green since you toss the third coin

The models that satisfies all the formulas are 4

$$\{B, Y\} \qquad \{B, G\} \qquad \{W, R\} \qquad \{W, G\}$$

If we associate the following weights:

$$w(B) = p \quad w(W) = 1 - p \quad w(Y) = q \quad w(G) = 1 - q \quad w(R) = r \quad w(G) = 1 - r$$

In this wey however we assign two weights to the same atom G. Indeed we have to distinguish when G is obtained by the second or by the third coin. To this purpose we introduce two new atoms

$$G_2 \leftrightarrow B \wedge G \qquad G_3 \leftrightarrow W \wedge G$$

Adding these new propositions does not change the number of models since they are fully defined in terms of the previous propositions. We still have 4 models by they are:

$$\{B, Y\} \qquad \{B, G, G_2\} \qquad \{W, R\} \qquad \{W, G, G_3\}$$

and update the weights for G to

$$w(G_2) = 1 - q \qquad w(G_3) = 1 - r$$

$$w(\{B, Y\}) = pq \quad w(\{B, G, G_2\}) = p(1 - q) \quad w(\{W, R\}) = (1 - p)r \quad w(\{W, G, G_3\}) = (1 - p))(1 - r)$$

Notice that the weight of all the models sum up to 1, and therefore they can be considered to be probabilities of the outcomes. Finally, the probability of in the result we have either red or green, is equal to the probability of the formula $R \vee G$ which can be computed by the sum of the probabilities of the models that satisfies $R \vee G$, i.e., $p(1-q)+(1-p)r+(1-p)(1-r) = p(1-q)+1-p = 1-pq$. $\square$

Exercise 5 (4 points) *Describe all the models of the following set of formulas in the domain* $\{1, 2, 3\}$.

$$\forall x \neg R(x, x) \tag{3}$$
$$\forall x \forall y (R(x, y) \rightarrow R(y, x)) \wedge \tag{4}$$
$$\forall x (A(x) \rightarrow \exists y (R(x, y) \wedge A(y))) \tag{5}$$

Solution

$\mathcal{I}(A)$	$\mathcal{I}(R)$
$\emptyset$	any symmetric relation on $\{1,2,3\}$
$\{1,2\}$	$\{(1,2),(2,1)\}$
$\{1,3\}$	$\{(1,3),(3,1)\}$
$\{2,3\}$	$\{(2,3),(3,2)\}$
$\{1,2\}$	$\{(1,2),(2,1),(1,3),(3,1)\}$
$\{1,2\}$	$\{(1,2),(2,1),(2,3),(3,2)\}$
$\{1,2\}$	$\{(1,2).(2,1),(1,3),(3,1),(2,3),(3,2))\}$
$\{1,3\}$	$\{(1,3),(3,1),(1,2),(2,1)\}$
$\{1,3\}$	$\{(1,3),(3,1),(3,2),(2,3)\}$
$\{1,3\}$	$\{(1,2).(2,1),(1,3),(3,1),(2,3),(3,2))\}$
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$\{2,3\}$	$\{(1,2).(2,1),(1,3),(3,1),(2,3),(3,2))\}$
$\{1,2,3\}$	$\{(1,2),(2,1),(1,3),(3,1))\}$
$\{1,2,3\}$	$\{(1,2),(2,1),(2,3),(3,2))\}$
$\{1,2,3\}$	$\{(1,3),(3,1),(2,3),(3,2))\}$
$\{1,2,3\}$	$\{(1,2).(2,1),(1,3),(3,1),(2,3),(3,2))\}$

$\square$

Exercise 6 (5 points) *Consider the following statements*

> *A grandparent of a person is a parent of a parent of the person*

Translate the above facts in FOL using the following symbols:

$$P(x,y) = x \textit{ is a parent of } y,\ P \textit{ is a binary predicate}$$
$$G(x,y) = x \textit{ is a grandparent of } y,\ G \textit{ is a binary predicate}$$

Then use resolution to show that if x and y have the same parents they also have the same grandparents. Only formulate the problem in clausal form without doing the resolution proof.

Solution The definition of grandparent can be obtrained by formalizing the sentence: "x is a granparent of y if there is a z that has x as parent and is the parent of z. In FOL

$$\forall x \forall y (G(x,y) \leftrightarrow \exists z P(x,z) \wedge P(z,y))$$

in clausal form after Skolemization:

$$\{\neg G(x,y), P(x,f(x,y))\}$$
$$\{\neg G(x,y), P(f(x,y),y)\}$$
$$\{\neg P(x,f(x,y)), \neg P(f(x,y),y), G(x,y)\}$$

The goal states that if two people has the same parents, then they have the same grandparents. Let us first formalize the sentence x and y has the same parents. This can be formalized by $\forall z(P(z,x) \leftrightarrow P(z,y))$ Similarly "x, y having the same grapdparents" can be formalized with the formula $\forall z(G(z,x) \leftrightarrow G(z,y))$. The entire statement is the implication between the two formulas for every x and y. i.e.,

$$\forall x \forall y (\forall z(P(z,x) \leftrightarrow P(z,y)) \rightarrow \forall z(G(z,x) \leftrightarrow G(z,y)))$$

Negate the goal and transform in CNF

$$\begin{aligned}
\neg\forall x \forall y (\forall z(P(z,x) \leftrightarrow P(z,y)) \rightarrow \forall z(G(z,x) \leftrightarrow G(z,y))) \\
\exists x \exists y (\forall z(P(z,x) \leftrightarrow P(z,y)) \wedge \exists z \neg(G(z,x) \leftrightarrow G(z,y))) \\
\forall z(P(z,a) \leftrightarrow P(z,b)) \wedge \neg(G(c,a) \leftrightarrow G(c,b))) \\
\{\neg P(z,a), P(z,b)\}, \{\neg P(z,b), P(z,a)\}, \{\neg G(c,a), \neg G(c,b)\}, \{G(c,a), G(c,b)\},
\end{aligned}$$

□

Exercise 7 (4 points) *Using the formula for first order model counting compu¯te the number of models of the FO²-formula*

$$\forall xy(R(x,x) \rightarrow (R(x,y) \rightarrow R(y,x)))$$

Solution Let us first determine which are the 1-types and the 2-tables. The 1-types are

$$\begin{aligned}
1(x) &\triangleq R(x,x), \\
2(x) &\triangleq \neg R(x,x),
\end{aligned}$$

and the following 2-tables

$$\begin{aligned}
1(x,y) &\triangleq R(x,y) \wedge R(y,x) \wedge x \neq y \\
2(x,y) &\triangleq R(x,y) \wedge \neg R(y,x) \wedge x \neq y \\
3(x,y) &\triangleq \neg R(x,y) \wedge R(y,x) \wedge x \neq y \\
4(x,y) &\triangleq \neg R(x,y) \wedge \neg R(y,x) \wedge x \neq y
\end{aligned}$$

Now let us compute n_{11}, n_{12} and n_{22}. To do so we have to do the grounding of the formula obtaining:

$$\begin{aligned}
&(R(c,c) \rightarrow (R(c,c) \rightarrow R(c,c))) \wedge \\
&(R(c,c) \rightarrow (R(c,d) \rightarrow R(d,c))) \wedge \\
&(R(d,d) \rightarrow (R(d,c) \rightarrow R(c,d))) \wedge \\
&(R(d,d) \rightarrow (R(d,d) \rightarrow R(d,d)))
\end{aligned}$$

which can be simplified in

$$(R(c,d) \rightarrow (R(c,d) \rightarrow R(d,c))) \;\wedge (R(d,d) \rightarrow (R(d,d) \rightarrow R(c,d))) \tag{6}$$

Let us now construct the truth table

2-type	$R(c,c)$	$R(d,d)$	$R(c,d)$	$R(d,c)$	(6)
$111(c,d)$	T	T	T	T	T
$112(c,d)$	T	T	T	F	F
$113(c,d)$	T	T	F	T	F
$114(c,d)$	T	T	F	F	T
$121(c,d)$	T	F	T	T	T
$122(c,d)$	T	F	T	F	T
$123(c,d)$	T	F	F	T	T
$124(c,d)$	T	F	F	F	T
$221(c,d)$	F	F	T	T	T
$222(c,d)$	F	F	T	F	T
$223(c,d)$	F	F	F	T	T
$224(c,d)$	F	F	F	F	T

from which we have that $n_{11} = 2$, $n_{12} = 3$ and $n_{22} = 4$ We can then replace in the formula for FOMC

$$\sum_{k=1}^{4} \binom{4}{k} n_{11}^{\frac{k(k-1)}{2}} n_{12}^{k(4-k} n_{22}^{\frac{(4-k)(3-k)}{2}} = 4^6 + 4(3^3 4^3) + 6(2^1 3^4 4^1) + 4(2^3 3^3) + 2^6$$

$\square$