Logic for Knowledge Representation, Learning, and Inference

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Contents

CHAPTER 1

First Order Model Counting

In the same manner in which first-order logic is a generalization of propositional logic, one can ask him/herself how propositional model counting generalizes to first-order model counting. This amounts to making sense of the questions: how many Σ structures satisfy a first-order formula? Without further specifications, the answer is: a formula either has 0 models, if it is not satisfiable, or it has an infinite number of models if it is satisfiable. Indeed if a formula has one model $\mathcal{I}$ with domain $\Delta^{\mathcal{I}}$ then it has an infinite set of models with domains isomorphic to $\Delta^{\mathcal{I}}$. If we change our question to: is it possible to count the models of a first-order formula on a given domain? This makes more sense. However, if a formula is satisfiable by an infinite domain there are still infinite models. Indeed if $\Delta^{\mathcal{I}}$ is infinite and π is a one-to-one function from $\Delta^{\mathcal{I}}$ to $\Delta^{\mathcal{I}}$ we can define an interpretation $\mathcal{I}^{\pi}$ that satisfies the same formulas of $\mathcal{I}$ (proof by exercise). If $\Delta^{\mathcal{I}}$ is infinite then when $\mathcal{I} \models \phi$ there might be infinite $\mathcal{I}^{\pi}$ that satisfy ϕ. For this reason, we concentrate on first-order model counting on a finite domain that contains n elements. Without loss of generality, we concentrate on the domain of the first n integers $\{1, \ldots, n\}$ also denoted by $[n]$.

Definition 1.1 (First order model counting). *The problem of* first order model counting *is the problem of computing the number of Σ-structures that satisfy a first-order sentence ϕ on a given finite domain of $n \geq 2$ elements. The problem is denoted as*

$$\textsc{fomc}(\phi, n)$$

Remark 1. *Notice that in first-order model counting, we are interested in the case in which the domain contains at least 2 elements. This is because if $n = 1$ the problem reduces to propositional model counting. Indeed, if $n = 1$ we have that the formula $\forall x \phi(x) \leftrightarrow \exists x \phi(x)$ and $Q_1 x_1, \ldots, Q_n x_n \phi(x_1, \ldots, x_n)$ is equivalent to $\forall x \phi(x, \ldots, x)$. This amounts in propositional model counting $\phi(a, \ldots, a)$ for some constant a.* $\textsc{fomc}(\phi, [1]) = \#\textsc{sat}(\mathit{Ground}(\phi, \{a\}))$.

1. Formalizing Counting Problems in fomc

First-order model counting provides a general methodology for solving a problem of counting a set of items w.r.t. some integer parameter n. Such a methodology is based on three main steps.

(1) Define a FOL signature Σ such that the items to be counted are mapped one-to-one is a set S of Σ-structures on a domain of n elements;
(2) Provide a complete axiomatization of S in terms of a finite set of first order formula $\phi_1, \ldots, \phi_k$. This means $\mathcal{I} \models \phi_1 \wedge \cdots \wedge \phi_k$ if and only if $\mathcal{I}$ corresponds to an element of S.
(3) Compute $\textsc{fomc}(\phi_1 \wedge \cdots \wedge \phi_k, n)$.

In the following, we provide a few examples of the formalization of counting problems in FOMC.

EXAMPLE 1.1. *The number of undirected graphs with n nodes can be obtained by* FOMC(UG, n)

$$UG \triangleq \forall x \forall y (\neg R(x,x) \wedge (R(x,y) \leftrightarrow R(y,x)))$$

EXAMPLE 1.2. *The number of undirected graphs with n nodes with at most k arks can be obtained by* FOMC$(UG \wedge |R| \leq k, n)$ *were*

$$|R| \leq k \triangleq \forall x_1 \dots \forall x_{k+1} \forall y_1 \dots \forall y_{k+1} \left(\bigwedge_{i=1}^{k+1} R(x_i, y_i) \rightarrow \bigvee_{i<j=1}^{k+1} (x_i = x_j \wedge y_i = y_j) \right)$$

EXAMPLE 1.3. *The number of 3-coloured undirected graphs with n nodes is equal to* FOMC$(UG \wedge 3C, n)$

$$3C \triangleq \forall x \forall y ((C_1(c) \veebar C_2(x) \veebar C_3(x)) \wedge R(x,y) \rightarrow \bigwedge_{i=1}^{3} (\neg C_i(x) \wedge C_i(y)))$$

In the formula, we use the connective $\veebar$ for exclusive or, which is definable in terms of the other connectives. Namely, $a \veebar b \triangleq (a \vee b) \wedge \neg(a \wedge b)$.

EXAMPLE 1.4. *the number of graphs with n vertexes, and such that every pair of nodes are connected with a path with length $\leq k$. To encode this the problem we have to extend the signature with k new binary symbols $R_1 \dots R_k$. Intuitively $R_i(x,y)$, x is connected with y with a path of length i. We can solve this counting problem my computing:* FOMC$(UG \wedge R_{\leq k}, n)$

$$\begin{aligned} R_{\leq k} \triangleq\ & \forall x \forall y\, R_{\leq k}(x,y) \\ & \wedge \forall x \forall y\, (R_{\leq 1}(x,y) \leftrightarrow R(x,y)) \\ & \wedge \forall x \forall y (\bigwedge_{i=2}^{k-1} (R_{\leq i}(x,y) \leftrightarrow (R_{\leq i-1}(x,y) \vee \exists z (R_{\leq i-1}(x,z) \wedge R(z,y))) \end{aligned}$$

EXAMPLE 1.5. *Compute the number of configurations of a group of n people composed of PhD students and professors knowing that every student has a supervisor that is a professor, every professor supervises at least one student. To formalize the problem, we introduce two unary predicates $\mathsf{Prof}/1$ and $\mathsf{Stud}/1$ that represents the professors and the students respectively, and a binary predicate $\mathsf{Super}/2$ that represent the relations between a student and his/her supervisor. To compute the number of the configuration described above we can compute* FOMC(SP, n) *where SP is the following set of formulas:*

$$SP = \left\{ \begin{array}{l} \mathsf{Prof}(x) \veebar \mathsf{Stud}(x) \\ \mathsf{Super}(x,y) \rightarrow \mathsf{Stud}(x) \wedge \mathsf{Prof}(y) \\ \mathsf{Stud}(x) \rightarrow \exists y (\mathsf{Prof}(y) \wedge Super(x,y)) \end{array} \right\}$$

2. Solving FOMC for specific FOL formulas

Before considering a systematic and general enough method to solve FOMC(ϕ, n) for every formula ϕ in (a subclass of) first-order language. let us see some examples on the solution of FOMC(ϕ, n) for specific formulas ϕ.

EXAMPLE 1.6. *To compute* FOMC(ϕ, n) *when* ϕ *is*

$$\exists x \exists y (A(x) \wedge R(x, y) \wedge B(y))$$

We can reason as follows: if A is interpreted in a subset of a elements and B is interpreted in a subset of b elements. then R cannot be interpreted in any subset of $[n] \setminus \mathcal{I}(A) \times [n] \cup [n] \times [n] \setminus \mathcal{I}(B)$. Since there fre $2^{na} + 2^{nb} - 2^{ab}$ such a subsets, we have that

$$\text{FOMC}(\phi, n) = 2^{n^2+2n} - \sum_a \sum_b \binom{n}{a}\binom{n}{b} 2^{an+bn-ab}$$

EXAMPLE 1.7. *To count the models of* FOMC$(\forall x R(a, x), n)$ *we can proceed as follows. We have n possibilities to interpret the constant a and for all the other $n-1$ can be connected via R with any subset of $[n]$ which means that there are $2n(n-1)$ possible choices. We, therefore, have that*

$$\text{FOMC}(\forall x R(a, x)) = n 2^{n(n-1)} \tag{1}$$

EXAMPLE 1.8. *Formulas which are largely used in knowledge and ontology engineering have one of the following two forms:*

$$\phi_1 \triangleq \forall x (A(x) \to \forall y (R(x, y) \to B(y)))$$
$$\phi_2 \triangleq \forall x (A(x) \to \exists y (R(x, y) \wedge B(y)))$$

ϕ_1 can be rewritten in $\forall x \forall y (A(x) \wedge \neg B(y) \to \neg R(x, y))$. If A is interpreted in a elements and B in b elements then $\neg R$ should contain $\mathcal{I}(A) \times \mathcal{I}(\neg B)$ plus some subset of $\mathcal{I}(\neg A) \times \mathcal{I}(\neg B) \cup \mathcal{I}(\neg A) \times \mathcal{I}(B) \cup \mathcal{I}(A) \times \mathcal{I}(B)$. Therefore there are $2^{n^2-an+ab}$ possible interpretation of $\not R$ and therefore also prossible interpretations of R. Therefore the total number of interpretations that satisfies ϕ_2 is

$$\text{FOMC}(\phi_1, n) = \sum_a \sum_b \binom{n}{a}\binom{n}{b} 2^{n^2-an+ab}$$

About ϕ_2, if $\mathcal{I}(A)$ contains a elements and $\mathcal{I}(B)$ b elements then, for every element in $\mathcal{I}(A)$ we have to select a non empty subset of $\mathcal{I}(B)$ and any subset of $\mathcal{I}(\neg B)$. Therefore for every element of $\mathcal{I}(A)$ we have $(2^b - 1)2^{n-b}$ possibilities. For the elements not in $\mathcal{I}(A)$ we can select any subset of the n element having $2^{n(n-a)} = 2^{n^2-an}$ possibilities.

$$\text{FOMC}(\phi_2, n) = \sum_a \sum_b \binom{n}{a}\binom{n}{b} (2^b - 1)^a 2^{n^2-ab}$$

EXAMPLE 1.9. *Counting the number of transitive relation on a set of n elements has been the object of study in discrete mathematics. In first-order model counting terms this means finding a formula for* FOMC$(\mathit{Trans}(R), n)$ *where*

$$\mathit{Trans}(R) \triangleq \forall x \forall y \forall z (R(x, y) \wedge R(y, z) \to R(x, z))$$

Mala 2022 proves that any formula for the number of transitive relations on a set cannot be a polynomial. At the same time it provides some interesting recursive lower and upper bounds for FOMC$(\mathit{Trans}(R), n)$. *To have an idea of how*

FOMC(*Trans*$(R), n)$ *behaves w.r.t.* n *we report here the sequence reported by the On-Line Encyclopedia of Integer Sequences (OEIS) OEIS Foundation Inc. n.d.*

n	FOMC(*Trans*$(R), n)$	n	FOMC(*Trans*$(R), n)$
1	2	10	7307450299510288
2	13	11	3053521546333103057
3	171	12	1797003559223770324237
4	3994	13	1476062693867019126073312
5	154303	14	1679239558149570229156802997
6	9415189	15	2628225174143857306623695576671
7	878222530	16	5626175867513779058707006016592954
8	122207703623	17	16388270713364863943791979866838296851
9	24890747921947	18	64662720846908542794678859718227127212465

3. FOMC via grounding

A first naïve idea to develop a general procedure to compute FOMC(ϕ, n) can be obtained by grounding the formula with n constants, which results in a propositional formula, and then apply propositional model counting. Since we are dealing with finite domains we can reduce first order formulas to equivalent propositional formulas by grounding quantifiers:

DEFINITION 1.2. *For every First Order sentence (= formula with no free variables)* ϕ *on a signature* Σ*, and set of constants* C*,* $Ground(\phi, C)$ *is recursively defined as follows:*

(1) $Ground(\phi, C) = \phi$ *if* ϕ *does not contain quantifiers;*
(2) $Ground(\forall x.\phi(x), C) = \bigwedge_{c \in C} Ground(\phi(c), C)$
(3) $Ground(\exists x.\phi(x), C) = \bigvee_{c \in C} Ground(\phi(c), C)$
(4) $Ground(\phi \circ \psi, C) = Ground(\phi, C) \circ Ground(\phi, C)$ *for every connective* $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$*;*
(5) $Ground(\neg\phi, C) = \neg Ground(\phi, C)$

In other words the operation of grounding a firt order formula w.r.t, a set of constants C replaces the universal quantifier $\forall x$ with a big conjunction where the variable x is replaced with each constant $c \in C$ and each existential quantifier $\exists x$ is replaced by a big disjunciton where c is replaced with each of the constant in C.

EXAMPLE 1.10. $Ground(\forall x(A(x) \rightarrow \exists y(R(x, y) \wedge B(y))), \{a, b\})$

$$\begin{array}{c} A(a) \rightarrow (R(a,a) \wedge B(a)) \vee (R(a,b) \wedge B(b)) \wedge \\ A(b) \rightarrow (R(b,a) \wedge B(a)) \vee (R(b,b) \wedge B(b)) \end{array}$$

EXAMPLE 1.11. $Ground(\forall x, y.(R(x, y) \rightarrow R(y, x)), C) =$

$$\bigwedge_{c \in C} \bigwedge_{c' \in C} R(c, c') \rightarrow R(c', c)$$

Let us now shw the circumstances under which FOMC(ϕ, n) can be translated in propositional model counting.

PROPOSITION 1.1. *If* ϕ *is a first order sentence on a signature* Σ *containing only predicate symbols (i.e., no constant and function symbols), then*

$$\text{FOMC}(\phi, n) = \#\text{SAT}(Ground(\phi, \{c_1, \dots, c_n\}), \mathcal{HB}_{\Sigma \cup \{c_1, \dots, c_n\}})$$

where for every signature Σ, $\mathcal{HB}_\Sigma$ *denotes the Herbrand's base (i.e. the set of ground atomc) that can be built from* Σ.

Proof Outline. For every model $\mathcal{I}$ of ϕ on the domian of $\{1,\dots,n\}$ We define the following bijection:

$$\mathcal{I}_{FOL} \models p(x_1,\dots,x_n)[a_{x_1 \leftarrow d_1,\dots,x_n \leftarrow d_n}] \text{ iff } \mathcal{I}_{PROP}(p(c_{d_1},\dots,c_{d_n})) = 1$$

One can easily show that this mapping is an isomorphism between the set fo FOL interrpetatins on $\{1,\dots,n\}$ and the propositional assignment $\mathcal{I}_{PROP}$ and thast $\mathcal{I}_{FOL} \models \phi$ if and only if $\mathcal{I}_{PROP} \models ground(\phi, \{c_1 \dots, c_n\})$ $\square$

Notice that Proposition 1.1 requires that the formula does not contain neither constants nor functional symbols. If these symbols are there one have to provide also an interpretations of constants and function symbols. The rewriting is still possible but a bit more convoluted. Consider for instance the face that ϕ is the formula $\forall x R(a,x)$ for some constant a. The grounding is $R(a,c_1) \wedge \dots \wedge R(a,c_n)$. Notice that $\mathcal{HB}_{\{R,a,c_1,\dots,c_n\}}$ contains $(n+1)^2$ distinct propositional atoms, and only n of them occour in the grounding. This implies that $\#\text{SAT}(R(a,c_1) \wedge \dots \wedge R(a,c_n), \mathcal{HB}_{R,a,c_1,\dots,c_n}) = mc(R(a,c_1) \wedge \dots \wedge R(a,c_n)) \cdot 2^{((n+1)^2)-n} = 2^{n^2+n+1}$. The difference is due to the fact that on the domain of n element a is interpreted in one of the elements of the domain, and therefore $a = c_i$ is true for at least one c_i, Furthermore, when a is interpreted in the same element than c_i then the propositional variable $R(a,c_j)$ is equivalent to $R(c_i,c_j)$ therefore the two propositon cannot be interpreted independently.

The positive aspect of the metod of grounding in that it is a general method for FOMC which works for every first order sentence (without constants and function symbols). However, it has one major drawback, which is the fact that the grounding operation has the undesirable effect of exponentially exploding the formula. For instance the grounding of the formula $Q_1x_1,\dots,Q_nx_kP(x_1,\dots,x_k)$ on a domain of n elements generates a conjunction of n^k formulas $\psi(c_1,\dots,c_k)$ where $n = |C|$. This conjunction will contain a polinomially large (in n) number of propositional variables and we know that model counting algorithm take exponential time in the number of propositional variables. This means that the complexity of this method will grow exponentially with the number of domain elements.

4. Liftability in FOMC

The notion of *liftability* has been introduced in Statistical Relational Learning models Poole 2003 as the capability of carry out probabilistic inference without grounding a probabilistic model to every single instance in the domain, assuming that objects are undistinguished. Since, one of the most important motivation for developing first order model counting is to develop liftable methods for probabilistic inference, the notion of liftability is very central in FOMC.

Definition 1.3 (Liftable class of formulas). *A class* $\mathcal{C}$ *of first order formulas are* liftable *(for FOMC) if for every sentence* $\phi \in \mathcal{C}$ *there is an algorithm to compute* $\text{FOMC}(\phi, n)$ *that runs in time polynomial in* n.

The work Jaeger and Van den Broeck 2012 the authors provides a set of positive and negative results on liftability of certain classes of first order logic formulas. Here we concentrate with one of the most well known classes of first order logic formulas for which FOMC has been shown to be liftable. This is the C^2 class. In the rest of the capter we concentrate on this class.

5. The Two-Variable Fragments: $\mathcal{L}^2$

FO^2 is the class of first order logical formulas that contains only two variables. Conventionally, these two variables are x and y. chapter we will mainlybe dealing with this fragment and its various extensions. From now on,

Definition 1.4. *For every $k \geq 1$ the language $\mathcal{L}^k$ contains all the first order formulas that can be build using only k individual variables.*

Example 1.12. *The following are formulas of FO^2;*

- $\forall x \exists y (R(x, y) \wedge A(x) \wedge B(y) \wedge \neg x = y)$
- $\exists x (A(x) \wedge \forall y (R(x, y) \rightarrow \exists x R(y, x) \wedge B(x))$

Example 1.13. $\forall x, y, z. R(x, y) \wedge R(y, z) \rightarrow R(x, z)$ *is a formula in $\mathcal{L}^3$, that formalizes the fact that R is a transitive relation. Such a condition cannot be expressed in $\mathcal{L}^2$.*

From now on we will concentrate on $\mathcal{L}^2$.

5.1. Types and tables. In the following, we introduce the notion of 1-type. A 1-type describe one of the possible configuration of any element of a domain. A 1-type is a combination of all the unary properties an individual can have, e.g., “being red”, “being italian”, “not being male”, ...

Definition 1.5 (1-type). *Given a FOL signature Σ a 1-type is a conjunction of maximally consistent set of literals containing exactly one variable and no constants.*

Example 1.14 (1-type). *Let $\Sigma = \{A/1, R/2, S/3\}$ (the notation X/n means that X is a predicate with arity equal to n) The set of 1-types of Σ are:*

$$
\begin{array}{ll}
A(x) \wedge R(x,x) \wedge S(x,x,x) & A(x) \wedge R(x,x) \wedge \neg S(x,x,x) \\
A(x) \wedge \neg R(x,x) \wedge S(x,x,x) & A(x) \wedge \neg R(x,x) \wedge \neg S(x,x,x) \\
\neg A(x) \wedge R(x,x) \wedge S(x,x,x) & \neg A(x) \wedge R(x,x) \wedge \neg S(x,x,x) \\
\neg A(x) \wedge \neg R(x,x) \wedge S(x,x,x) & \neg A(x) \wedge \neg R(x,x) \wedge \neg S(x,x,x)
\end{array}
$$

Notice that in a 1-type we have also atoms with binary, ternary, and more in general n-ary predicates. The key point is that these predicates are applied only to a (n-tuple) of a single variable.

Proposition 1.2. *If Σ contains n predicates there are 2^n 1-types.*

We use natural number $1(x)$, $2(x)$, ..., $u(x)$ to denote the 1-types. u is used to denot the last 1-type and the total number of 1-types. The notation $i(y)$, where $i(x)$ is 1-type and y a variable is the result of replacing x with y in $i(x)$. We use a similar notation for constants c where $i(c)$ denotes the replacement of x with c in the 1-type $i(x)$.

Analogously to 1-types, which describe the values of all the boolean properties of an individual, we want to have a similar notion that describes the type of relationship between two individuals. e.g “x is the boss of y”, “x is older than y” “x is a friend of y”, “x and y share the same office”, ... Notice that x and y must stay for two distinct individuals since the case in which x and y denotes the same individual is already part of the 1-type of x (e.g., "x is the boss of x"). THis is the notion of 2-table

DEFINITION 1.6 (2-table). *A* 2-table *of a FOL signature* Σ *is any conjunction of a maximally consistent set of literals containing exactly two distinct variables* x, y *and the literal* $x \neq y$.

EXAMPLE 1.15 (2-table). *Let* $\Sigma = \{R/2, S/2\}$

$$
\begin{array}{l}
R(x,y) \wedge R(y,x) \wedge S(x,y) \wedge S(x,y) \wedge x \neq y \\
R(x,y) \wedge R(y,x) \wedge S(x,y) \wedge \neg S(y,x) \wedge x \neq y \\
R(x,y) \wedge R(y,x) \wedge \neg S(x,y) \wedge S(y,x) \wedge x \neq y \\
R(x,y) \wedge R(y,x) \wedge \neg S(x,y) \wedge \neg S(y,x) \wedge x \neq y \\
R(x,y) \wedge \neg R(y,x) \wedge S(x,y) \wedge S(y,x) \wedge x \neq y \\
R(x,y) \wedge \neg R(y,x) \wedge S(x,y) \wedge \neg S(y,x) \wedge x \neq y \\
R(x,y) \wedge \neg R(y,x) \wedge \neg S(x,y) \wedge S(y,x) \wedge x \neq y \\
R(x,y) \wedge \neg R(y,x) \wedge \neg S(x,y) \wedge \neg S(y,x) \wedge x \neq y \\
\neg R(x,y) \wedge R(y,x) \wedge S(x,y) \wedge S(y,x) \wedge x \neq y \\
\neg R(x,y) \wedge R(y,x) \wedge S(x,y) \wedge \neg S(y,x) \wedge x \neq y \\
\neg R(x,y) \wedge R(y,x) \wedge \neg S(x,y) \wedge S(x,y) \wedge x \neq y \\
\neg R(x,y) \wedge R(y,x) \wedge \neg S(x,y) \wedge \neg S(y,x) \wedge x \neq y \\
\neg R(x,y) \wedge \neg R(y,x) \wedge S(x,y) \wedge S(y,x) \wedge x \neq y \\
\neg R(x,y) \wedge \neg R(y,x) \wedge \neg S(x,y) \wedge S(y,x) \wedge x \neq y \\
\neg R(x,y) \wedge \neg R(y,x) \wedge \neg S(x,y) \wedge \neg S(y,x) \wedge x \neq y \\
\neg R(x,y) \wedge \neg \wedge S(x,y) \wedge \neg S(y,x) R(y,x) \wedge x \neq y
\end{array}
$$

A special case of 2-tables happens when the signature contains only unary predicates. In this case there is only a single 2-table which is $x \neq y$.

Similarly to what we have done for 1-types, we use the notation $1(x,y), 2(x,y), \ldots b(x,y)$ to denote 2-table of a FOL signature Σ, and b denotes the number of the 2-tables.

PROPOSITION 1.3. *if* Σ *contains* n_i *predicates with arity equal to* i, *then there are* $2^{\sum_i n_i(2^i-2)}$

We assume an arbitrary order on 1-types and 2-table. Finally we define 2-type that is a full description of the properties of two distinct domain elements and their relations.

DEFINITION 1.7 (2-type). *Given a FOL signature* Σ *a* 2-type *is the conjunction of a maximally consistent set of literals containing at most two distinct variables* x, y *and no constants and the literal* $x \neq y$.

Notice that a 2-type is the conjunction of two one types one for x and another for y and a 2-table. Therefore we denote 2-types with three numbers $ijl(x,y)$ where i and j are the 1-types of x and y respectively and l is the 2-table of x and y. Formally we have that $ijl(x,y)$ is equal to $i(x) \wedge j(y) \wedge l(x,y)$.

EXAMPLE 1.16. *The set of 2-types of the FOL signature* $\Sigma = \{R/2\}$ *are*

$$R(x,x) \wedge R(y,y) \wedge R(x,y) \wedge R(y,x) \wedge x \neq y$$
$$R(x,x) \wedge R(y,y) \wedge R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$
$$R(x,x) \wedge R(y,y) \wedge \neg R(x,y) \wedge R(y,x) \wedge x \neq y$$
$$R(x,x) \wedge R(y,y) \wedge \neg R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$
$$R(x,x) \wedge \neg R(y,y) \wedge R(x,y) \wedge R(y,x) \wedge x \neq y$$
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$$R(x,x) \wedge \neg R(y,y) \wedge \neg R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$
$$\neg R(x,x) \wedge R(y,y) \wedge R(x,y) \wedge R(y,x) \wedge x \neq y$$
$$\neg R(x,x) \wedge R(y,y) \wedge R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$
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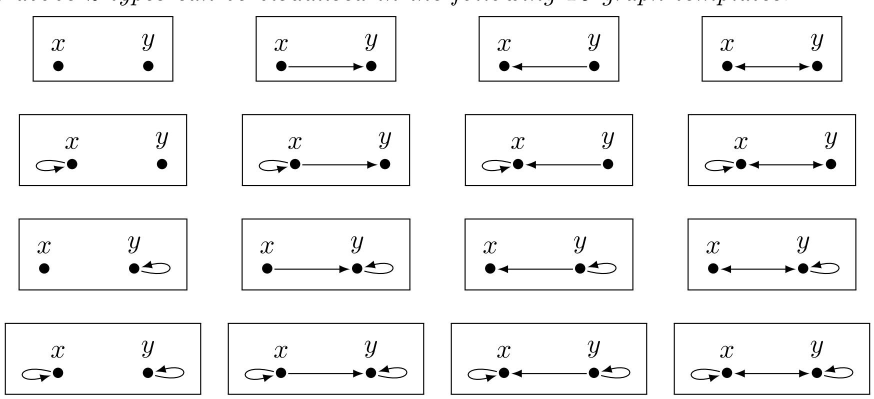
$$\neg R(x,x) \wedge \neg R(y,y) \wedge \neg R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$

The above 2-types can be visualised in the following 16 graph templates:



DEFINITION 1.8. *For every* Σ*-structure* $\mathcal{I}$

(1) *a constant* c realizes *a 1-type* i *if* $\mathcal{I} \models i(c)$*;*
(2) *every set of two constants* $\{c, d\}$ realizes *a 2-type* $ijl(x, y)$*, with* $i < j$ *if either* $\mathcal{I} \models ijl(c, d)$ *or* $\mathcal{I} \models ijl(d, c)$*;*
(3) *every set of two constants* $\{c, d\}$ realizes *a 2-type* $iil(x, y)$*, if* $c < d$ *implies that* $\mathcal{I} \models iil(c, d)$*.*

1-types and 2-types are exclusive in the sense that a domain element realizes one and only one 1-type; and a pair of domain elements realizes one and only one 2-type. This is formally stated by the following proposition:

PROPOSITION 1.4. *For every interpretation* $\mathcal{I}$*:*

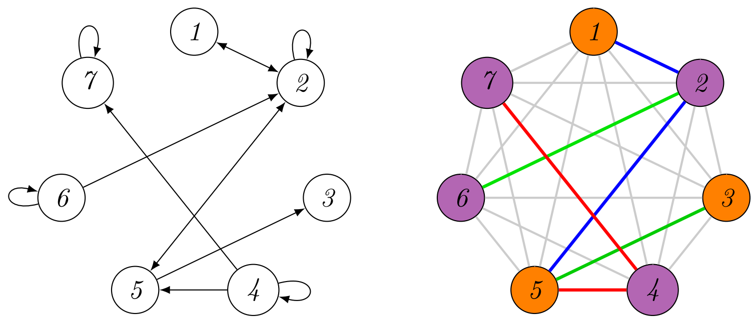


FIGURE 1. The left graph shows an interpretation on $\Sigma = \{R/2\}$, the graph on the right highlight with the corresponding colors q'the 1-type of every node and the 2-tables for every pair of nodes.

(1) Every domain element realizes a single 1-type i;
(2) Every unordered pair of domain elements (i.e, any set of two domain elements) realizes a single 2-type ijl with $i \leq j$.

PROOF. Let's start by proving that every domain element realizes a one and only une 1-type. For every n-ary predicate P and every constant we have that either $\mathcal{I} \models P(c, \ldots, c)$ or $\mathcal{I} \models \neg P(c, \ldots, c)$, but not both. Therefore $\mathcal{I} \models i(c)$ only for the 1-type

$$\bigwedge_{\mathcal{I} \models P(c,\ldots,c)} P(x, \ldots, x) \wedge \bigwedge_{\mathcal{I} \not\models P(c,\ldots,c)} \neg P(x, \ldots, x).$$

Similar reasoning can be done for 2-tables and 2-types. □

EXAMPLE 1.17. *Let $\Sigma = \{R/2\}$ then we have the following 1-types*

$$1(x) \triangleq R(x, x),$$
$$2(x) \triangleq \neg R(x, x),$$

and the following 2-tables

$$1(x, y) \triangleq R(x, y) \wedge R(y, x) \wedge x \neq y$$
$$2(x, y) \triangleq R(x, y) \wedge \neg R(y, x) \wedge x \neq y$$
$$3(x, y) \triangleq \neg R(x, y) \wedge R(y, x) \wedge x \neq y$$
$$4(x, y) \triangleq \neg R(x, y) \wedge \neg R(y, x) \wedge x \neq y$$

Suppose that we have an interpretation $\mathcal{I}$ as shown in the left part of Figure 1. This interpretation can be equivalently represented by associating to every element of the domin $\{1, \ldots, 7\}$ one specific 1-type and to every pair of elements a 2-table, as shown in the right graph of Figure 1.

In associating the 2-table one have to pay attention to the order of nodes, indeed if (c, d) realizes the 2-table $l(x, y)$ it is possible that d, c realizes a different 2-table. For instance we have that in the above example $\mathcal{I} \models 2(c, d)$ if and only if $\mathcal{I} \models 3(d, c)$. In order to maintain the fact that one pair of nodes realizes a single 2-table, we consider the following order:

- if $\mathcal{I} \models i(c) \wedge j(d)$ we consider the order (c, d) if $i < j$ and (d, c) if $j > i$;
- if $\mathcal{I} \models i(c) \wedge i(d)$ we consider (c, d) if $c < d$ otherwise we consider (d, c).

With this ordering every $\{c, d\}$ contained in the domain is associated with a single 2-type $ijl(x, y)$ with $i \leq j$.

5.2. Cardinality vectors. The *1-type cardinality vector* of a Σ-structure $\mathcal{I}$ is a vector $\boldsymbol{k} = (k_1, \dots, k_u)$ where u is the number of 1-types of Σ. where k_i is the number of elements of the domain of $\mathcal{I}$ that realize the i-th 1-type. Since every element of the domain realizes one and only one 1-type $\sum_i k_i = n$ the size of the domain of $\mathcal{I}$.

A *2-table cardinality vector* of an interpretation is a vector $\boldsymbol{h} = (\boldsymbol{h}^{ij})_{i \leq j}$ of vectors, such that n for every pair of 1-types $i \leq j$ the vector of integers $\boldsymbol{h}^{ij} = (h^{ij}_1, \dots, h^{ij}_b)$ si such that $\boldsymbol{h}^{ij} = (h^{ij}_1, \dots h^{ij}_b)$, where b is the number of 2-tables of Σ and h^{ij}_l contains the number of pairs of domain elements that realize the 2-type ijl.

EXAMPLE 1.18. *Let us consider the interpretation shown in Figure 1. The 1-type cardinality vectors is*

$$\boldsymbol{k} = (4, 3)$$

Indeed we hae that 2, 4, 6 *and* 7 *realize the 1-type* $1(x)$*, and* 1, 3 *and* 5 *realize the 1-type* $2(x)$*. The 2-type cardinality vector is*

$$\boldsymbol{h}^{11} = (0, 1, 1, 4)$$
$$\boldsymbol{h}^{12} = (2, 1, 0, 9)$$
$$\boldsymbol{h}^{22} = (0, 0, 1, 2)$$

Let us summarise some equality about cardinality vectors.

- $\sum \boldsymbol{k} = \sum_{i=1}^{u} k_i = n$. This follows directly from the fact that every element of the domain realizes one and only one 1-type.
- $\sum \boldsymbol{h}^{ii} = \sum_l h^{ii}_l = \frac{k_i(k_i-1)}{2}$ this derives from the fact that if $\{c, d\}$ realizes iil then both c and d realizes i and therefore there are $\binom{k_i}{2=\frac{k_i(k_i-1)}{2}}$ subsets of two elements of a set of k_i elements.
- $\sum \boldsymbol{h}^{ij} = \sum_l h^{ij}_l = k_i \cdot k_j$ (if $i \neq j$) This is a consequence of the fact that every subset $\{a, b\}$ realizes one and only one 2-type ijl with $i \leq j$, and that a and b realize i and j respectively. Therefore there is a total of $k_i k_j$ sets that realizes some 2-table ijl for some l.
- $\sum \boldsymbol{h} = \sum_{i \leq j} \sum_{l=1}^{b} k^{ij}_l = \frac{n(n-1)}{2}$. This is the consequence of the fact that there are $\binom{n}{2}$ subsets of 2 elements of a set of n elements.

6. FOMC of universal formulas

In this section we provide a mathematical formula (a polinomial) that allows to compute the first order model counting of a restricted class of formulas of $\mathcal{L}^2$. They are universal formula that contains no constant and function symbols and only the two variables $x.y$. In other words they are formulas of the form

$$\forall x \forall y \phi(x, y) \tag{2}$$

We start by observing that, for every cardinality vector $(\boldsymbol{k}, \boldsymbol{h})$ there are

$$\binom{n}{\boldsymbol{k}} \prod_i \binom{\frac{k_i(k_i-1}{2}}{\boldsymbol{h}^{ii}} \prod_{i<j} \binom{k_i k_j}{\boldsymbol{h}^{ij}} \tag{3}$$

distinct interpretations that have the cardinality vector $(\boldsymbol{k}, \boldsymbol{h})$ where for every positive integers $a, b_1, \dots, b_m$ with $\sum_i b_i = a$

$$\binom{a}{b_1, \dots, b_m} = \frac{a!}{b_1! \cdot b_2! \cdots b_n!}$$

Suppose that for some cardinality vectors $(\boldsymbol{k}, \boldsymbol{h})$, we have that $h_l^{ij} \neq 0$, then every interpretation with this cardinality vector should have a pair $\{c, d\}$ that realizes ijl. If $\mathcal{I}$ satisfies also $\forall xy\phi(x, y)$, then $\mathcal{I}$ should also satisfy the grounding of $\phi(x, y)$ with $\{c, d\}$. I.e., $\phi(c, c) \wedge \phi(d, d) \wedge \phi(c, d) \wedge \phi(d, c)$. But this is only possible if the formula

$$\phi(c, c) \wedge \phi(d, d) \wedge \phi(c, d) \wedge \phi(d, c) \wedge i(c) \wedge j(d) \wedge l(c, d)$$

Let us introduce this notion formally

Definition 1.9. *A 2-type $ijl(x, y)$ is consistent with a universal formula $\forall x \forall y \phi(x, y)$, if and only if the propositional formula*

$$ijl(c, d) \wedge \mathit{Ground}(\phi(x, y), \{c, d\}) \tag{4}$$

for a pair of distinct constants c and d is satisfiable. $2t(\phi)$ denotes The set of 2-types consistent with $\forall x \forall y\, \phi(x, y)$.

Notice that the first part of formula (4) i.e., $ilj(c, d)$, is a conjunction of literals, and contains all the atoms that appears in $\phi(c, d)$. This implies that if (4) is consistent then the only assignment $\mathcal{I}$ that satisfies $ijl(c, d)$, satisfies $\phi(c, c)$, $\phi(c, d)$, $\phi(d, c)$ and $\phi(d, d)$.

A simple method for computing the set $2t(\phi)$ is via truth table

Example 1.19. *Consider the formula. $\forall x \forall y (R(x, x) \wedge x \neq y \wedge R(x, y) \rightarrow \neg R(y, x)))$ Let us compute the grounding of this formula w.r.t., the constants c, d. it is*

$$\begin{aligned} \mathit{Ground}(\forall x \forall y \phi(x, y), \{c, d\}) = {} & R(c, c) \wedge c \neq c \wedge R(c, c) \rightarrow \neg R(c, c) \\ & \wedge R(d, d) \wedge d \neq d \wedge R(d, d) \rightarrow \neg R(d, d) \\ & \wedge R(c, c) \wedge c \neq d \wedge R(c, d) \rightarrow \neg R(d, c) \\ & \wedge R(d, d) \wedge d \neq c \wedge R(d, c) \rightarrow \neg R(c, d) \end{aligned}$$

We have that $c \neq c$ is always false while $c \neq d$ is always true. This allow to simplify the above formula as follows:

$$(R(c, c) \wedge R(c, d) \rightarrow \neg R(d, c)) \wedge (R(d, d) \wedge R(d, c) \rightarrow \neg R(c, d))$$

2-type	$R(c,c)$	$R(d,d)$	$R(c,d)$	$R(d,c)$	$((R(c,c) \wedge R(c,d)) \rightarrow \neg R(d,c)) \wedge ((R(d,d) \wedge R(d,c)) \rightarrow \neg R(c,d))$										
$111(c,d)$	T	T	T	T	T	T	T	F	F	F	T	T	T	F	F
$112(c,d)$	T	T	T	F	T	T	T	T	T	T	T	F	F	T	F
$113(c,d)$	T	T	F	T	T	F	F	T	F	T	T	T	T	T	T
$114(c,d)$	T	T	F	F	T	F	F	T	T	T	T	F	F	T	T
$121(c,d)$	T	F	T	T	T	T	T	F	F	F	F	F	T	T	F
$122(c,d)$	T	F	T	F	T	T	T	T	T	T	F	F	F	T	F
$123(c,d)$	T	F	F	T	T	F	F	T	F	T	F	F	T	T	T
$124(c,d)$	T	F	F	F	T	F	F	T	T	T	F	F	F	T	T
$221(c,d)$	F	F	T	T	F	F	T	T	F	T	F	F	T	T	F
$222(c,d)$	F	F	T	F	F	F	T	T	T	T	F	F	F	T	F
$223(c,d)$	F	F	F	T	F	F	F	T	F	T	F	F	T	T	T
$224(c,d)$	F	F	F	F	F	F	F	T	T	T	F	F	F	T	T

Therefore the set $2t(\phi)$ contains the 3-types which evaluates the formula ϕ to be true (i.e.,they are consistent with the formula).

If ijl is not consistent with $\forall xy(\phi(x,y)$, i.e., if $ijl \not\in t2(\phi)$, then any interpretation that contains at least one (c,d) that realizes the 2-type ijl should be excluded from the count of the model. All the remaining interpretations will be models of $\forall xy\phi(x,y)$. We can therefore modify equation (3) bu adding an indicator function that excludes these models from the summation.

PROPOSITION 1.5. *A pure universal formula $\forall x \forall y\, \phi(x,y)$ is equivalent to*

$$\forall x \forall y \left(x \neq y \rightarrow \bigvee_{i \leq j} \bigvee_{iji \in 2t(\phi)} ijl(x,y) \right) \tag{5}$$

on the class of models that contains at least 2 elements.

PROOF OUTLINE. Let $\mathcal{I}_1 \ldots, \mathcal{I}_k$ be the models of $\forall x \forall y.\phi(x,y)$. For every $\mathcal{I}_j$, every $\{a,b\} \subseteq [n]$ realizes exaclty 1 2-type, ijl which implies that ijl is consistent with $\forall x \forall y\, \phi(x,y)$. Therefore $ijl \in 2t(\phi)$. This implies that $\mathcal{I}_i \models (5)$.

Viceversa suppose that $\mathcal{I} \not\models \forall x \forall y\, \phi(x,y)$. Then either $\mathcal{I} \not\models \phi(a,b)$ for $a \neq b$ (case (1)) or $\mathcal{I} \not\models \phi(a,a)$ for some a (case (2))

(1) If $\mathcal{I} \not\models \phi(a,b)$. Let ijl be the two type realized by c,d in $\mathcal{I}$, we have that $\phi(c,d) \wedge ijl(c,d)$ is not consistent and therefore $ijl(c,d) \not\in 2t(\phi)$. Since $\{c,d\}$ can realize only a single 2-type we have that $\mathcal{I} \not\models \bigvee_{ijl \in 2t(\phi)} ijl(a,b)$ and therefore $\mathcal{I} \not\models (5)$

(2) If $\mathcal{I} \not\models \phi(a,a)$ let c be another element of the domain. This c exists since we have at least two elements. Suppose that $i \leq j$ (the proof of the other case is analogous) Let ijl be the 2-type realized by $\{a,c\}$ in $\mathcal{I}$ then we have that $\mathcal{I} \not\models \phi(a,a) \wedge \phi(c,c) \wedge \phi(c,a) \wedge \phi(a,c) \wedge ijl(a,b)$. We are now back to case (1).

$\square$

EXAMPLE 1.20. $\forall x \forall y(R(x,x) \wedge R(x,y) \rightarrow R(y,y))$ *is equivalent to:*

$$\begin{aligned} \forall x \forall y\, (x \neq y \rightarrow\ & 111(x,y) \vee 112(x,y) \vee 113(x,y) \vee 114(x,y) \vee \\ & 123(x,y) \vee 114(x,y) \\ & 221(x,y) \vee 222(x,y) \vee 223(x,y) \vee 224(x,y) \vee \end{aligned}$$

Property 1.5 allow us to transform the problem of counting the models of $\forall x \forall y \phi(x, y)$ in the problem of counting the models of (5). Notice that an interpretation $\mathcal{I}$ with cardinality vectors $(\boldsymbol{k}, \boldsymbol{h})$ is a model of (5) iff for

$$h_l^{ij} \neq 0 \Rightarrow ijl \in 2t(\phi) \tag{6}$$

As a consequence all the models of $\forall x \forall y\, \phi(x, y)$ are those that have a cardinality vector that satisfies the condition f (6). Finally notice that condition (6) can be represented with

$$\mathbb{1}_{ijl \in 2t(\phi)}^{h_l^{ij}} = \begin{cases} 1 & \text{if } h_l^{ij} = 0 \text{ or } n_{ij} \neq 0 \\ 0 & \text{Otherwise} \end{cases}$$

where $\mathbb{1}_{ijl \in 2t(\phi)}$ is the indicator function for the set $2t(\phi)$. We can therefore conclude the following:

$$\begin{aligned}
\text{FOMC}(\forall x, y.\phi(x, y), n) &= \sum_{\boldsymbol{k},\boldsymbol{h}} \binom{n}{\boldsymbol{k}} \prod_{i \le j=1}^{u} \binom{\boldsymbol{k}(i,j)}{\boldsymbol{h}^{ij}} \prod_{l} \mathbb{1}_{ijl \in 2t(\phi)}^{h_l^{ij}} \\
&= \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{i \le j=1}^{u} \sum_{\boldsymbol{k}} \binom{\boldsymbol{k}(i,j)}{\boldsymbol{h}^{ij}} \prod_{l} \mathbb{1}_{ijl \in 2t(\phi)}^{h_l^{ij}} \\
&= \sum_{\boldsymbol{k},\boldsymbol{h}} \binom{n}{\boldsymbol{k}} \prod_{i \le j=1}^{u} \left(\sum_{l=1}^{b} \mathbb{1}_{ijl \in 2t(\phi)} \right)^{\boldsymbol{k}(i,j)} \\
&= \sum_{\boldsymbol{k},\boldsymbol{h}} \binom{n}{\boldsymbol{k}} \prod_{i \le j=1}^{u} n_{ij}^{\boldsymbol{k}(i,j)}
\end{aligned}$$

with

$$n_{ij} = \sum_{l=1}^{b} \mathbb{1}_{ijl \in 2t(\phi)}$$

Theorem 1.1. *Let $\phi(x, y)$ a quantifier free formula that contains p predicate symbols and the two free variables x and y and no constant and functional symbolss;*

$$\text{FOMC}(\forall x, y.\phi(x, y), n) = \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{1 \le i \le j \le u} n_{ij}^{\boldsymbol{k}(i,j)} \tag{7}$$

$$\text{FOMC}(\forall x, y.\phi(x, y), n) = \sum_{\boldsymbol{k},\boldsymbol{h}} \binom{n}{\boldsymbol{k}} \prod_{i \le j=1}^{u} \binom{\boldsymbol{k}(i,j))}{\boldsymbol{h}^{ij}} \prod_{l} \mathbb{1}_{ijl \in 2t(\phi)}^{h_l^{ij}} \tag{8}$$

- $\boldsymbol{k} = (k_1, k_1, \dots, k_u)$, *s.t.*, $\sum_{i=1}^{u} k_i = n$;
- $n_{ij} = \#\text{SAT}(Ground(\forall x \forall y \phi(x, y), [2]) \wedge i(1) \wedge j(2))$
- $\boldsymbol{k}(i, j) = \begin{cases} \frac{k_i \cdot (k_j - 1)}{2} & \textit{if } i = j \\ k_i \cdot k_j & \textit{Otherwise} \end{cases}$

Theore 1.1 provides two formulas for computing the first order model counting of a pure universal formula. The first formula require to consider only the cardinality vector for the 1-types (i.e., $\boldsymbol{k}$). This formula is simpler but as we will see later considering only the cardinality of the 1-types could not be enough to perform *weighted first order model counting*. The second and more complete formula consider also the cardinality vectors for the 2-tables (i.e., $\boldsymbol{h}$). Considering alse these vectors will become essential when weights of the models are specified as weight on binary predicates.

EXAMPLE 1.21. *Consider the formula* $\Phi = \forall x(\neg R(x,x) \wedge (A(x) \wedge R(x,y) \rightarrow A(y))$. *Let us compute the first order model counting in the domain of 4 elements, i.e* FOMC$(\Phi, 4)$.

- *Let us first determine which are the 1-types and the 2-table for this formula*

1-types	**2-tables**
$1(x) = A(x) \wedge R(x,x)$	$1(x,y) = R(x,y) \wedge R(y,x)$
$2(x) = A(x) \wedge \neg R(x,x)$	$2(x,y) = R(x,y) \wedge \neg R(y,x)$
$3(x) = \neg A(x) \wedge R(x,x)$	$3(x,y) = \neg R(x,y) \wedge R(y,x)$
$4(x) = \neg A(x) \wedge \neg R(x,x)$	$4(x,y) = \neg R(x,y) \wedge \neg R(y,x)$

- *then we have to compute* $2t(\phi$ *i.e., the 2-types which are consistent with* ϕ. *For this we conside rthe formula*

$$\begin{aligned} Ground(\Phi, \{c,d\}) = {} & \neg R(c,c) \wedge \neg R(d,d) \\ & \wedge (A(c) \wedge R(c,c) \rightarrow A(c)) \\ & \wedge (A(c) \wedge R(c,d) \rightarrow A(d)) \\ & \wedge (A(d) \wedge R(d,c) \rightarrow A(c)) \\ & \wedge (A(d) \wedge R(d,d) \rightarrow A(d)) \end{aligned}$$

That can be simplified in consistent with ϕ. *For this we conside rthe formula*

$$\neg R(c,c) \wedge \neg R(d,d) \wedge (A(c) \wedge R(c,d) \rightarrow A(d)) \wedge (A(d) \wedge R(d,c) \rightarrow A(c)) \tag{9}$$

For this we could compute the truth table for all the 2-types, however this will require a truth table with $6 \cdot 4 = 24$. *We can simplify this computation by observing that all the 1-types that contains* $R(x,x)$ *are not consistent with* (9). *So it is enoug to consider the 1-types* $2(x)$ *and* $4(x)$.

	$A(c)$	$A(d)$	$R(c,c)$	$R(d,d)$	$R(c,d)$	$R(d,c)$	(9)
$221(c,d)$	T	T	F	F	T	T	T
$222(c,d)$	T	T	F	F	T	F	T
$223(c,d)$	T	T	F	F	F	T	T
$224(c,d)$	T	T	F	F	F	F	T
$241(c,d)$	T	F	F	F	T	T	F
$242(c,d)$	T	F	F	F	T	F	F
$243(c,d)$	T	F	F	F	F	T	T
$244(c,d)$	T	F	F	F	F	F	T
$441(c,d)$	F	F	F	F	T	T	T
$442(c,d)$	F	F	F	F	T	F	T
$443(c,d)$	F	F	F	F	F	T	T
$444(c,d)$	F	F	F	F	F	F	T

Therefore we have that $2t(\phi) = \{221, 222, 223, 224, 243, 244, 441, 442, 443, 444\}$

- *from* $2t(\phi$ *we can compute* n_{ij} *which is the number of* $ijl \in 2t(\phi)$

$$n_{22} = 4 \qquad n_{24} = 2 \qquad n_{44} = 4$$

All the others are equal to 0*. We now have all the elements to compute formula* (7)*.*

$$
\begin{aligned}
\text{FOMC}(\Phi, 4) = & \binom{4}{4,0,0,0} n_{11}^6 \\
& + \binom{4}{3,1,0,0} n_{11}^3 n_{12}^3 \\
& + \dots \\
& + \binom{4}{0,4,0,0} n_{22}^6 \\
& + \binom{4}{0,3,1,0} n_{22}^3 n_{23}^3 \\
& + \dots
\end{aligned}
$$

Notice however that all the n_{ij} where i or j are equal to 1 *or* 3 *are equal to* 0*, and therefore when k_i ore k_j is different form there the resulting term will be equal to* 0*, not contributing to the sum This means that we can concentrate only on k_2 and k_4. We can therefore simplify the formula in:*

$$
\begin{aligned}
\text{FOMC}(\Phi, 4) &= \sum_{k_2+k_4=4} \binom{4}{k_2, k_4} n_{22}^{\frac{k_2(k_2-1)}{2}} n_{24}^{k_2 k_4} n_{44}^{\frac{k_4(k_4-1)}{2}} \\
&= \sum_{k_2=0} \binom{4}{k_2} 4^{\frac{k_2(k_2-1)}{2}} 2^{k_2(4-k_2)} 4^{\frac{(4-k_2)(3-k_2)}{2}} \\
&= \sum_{k_2=0}^{4} \binom{4}{k_2} 2^{k_2(k_2-1)+k_2(4-k_2)+(4-k_2)(3-k_2)} \\
&= \sum_{k_2=0}^{4} \binom{4}{k_2} 2^{k_2^2-4k_2+12} \\
&= 2^{12} + 4 \cdot 2^9 + 6 \cdot 2^8 + 4 \cdot 2^9 + 2^{12} \\
&= 3 \cdot 2^{12} + 6 \cdot 2^8 = 13824
\end{aligned}
$$

In the above equations we use the simplified notation $\binom{n}{k_2}$ in place of $\binom{n}{k_1,k_2}$. This is the standard notation for the binomial coefficient where for every pair of integers $a \geq b$. $\binom{a}{b} = \binom{a}{b,a-b} = \frac{a!}{b!(a-b)!}$

7. Cardinality Constraints

A *cardinality constraint* is an arithmetic expression that imposes restrictions on the number of (pairs of) individual objects that belong to the interpretation of a certain predicate. In other words, a cardinality constraint imposes some restriction on the size of $\mathcal{I}(P_1), \dots \mathcal{I}(P_k)$ for some predicates $P_1, \dots, P_k$. A simple example of a cardinality constraint is $|A| = m$, for some unary predicate A and positive integer m. This cardinality constraint is satisfied by any interpretation $\mathcal{I}$ in which $\mathcal{I}(A)$ contains exactly m distinct individual objects. A more complex example of a cardinality constraint could be: $|A| + |B| \leq |C|$, where A, B and C are some predicates in the language.

Notice that, the fact that an interpretation $\mathcal{I}$ satisfies a cardinality constraint γ depends only from its cardinality vector $(\boldsymbol{k}, \boldsymbol{h})$ of the interpretation. Indeed the cardinality of unary and binary predicates of an interpretation $\mathcal{I}$ can be directly computed starting from the cardinality vector of $\mathcal{I}$.

DEFINITION 1.10 (Satisfiability of a cardinality constraint). *For every predicate P we can compute $|\mathcal{I}(P)|$ from the cardinality vectors $\boldsymbol{k}, \boldsymbol{h}$ of $\mathcal{I}$ as follows, where A is a unary predicate and R a binary predicate.*

$$
\begin{aligned}
|\mathcal{I}(A)| = \boldsymbol{k}(A) &= \sum_{i=1}^{u} |\{A(x)\} \cap i(x)| \cdot k_i \\
\boldsymbol{k}(R) &= \sum_{i=1}^{u} |\{R(x,x)\} \cap i(x)| \cdot k_i \\
\boldsymbol{h}(R) &= \sum_{l=1}^{b} |\{R(x,y), R(y,x)\} \cap l(x)| \cdot h_l \\
|\mathcal{I}(R)| = (\boldsymbol{k}, \boldsymbol{h})(R) &= \boldsymbol{k}(R) + \boldsymbol{h}(R)
\end{aligned}
$$

where, for every 2-table l, $h_l = \sum_{i \leq j=1}^{u} h_l^{ij}$. If γ is a cardinality constraint then $\mathcal{I} \models \gamma$ holds if the expression obtained replacing $|A|$ with the value of $\boldsymbol{k}(A)$ and $|R|$ with the value of $(\boldsymbol{k}, \boldsymbol{h})(R)$ is true, where $\boldsymbol{k}, \boldsymbol{h}$ is the cartinality vectors of $\mathcal{I}$.

EXAMPLE 1.22. *Consider the formumrla $\Phi = \forall x \forall y\, A(x) \wedge R(x,y) \rightarrow A(y)$. This formula has the following 1-types and 2-tables:*

1-types	**2-tables**
$1(x) = A(x) \wedge R(x,x)$	$1(x,y) = R(x,y) \wedge R(y,x)$
$2(x) = A(x) \wedge \neg R(x,x)$	$2(x,y) = R(x,y) \wedge \neg R(y,x)$
$3(x) = \neg A(x) \wedge R(x,x)$	$3(x,y) = \neg R(x,y) \wedge R(y,x)$
$4(x) = \neg A(x) \wedge \neg R(x,x)$	$4(x,y) = \neg R(x,y) \wedge \neg R(y,x)$

With the following n_{ij}

$n_{11} = 4$	$n_{12} = 4$	$n_{13} = 2$	$n_{14} = 2$
	$n_{22} = 4$	$n_{23} = 2$	$n_{24} = 2$
		$n_{33} = 4$	$n_{34} = 4$
			$n_{44} = 4$

Suppose that we are interested in counting the models of Φ on a domain of 5 elements with the cardinality constraint $|A| = 3$, i.e., FOMC$(\Phi \wedge |A| = 3, 5)$. *Since the cardinality constraints involves only a unary predicate, we can adopt the formula (7) that sum over all possible cardinality vectors for unary predicates, and restrict* *the cardinality vectors that satisfies $\boldsymbol{k}(A) = 3$ i.e., $k_1 + k_2 = 3$.*

$$\begin{aligned}\textsc{fomc}(\forall x, y.\Phi \wedge |A| = 3, 5) &= \sum_{\substack{k_1+k_2+k_3+k_4=5\\ k_1+k_2=3}} \binom{5}{k_1, k_2, k_3, k_4} \prod_{1 \le i \le j \le u} n_{ij}^{\boldsymbol{k}(i,j)} \\ &= \sum_{\substack{k_1+k_2=3\\ k_3+k_4=2}} \binom{3}{k_1}\binom{2}{k_3} \prod_{1 \le i \le j \le u} n_{ij}^{\boldsymbol{k}(i,j)}\end{aligned}$$

I f we want for instance to impose an additional cardinality constraint $|R| = 2$ on the binary predicate R, then we have to consider the expanded version of the formula for FOMC, i.e., formula (8) and additionally restrict the $\boldsymbol{h}$ vector to satisfy $\boldsymbol{k}, \boldsymbol{h}(R) = 2$

$$\textsc{fomc}(\forall x, y.\Phi \wedge |A| = 3 \wedge |R| = 2, 5) =$$
$$\sum_{\substack{k_1+k_2=3\\ k_3+k_4=2}} \binom{3}{k_1}\binom{2}{k_3} \sum_{\substack{\boldsymbol{h}\\ 2\cdot h_1+h_2+h_3+k_1+k_2=2}} \prod_{i \le j=1}^{u} \binom{\boldsymbol{k}(i,j)}{\boldsymbol{h}^{ij}} \prod_{l} \mathbb{1}_{ijl \in 2t(\phi)}^{h_l^{ij}}$$

8. Dealing with Existential Quantifiers

In order to perform model counting of formulas that contain existential quantifier we suppose that the formula is on a special form called Scott's Normal form. In the following subsection we introduce such a form and show how every formula can be transformed in Scott's normal form which is counting-equivalent, i.e., the resultinf formula has the same number of models of the original formula.

8.1. Scott's Normal Form.

Theorem 1.2 (Scott's Normal Form Scott 1962 and Kuusisto and Lutz 2018). *Every FO^2 sentence Φ in the signature Σ can be transformed in a formula*

$$\Phi' = \forall xy.\phi(x, y) \wedge \bigwedge_{i=1}^{m} \forall x \exists y.\psi_i(x, y) \tag{10}$$

where ϕ and ψ_i are quantified free formulas in the signature $\Sigma' = \Sigma \cup \{P_1, \dots, P_m\}$ for m new unary predicates P_i, such that every Σ-structure $\mathcal{I}$ that satisfies Φ can be extended in a unique way in a Σ'-structure that satisfies Φ'.

Proof Outline. To transform a formula Φ in Scott's normal form you have to apply the following transformations until the formula does not contain subformulas of the form $Qy.\alpha(x, y)$ for some quantifier $Q \in \{\forall, \exists\}$

- If $Qy.\alpha(x, y)$ is a subformula of Φ and $\alpha(x, y)$ does not contain quantifiers, then replace it with a new predicate $P(x)$ and *define* $P(x)$ as $Qy.\alpha(x, y)$; Collect all the definition of the predicates $P(x)$ in Γ.

$$\begin{aligned}\Phi &\Longrightarrow \Phi[Qy.\alpha(x, y)/P(x)] \\ \Gamma &\Longrightarrow \Gamma \wedge \forall x.(A(x) \leftrightarrow Qy.\alpha(x, y))\end{aligned}$$

- Transform the formula $\forall x P(x) \leftrightarrow \forall y \alpha(x,y))$ that belongs to Γ in the following way

$$\begin{aligned}\forall x(P(x) \leftrightarrow Qy.\alpha(x,y)) \Longrightarrow & \forall x(P(x) \rightarrow Qy.\alpha(x,y)) \wedge \\ & \forall x(\neg P(x) \rightarrow \bar{Q}y.\neg\alpha(x,y))\end{aligned}$$

  where $\bar{Q}$ is the dual quantifier than Q, (i.e., if Q is $\forall$ then $\bar{Q}$ is $\exists$ and viceversa).
- then transform each implication as follows:

$$\begin{aligned}\forall x(P(x) \rightarrow \forall y \alpha(x,y)) &\Longrightarrow \forall x \forall y.(P(x) \rightarrow \alpha(x,y)) \\ \forall x(\neg P(x) \rightarrow \exists y \neg\alpha(x,y)) &\Longrightarrow \forall x \exists y(\neg P(x) \rightarrow \neg\alpha(x,y)) \\ \forall x(P(x) \rightarrow \exists y \alpha(x,y)) &\Longrightarrow \forall x \exists y(P(x) \rightarrow \alpha(x,y)) \\ \forall x(\neg P(x) \rightarrow \forall y.\neg\alpha(x,y)) &\Longrightarrow \forall x \forall y(\neg P(x) \rightarrow \neg\alpha(x,y))\end{aligned}$$

□

EXAMPLE 1.23. *Consider the formula*

$$\forall x(A(x) \rightarrow \exists y(R(x,y) \wedge \forall x(S(y,x) \rightarrow B(x)))) \tag{11}$$

We start by replacing the subformula $\forall x(S(y,x) \rightarrow B(x)$ with $P_1(y)$ and we add the definition of P_1 obtaining

$$\begin{aligned}&\forall x(A(x) \rightarrow \exists y(R(x,y) \wedge P_1(y))) \\ &\quad \wedge \forall x(P_1(x) \leftrightarrow \forall y(S(x,y) \rightarrow B(y)))\end{aligned}$$

then we replace the formula $\exists y(R(x,y) \wedge P_1(y))$ with $P_2(x)$ and add the definition of P_2 obtaining:

$$\begin{aligned}&\forall x(A(x) \rightarrow P_2(x)) \\ &\quad \wedge \forall x(P_1(x) \leftrightarrow \forall y(S(x,y) \rightarrow B(y))) \\ &\quad \wedge \forall x(P_2(x) \leftrightarrow \exists y(R(x,y) \wedge P_1(y)))\end{aligned}$$

Finally we replace the equivalence with the implication and move out the quantifiers obtaining

$$\begin{aligned}&\forall x(A(x) \rightarrow P_2(x)) \\ &\quad \wedge \forall x \forall y(P_1(x) \rightarrow (S(x,y) \rightarrow B(y))) \\ &\quad \wedge \forall x \exists y(\neg P_1(x) \rightarrow \neg(S(x,y) \rightarrow B(y))) \\ &\quad \wedge \forall x \exists y(P_2(x) \rightarrow (R(x,y) \wedge P_1(y))) \\ &\quad \wedge \forall x \forall y(\neg P_2(x) \rightarrow \neg(R(x,y) \wedge P_1(y)))\end{aligned}$$

8.2. Inclusion-Exclusion principle. In this section, we provide a proof for model counting of formulas in Scott's normal form by meking explicit use of the principle of inclusion-exclusion. A corollary of the principle of inclusion-exclusion that will be used for preforming FOMC is the following:

COROLLARY 1 (Wilf 2005 section 4.2). *Let Ω be a set of objects and let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a set of subsets of Ω. For every $\mathcal{Q} \subseteq \mathcal{S}$, let $N(\supseteq \mathcal{Q})$ be the count of objects in Ω that belong to all the subsets $S_i \in \mathcal{Q}$, i.e., $N(\supseteq \mathcal{Q}) = \left|\{\bigcap_{S_i \in Q} S_i\}\right|$.*

For every $0 \leq l \leq m$, let $s_l = \sum_{|\mathcal{Q}|=l} N(\supseteq \mathcal{Q})$ and let e_0 be count of objects that d onot belong to any of the S_i in $\mathcal{S}$, then

$$e_0 = \sum_{l=0}^{m} (-1)^l s_l \tag{12}$$

THEOREM 1.3. *For an FO^2 formula in Scott's Normal Form as given in* (10), *let $\Phi' = \forall xy.(\Phi(x,y) \wedge \bigwedge_{i=1}^{q} P_i(x) \rightarrow \neg\Psi_i(x,y))$ where P_i's are fresh unary predicates, then:*

$$\text{FOMC}((10), n) = \sum_{(\boldsymbol{k},\boldsymbol{h})} (-1)^{\sum_i \boldsymbol{k}(P_i)} \text{FOMC}(\Phi', (\boldsymbol{k}, \boldsymbol{h})) \tag{13}$$

PROOF. TO BE REVISED Let Ω be the set of models of $\forall xy.\Phi(x,y)$ over the language of Φ and $\{\Psi_i\}$ (i.e., the language of Φ' excluding the predicates P_i) and on a domain Δ consisting of n elements. Let $\mathcal{S} = \{\Omega_{ci}\}_{c \in \Delta,\ 1 \leq i \leq q}$ be the set of subsets of Ω where Ω_{ci} is the set of ω such that $\omega \models \forall y.\neg\Psi_i(c,y)$. For every model ω of (10), $\omega \not\models \forall y \neg\Psi_i(c,y)$ for any pair of i and c i.e. ω is not in any Ω_{ci}. Also, for every $\omega \in \Omega$, if $\omega \notin \Omega_{ci}$ for any pair of i and c, then $\omega \models \exists y.\Psi_i(c,y)$ for all i and for all $c \in \Delta$ i.e., $\omega \models \bigwedge_{i=1}^{q} \forall x \exists y.\Psi_i(x,y)$. Hence, $\omega \models$ (10) if and only if $\omega \notin \Omega_{ci}$ for all c and i. Therefore, the count of models of (10) is equal to the count of models in Ω which do not belong to any Ω_{ci}. Hence, If we are able to compute s_l (as introduced in Corollary 1), then we could use Corollary 1 for computing cardinality of all the models which do not belong to any Ω_{ci} and hence $\text{FOMC}((10), n)$.

For every $0 \leq l \leq n \cdot q$, let us define

$$\Phi'_l = \Phi' \wedge \sum_{i=1}^{q} |P_i| = l \tag{14}$$

We will now show that s_l is exactly given by $\text{FOMC}((14), n)$.

Every model of Φ'_l is an extension of an $\omega \in \Omega$ that belongs to at least l elements in $\mathcal{S}$. In fact, for every model ω of $\forall xy.\Phi(x,y)$ i.e. $\omega \in \Omega$, if $\mathcal{Q}'$ is the set of elements of $\mathcal{S}$ that contain ω, then ω can be extended into a model of Φ'_l in $\binom{|Q'|}{l}$ ways. Each such model can be obtained by choosing l elements in Q' and interpreting $P_i(c)$ to be true in the extended model, for each of the l chosen elements $\Omega_{ci} \in Q'$. On the other hand, recall that $s_l = \sum_{|\mathcal{Q}|=l} N(\supseteq Q)$. Hence, for any $\omega \in \Omega$ if $\mathcal{Q}'$ is the set of elements of $\mathcal{S}$ that contain ω, then there are $\binom{|\mathcal{Q}'|}{l}$ distinct subsets $\mathcal{Q} \subseteq \mathcal{Q}'$ such that $|\mathcal{Q}| = l$. Hence, we have that ω contributes $\binom{|\mathcal{Q}'|}{l}$ times to s_l. Therefore, we can conclude that

$$s_l = \text{FOMC}(\Phi'_l, n) = \sum_{|\mathcal{Q}|=l} N(\supseteq Q)$$

and by the principle of inclusion-exclusion as given in Corollary 1, we have that :

$$
\begin{aligned}
\textsc{fomc}((10), n) = e_0 &= \sum_{l=0}^{n \cdot q} (-1)^l s_l \\
&= \sum_{l=0}^{n \cdot q} (-1)^l \textsc{fomc}(\Phi'_l, n) \\
&= \sum_{l=0}^{n \cdot q} (-1)^l \sum_{(\boldsymbol{k},\boldsymbol{h}) \models \sum_i |P_i| = l} \textsc{fomc}(\Phi', (\boldsymbol{k}, \boldsymbol{h})) \\
&= \sum_{(\boldsymbol{k},\boldsymbol{h})} (-1)^{\sum_i \boldsymbol{k}(P_i)} \textsc{fomc}(\Phi', (\boldsymbol{k}, \boldsymbol{h}))
\end{aligned}
$$

□

9. Exercises

Exercise 1:
Show that if $|\Delta^{\mathcal{I}}| = 1$ then $\mathcal{I} \models \forall x \phi(x) \leftrightarrow \exists x \phi(x)$

Exercise 2:
Prove that $\textsc{fomc}(\phi, [1]) = \#\textsc{sat}(\text{Ground}(\phi, \{1\}))$.

Exercise 3:
List all the models of the formula

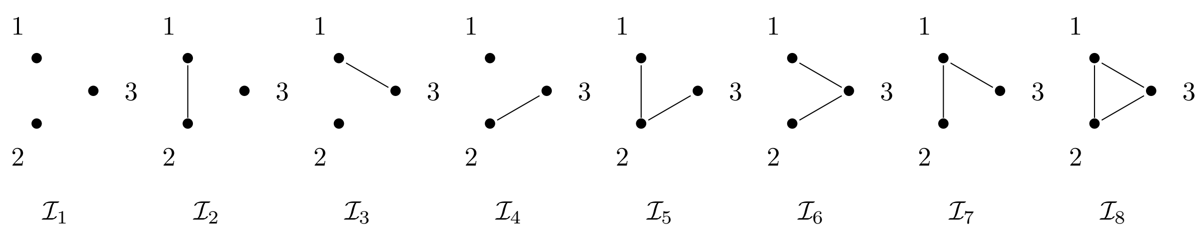
$$\forall x \neg R(x, x) \land \forall xy (R(x, y) \to R(y, x))$$

in the domain $\{1, 2, 3\}$.

Solution The intuitive reading of the formula is as follows:

- every object is not related with itself. (i.e., R is not reflexive)
- R is symmetric.

This means that we are interested in all the undirected graphs on the three edges 1, 2, and 3. Therefore for every subset of pairs of objects, there is a model. Which implies that the number of models are $2^{\binom{3}{2}} = 2^3 = 8$. A graphical representation of the models are shown in the following:a



□

Exercise 4:
List all the models of the formula $\forall x (A(x) \to B(x))$ on the interpretation domain $\{1, 2, 3\}$.

Exercise 5:

Let $\mathcal{I}$ be a first order interpretation and $\pi : \Delta^{\mathcal{I}} \to \Delta^{\mathcal{I}}$ be a isomorphism. Show that the interpretation $\mathcal{I}^\pi$ where

$$\mathcal{I}^\pi(a) \triangleq \pi(\mathcal{I}(a))$$
$$\mathcal{I}^\pi(f) \triangleq (d_t \dots d_n) \mapsto \pi(\pi^{-1}(d_1), \dots, \pi^{-1}(d_n))$$
$$\mathcal{I}^\pi(R) \triangleq \{(\pi(d_1), \dots, \pi(d_n) \mid (d_1, \dots, d_n) \in \mathcal{I}(R)\}$$

is such that $\mathcal{I} \models \phi$ if and only if $\mathcal{I}^\pi \models \phi$ for every first order sentence ϕ.

Exercise 6:

Provide an explicit mathematical formula to compute the number of models of $\forall x(A(x) \to B(x))$ in the domain $\{1, 2, 3, \dots, n\}$.

Solution The models of $forallx.A(x) \to B(x)$ are those in which B is interpreted in a subset of the interpretation of A. Notice that we have $\binom{n}{k}$ possible ways of interpreting A in a set of k objects. For any such interpretation B can take any subset of the interpretation of A, i.e, 2^k. therefore the set of interpretations are

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = (1+2)^n = 3^n$$

□

Exercise 7:

For the following formuls ϕ describe all the models in the domain $\{1, 2, 3\}$ and find an explicit mathematical formula that computes $\textsc{fomc}(\phi, n)$

$$\forall x(A(x) \leftrightarrow \forall y R(x, y))$$

Solution Notice that, in the above formula, the interpretation of R fully determines the interpretation of A, and furthermore R can be freely interpreted in any subset of pairs of elements of $\{1, 2, 3\}$. This means that the number of interpretations that satisfies the above formula coincides with the number of interpretations of R, which are $2^{3 \cdot 3} = 2^9$. □

Exercise 8:

Provide an explicit mathematical formula to compute the number of models of the formula of the previous exercise in the domain $\{1, 2, 3, \dots, n\}$.

Exercise 9:

For each formuls ϕ in the following list, describe all the models in the domain $\{1, 2, 3\}$ and find an explicit mathematical formula that computes $\mathsf{fomc}(\phi, n)$

(1) $\exists x A(x)$
(2) $\exists x \neg A(x)$
(3) $\neg \forall x \neg A(x)$
(4) $\forall x \exists y R(x, y)$
(5) $\exists x \forall y \neg R(x, y)$
(6) $\forall x(A(x) \to \exists y R(x, y))$
(7) $\forall x \exists y R(x, y) \wedge \forall xyz R(x, y) \wedge R(x, z) \to y = z$

(8) $\forall xy(R(x,y) \rightarrow A(x) \land \neg A(y))$

Solution

(1) $\exists x A(x)$: the set of models interpret A in a non empty subset of $\{1,2,3\}$. The number of non empty subsets of $\{1,2,3\}$ are 2^3-1.
(2) $\exists x \neg A(x)$ the set of models interpret A in a set different from the entire domain. The number of such sets is 2^3-1.
(3) $\neg\forall x \neg A(x)$: This formula is equivalent to $\exists x A(x)$. See item 1.
(4) $\forall x \exists y R(x,y)$. The set of models of such a formula are the interpretations that associates to every element $d \in \{1,2,3\}$ at least one element $d' \in \{1,2,3\}$ such that $(d,d') \in R^{\mathcal{I}}$. Therefore a model of this formula associates to every element $d \in \{1,2,3\}$ a non empty subset $D \subseteq \{1,2,3\}$ such that $(d,d') \in R^{\mathcal{I}}$ for all $d' \in D$. Since the number of non empty subsets of $\{1,2,3\}$ is 2^3-1 we have that the number of models of the formula is equal to $(2^3-1)^3 = 7^3 = 343$.
(5) $\exists x \forall y \neg R(x,y)$. This formula is equivalent to $\neg\forall x \exists y R(x,y)$. Therefore the number of models of this formula is the total number of interpretations (which is equal to 2^9) minus the numbr of models of $\forall x \exists y \neg R(x,y)$ which is equal to $(2^3-1)^3$ (see previous point). This means that the number of models of the formula is equal to $2^9-(2^3-1)^3$.
(6) $\forall x(A(x) \rightarrow \exists y R(x,y))$. An interpretation can associate to A any subset of $\{1,2,3\}$. Given an interpretation of A, for every element in the interpretation of A, R should associate a non empty set of elements of $\{1,2,3\}$. This means that, if A contains k elements, we have 2^3-1 possibilities. If A contains k elements there are $(2^3-1)^k$ possibilities for the other $n-k$ element R can be interpreted freely, allowing $2^{3(n-k)}$. Therefore in total there are $(2^3-1)^k 2^{3(3-k)}$ models where A is interpreted in a set of k elements. Since there are $\binom{3}{k}$ possible interpretations of A that contains k elements, the total number of interpretations are

$$\sum_{k=0}^{3} \binom{3}{k} ((2^3-1)^k + 2^{3(3-k)})$$

(7) $\forall x \exists y R(x,y) \land \forall xyz R(x,y) \land R(x,z) \rightarrow y = z$ This formula states that R is a total function on the domain $\{1,2,3\}$ i.e., for every element $d \in \{1,2,3\}$ it associates one and only one element $d' \in \{1,2,3\}$ such that $(d,d') \in R^{\mathcal{I}}$. The number of functions on a set of n elements are n^n, in this case we have $3^3 = 27$ models.
(8) $\forall xy(R(x,y) \rightarrow A(x) \land \neg A(y))$. The formula state that the interpretation of R must be a subset of $A^{\mathcal{I}} \times (\neg A)^{\mathcal{I}}$. If A is interpreted in a set of k k then R can be interpreted any subset of $k(n-k)$ pairs. So there are $2^{k(n-k)}$ possible interpretations of R. Since there are $\binom{n}{k}$ interpretations of A that contains k elements, the total number of models of the formula are:

$$\sum_{k=0}^{3} \binom{n}{k} 2^{k(n-k)}$$

□

Exercise 10:

Ground the formula $\forall x \exists y R(x, y)$ in the domain $\{a, b, c\}$.

Solution

$$\bigwedge_{x \in \{a,b,c\}} \bigvee_{y \in \{a,b,c\}} R(x, y)$$

□

Exercise 11:
Ground the formula $\forall x(A(x) \to \exists x R(x, x))$ in the domain $\{a, b, c\}$. **Solution**

$$\bigwedge_{d \in \{a,b,c\}} (A(d) \to \bigvee_{e \in \{a,b,c\}} R(d, e))$$

Extensively written is:

$$\begin{aligned} & (A(a) \to R(a, a) \vee R(a, b) \vee R(a, c)) \\ \wedge\ & (A(b) \to R(b, a) \vee R(b, b) \vee R(b, c)) \\ \wedge\ & (A(c) \to R(c, a) \vee R(c, b) \vee R(c, c)) \end{aligned}$$

□

Exercise 12:
How many ground atoms occour in a formula that contains $A(x)$, $B(y)$ and $R(x, y)$ when it is grounded with the set of constants $\{c_1, \dots, c_n\}$?

Exercise 13:
Compute the grounding of $\forall x(\exists y A(x, y) \to \forall z \neg B(z, w))$

Exercise 14:
Find a formula for computing $\textsc{fomc}(\forall x R(a, x), n)$ where a is a constant.

Exercise 15:
Find a formula for computing $\textsc{fomc}(\forall x(A(x) \to A(f(x))), n)$ where f is a function symbol.

Exercise 16:
Find a formula that computes $\textsc{fomc}(\forall xy(R(x, y) \to \exists z S(x, y, z)), n)$

Solution Suppose that $\mathcal{I}(R)$ contains r pairs then for every pair $(a, b) \in \mathcal{I}(R)$ we can associate a non empty subset $S_{a,b}$ such that $(a, b, c) \in \mathcal{I}(S)$ for all $c \in S_{a,b}$. This results in:

$$\sum_{r=0}^{n^2} \binom{n^2}{r} (2^n - 1)^r (2^n)^{n^2 - r} = (2^n - 1 + 2^n)^{n^2} = (2^{n+1} - 1)^{n^2}$$

□

Exercise 17:
List all the 1-types of the signature $\Sigma = \{A/1, B/1, R/2\}$.

Exercise 18:

List all the 1-types, and 2-tables of the signature $\Sigma = \{A/1, B/1, R/2, S/2\}$.

Exercise 19:
What is the cardinality vectors $\boldsymbol{k}, \boldsymbol{h}$ of the following Σ-structure on the domain [6] where $\Sigma = \{A/1, B/1, R/2\}$:

$$\mathcal{I}(A) = \{1, 3, 5\}$$
$$\mathcal{I}(B) = \{3, 4, 5\}$$
$$\mathcal{I}(R) = \{((2, 2), (3, 3), (3, 5), (6, 4), (5, 3), (3, 2)\}$$

Exercise 20:
Compute the set $2t(\phi)$ for the following formulas:

(1) $\forall x \forall y(A(x) \wedge R(x, y) \rightarrow A(y)$;
(2) $\forall x \forall y(R(x, y) \rightarrow A(x) \wedge B(y))$;
(3) $\forall x \forall y(A(x) \wedge B(y) \rightarrow x \neq y)$.

Exercise 21:
In the formula

$$\textsc{fomc}(\forall x, y.\phi_0(x, y), n) = \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{0 \leq i \leq 2^p - 1} n_{ij}^{k(i,j)}$$

first order model counting if $\Phi_0(x, y)$ is $A(x) \wedge R(x, y) \rightarrow \neg A(y)$, specify the values of:

(1) the length of $\boldsymbol{k}$
(2) the number of 1-types
(3) n_{ij} for every pair of 1-type $i \leq j$

Exercise 22:
Compute $\textsc{fomc}(\Phi, 4)$ for the following formulas using the formula (7) (in parenthesis the result)

(1) $\forall x \forall y(R(x, y) \rightarrow A(x) \wedge \neg A(y))$ (162)
(2) $\forall x \forall y(R(x, y) \vee R(y, x))$ (729)
(3) $\forall x \forall y(A(x) \wedge B(y) \leftrightarrow R(x, y))$ (256)

Exercise 23:
Using the formula for first order model counting of universal formulas in $\mathcal{L}^2$ compute $\textsc{fomc}(\Phi, 3)$ where Φ is the following fomrula:

(1) $\forall x \forall y(R(x, y) \rightarrow R(y, x))$;
(2) $\forall x \forall y(R(x, y) \rightarrow \neg R(y, x))$;
(3) $\forall x \forall y(R(x, y) \rightarrow \neg R(x, x))$;
(4) $\forall x \forall y(R(x, x) \rightarrow (R(x, y) \rightarrow R(y, z)))$;

Solution

(1) $\forall x \forall y(R(x,y) \to R(y,x))$; Let us first compute the n_{ij} using the truth table

$R(x,x)$	$R(y,y)$	$(R(x,x) \to R(x,x)) \wedge R(x,y) \to R(y,x)) \wedge R(y,x) \to R(x,y)) \wedge R(y,y) \to R(y,y))$
0	0	$n_{00} = 2$
0	1	$n_{01} = 2$
1	0	$n_{10} = 2$
1	1	$n_{11} = 2$

Notice that n_{ij} is the number of models of the formula $(R(x,x) \to R(x,x)) \wedge R(x,y) \to R(y,x)) \wedge R(y,x) \to R(x,y))$ when $R(x,x)$ is interpreted in i and $R(y,y)$ is interpreted in j. Here we have 1 unary predicate R which is applied to x (obtaining $R(x,x)$ and to y obtaining $R(y,y)$). This means that the dimension of $\boldsymbol{k}$ is equal to $2^p = 2$, i.e, $\boldsymbol{k} = (k_0, k_1)$. If we expan the fomula we obtain

$$\begin{aligned}
&\sum_{\boldsymbol{k}\in\{(0,3),(1,2),(2,1),(3,0)\}} \binom{3}{\boldsymbol{k}} n_{00}^{\boldsymbol{k}(0,0)} n_{01}^{\boldsymbol{k}(0,1)}, n_{11}^{\boldsymbol{k}(1.1)} \\
=&\sum_{\boldsymbol{k}\in\{(0,3),(1,2),(2,1),(3,0)\}} \binom{3}{\boldsymbol{k}} 2^{\boldsymbol{k}(0,0)+\boldsymbol{k}(0,1)+\boldsymbol{k}(1.1)} \\
=&\binom{3}{0} 2^{\frac{0(0-1)}{2}+0\cdot 3+\frac{3(3-1)}{2}} + \binom{3}{1} 2^{\frac{1(1-1)}{2}+1\cdot 2+\frac{2(2-1)}{2}} \\
+&\binom{3}{2} 2^{\frac{2(2-1)}{2}+2\cdot 1+\frac{1(1-1)}{2}} + \binom{3}{3} 2^{\frac{3(3-1)}{2}+3\cdot 0+\frac{0(0-1)}{2}} \\
=&2^3 + 3\cdot 2^3 + 3\cdot 2^3 + 2^3 = 2^3 2^3 = 2^6
\end{aligned}$$

(2) $\forall x \forall y(R(x,y) \to \neg R(y,x))$; Let us compute first n_{ij}.

$R(x,x)$	$R(y,y)$	$(R(x,x) \to \neg R(x,x)) \wedge R(x,y) \to \neg R(y,x)) \wedge R(y,x) \to R\neg(x,y)) \wedge R(y,y) \to \neg R(y,y))$
0	0	$n_{00} = 2$
0	1	$n_{01} = 0$
1	0	$n_{10} = 0$
1	1	$n_{11} = 0$

The dimentions are the same as in the previous formula since we have only the predicate R.

$$\begin{aligned}
&\sum_{\boldsymbol{k}\in\{(0,3),(1,2),(2,1),(3,0)\}} \binom{3}{\boldsymbol{k}} n_{00}^{\boldsymbol{k}(0,0)} n_{01}^{\boldsymbol{k}(0,1)}, n_{11}^{\boldsymbol{k}(1.1)} \\
=&\sum_{\boldsymbol{k}\in\{(0,3),(1,2),(2,1),(3,0)\}} \binom{3}{\boldsymbol{k}} 2^{\boldsymbol{k}(0,0)} 0^{\boldsymbol{k}(0,1)+\boldsymbol{k}(1.1)} \\
=&\binom{3}{0} 2^{\frac{0(0-1)}{2}} 0^{0\cdot 3+\frac{3(3-1)}{2}} + \binom{3}{1} 2^{\frac{1(1-1)}{2}} 0^{1\cdot 2+\frac{2(2-1)}{2}} \\
+&\binom{3}{2} 2^{\frac{2(2-1)}{2}} 0^{2\cdot 1+\frac{1(1-1)}{2}} + \binom{3}{3} 2^{\frac{3(3-1)}{2}} 0^{+3\cdot 0+\frac{0(0-1)}{2}} \\
=&1\cdot 0 + 3\cdot 0 + 3\cdot 0 + 1\cdot 2^3 = 8
\end{aligned}$$

(3) $\forall x \forall y (R(x, y) \rightarrow \neg R(x, x))$;
(4) $\forall x \forall y (R(x, x) \rightarrow (R(x, y) \rightarrow R(y, z)))$;

$\square$

Exercise 24:

The formula for first order model counting of formulas that contain exitantial quantifiers is.

$$\text{FOMC}\,(\forall x, y.\phi(x, y) \wedge \forall x \exists y \psi(x, y), n) = \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} (-1)^{\boldsymbol{k}(P)} \prod_{0 \leq i \leq j \leq 2^{p+1}-1} n_{ij}^{k(i,j)}$$

Answer the following questions about the elements of the above mathematical formula when: $\phi(x, y)$ be $R(x, y)$ and $\psi(x, y)$ equal to $Q(x, y)$

(1) what is P?
(2) what is the value of p
(3) what is the lenght of $\boldsymbol{k}$
(4) on which formula do you compute n_{ij}

Exercise 25:

Using the formula for first order model counting of formulas that contains exitantial quantifiers, compute $\text{FOMC}(\Phi, n)$ where Φ is one of the following fomrula:

- $\forall x.\exists y.R(x.y)$
- $\forall x.\exists y.(R(x, y) \vee R(y, x))$
- $\forall x, y.(R(x, y) \rightarrow R(y, x)) \wedge \forall x \neg R(x, x) \wedge \forall x.\exists y.R(x.y)$

Exercise 26:

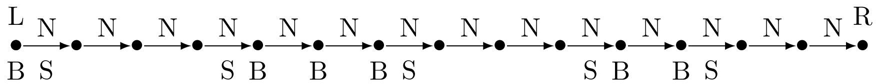
Formalize the following problem in FOL and formulate the solution in terms of FOMC (you don't need to actually compute the solution).

> Suppose that 6 boys and 9 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row GBGGB-BGBBGGGBGG we have S = 8. The average value of S (if all possible orders of these 15 people are considered) is closest to.

Solution Let Φ be the conjunction of the following formulas.

$$L(x) \leftrightarrow \forall y \neg N(y, x)$$
$$R(x) \leftrightarrow \forall x \neg N(x, y)$$
$$\forall x (\neg R(x) \rightarrow \exists y N(x, y))$$
$$S(x, y) \leftrightarrow N(x, y) \wedge (B(x) \leftrightarrow \neg B(y))$$
$$|L| = |R| = 1$$
$$|B| = 6$$
$$|N| = 14$$

Any model of Φ on the domain of 15 elements has the following structure



where the six labels B can be randomly assignet to any of the elements of the domain. $FOMC(\Phi, 15)$ therefore count how many of such configurations exists. For every $\boldsymbol{k}$ we can compute the cardinality of S, denoted by $\boldsymbol{k}(S)$. Therfore the problem can be solved by computing

$$\frac{\sum_{\boldsymbol{k}} \boldsymbol{k}(S)\binom{15}{\boldsymbol{k}} \prod_{i\leq j} n_{ij}^{\boldsymbol{k}(i,j)}}{\sum_{\boldsymbol{k}} \binom{15}{\boldsymbol{k}} \prod_{i\leq j} n_{ij}^{\boldsymbol{k}(i,j)}}$$

□

Exercise 27:

Formalize the following problem in FOL and formulate the solution in terms of FOMC (you don't need to actually compute the solution).

> A mission to Mars will consist of 4 astronauts selected from 14 available. Exactly 5 of the 14 are trained in exobiology. If the mission requires at least 2 trained in exobiology, how many different crews can be selected?

Solution We can easily formulate the problem as $FOMC(\Phi, 14)$ where Φ is the following formula.

$$|E| = 5 \wedge |M| = 4 \wedge |M \cap E| = 2$$

Since there are no FOL formulas, and no binary predicates, we have that have that $n_{ij} = 1$ for all ij which implies that the $FOMC$ formula reduces to

$$\sum_{\substack{\boldsymbol{k}(E)=2, \boldsymbol{k}(M)=5 \\ \boldsymbol{k}(M\cap E)=2}} \binom{14}{\boldsymbol{k}}$$

which is equal to

$$\binom{14}{7,3,2,2}$$

However this number include also all the possible choices of the experts in exobiology, which is known. We have therefore to devide it by all the possible subset of 5 expers among the 14 astronauts, i.e., $\binom{14}{5}$. The final result therefore is

$$\frac{\binom{14}{7,3,2,2}}{\binom{14}{5}}$$

□

Exercise 28:

Using the formula for FOMC

$$\text{FOMC}(\forall x, y.\phi(x, y), n) = \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{0\leq i\leq 2^p-1} n_{ij}^{k(i,j)}$$

compute $\text{FOMC}(\forall x, y(A(x) \wedge R(x, y) \rightarrow A(y)), 3)$

Solution We first have to compute the n_{ij}. Notice that we have 2 unary predicates $A(x)$ and $R(x,x)$. THerefore we have that $0 \leq i,j \leq 2^2-1=3$

$A(x)$	$R(x,x)$	$A(y)$	$R(y,y)$	n_{ij}
0	0	0	0	$n_{00}=4$
0	0	0	1	$n_{01}=4$
0	0	1	0	$n_{02}=2$
0	0	1	1	$n_{03}=2$
0	1	0	1	$n_{11}=4$
0	1	1	0	$n_{12}=2$
0	1	1	1	$n_{13}=2$
1	0	1	0	$n_{22}=4$
1	0	1	1	$n_{23}=4$
1	1	1	1	$n_{33}=4$

The expansion of the formula for $\text{FOMC}(\phi,n)$ for $n=3$ is the following.

$$\begin{array}{llllll}
\binom{3}{3,0,0,0}n_{00}^3 & + & \binom{3}{2,1,0,0}n_{00}n_{01}^2 & + & \binom{3}{2,0,1,0}n_{00}n_{02}^2 & + \\
\binom{3}{2,0,0,1}n_{00}n_{03}^2 & + & \binom{3}{1,2,0,0}n_{01}^2n_{11} & + & \binom{3}{1,1,1,0}n_{01}n_{02}n_{12} & + \\
\binom{3}{1,1,0,1}n_{01}n_{03}n_{13} & + & \binom{3}{1,0,2,0}n_{02}^2n_{22} & + & \binom{3}{1,0,1,1}n_{02}n_{03}n_{23} & + \\
\binom{3}{1,0,0,2}n_{22}n_{23}^2 & + & \binom{3}{0,3,0,0}n_{11}^3 & + & \binom{3}{0,2,1,0}n_{11}n_{12}^2 & + \\
\binom{3}{0,2,0,1}n_{11}n_{13}^2 & + & \binom{3}{0,1,2,0}n_{12}^2n_{22} & + & \binom{3}{0,1,1,1}n_{12}n_{13}n_{23} & + \\
\binom{3}{0,1,0,2}n_{13}^3n_{33} & + & \binom{3}{0,0,3,0}n_{22}^3 & + & \binom{3}{0,0,2,1}n_{22}n_{23}^2 & + \\
\binom{3}{0,0,1,2}n_{22}^3n_{33} & + & \binom{3}{0,0,0,3}n_{33}^3 & & &
\end{array}$$

The result can be obtaine by replacing the value of n_{ij}, in the above expression. □

Exercise 29:
Use the formula for first order model counting to compute:

$$\text{FOMC}(R(x,y) \to P(y,x), 4)$$

Solution Let us recall the formula for first order model counting for unviversally quantified formulas

$$\text{FOMC}(\forall x,y.\phi_0(x,y),n) = \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{0 \leq i \leq 2^p-1} n_{ij}^{k(i,j)}$$

- $\boldsymbol{k} = (k_0, k_1, \dots, k_{2^p-1})$, s.t., $\sum_{i=1}^{2^p-1} k_i = n$;
- $\binom{n}{\boldsymbol{k}} = \frac{n!}{k_0! \cdot k_1! \cdots k_{2^p-1}!}$
- $n_{ij} = \#\text{SAT}(Ground(\phi_0(x,y) \wedge \alpha_i(x) \wedge \alpha_j(y), \{a,b\})$
- $\alpha_i(x) = \bigwedge_{\substack{b=1 \\ i_b=0}}^{p} \neg A_b(x) \wedge \bigwedge_{\substack{b=1 \\ i_b=1}}^{p} A_b(x)$
- $k(i,j) = \begin{cases} \frac{k_i \cdot (k_j-1)}{2} & \text{if } i=j \\ k_i \cdot k_j & \text{Otherwise} \end{cases}$

Let us determine all the quantities contained in the formulas $p=2$, since we have the unary atoms $R(x,x)$ and $P(x,x)$. Therefore we have to compute n_{ij} for $0 \leq i \leq j \leq 3$. Let $\Phi(a,b)$ be the grounding of $R(x,y) \to P(y,x)$ in the domain

$\{a, b\}$, i.e.,

$$\begin{aligned}\Phi(a,b) = (R(a,a) \to P(a,a)) \wedge (R(b,b) \to P(b,b))\\ (R(a,b) \to P(b,a)) \wedge (R(b,a) \to P(a,b)))\end{aligned}$$

then the n_{ij} are the following:

$$\begin{aligned}
n_{00} &= \#\text{SAT}(\neg R(a,a) \wedge \neg R(b,b) \wedge \neg P(a,a) \wedge \neg P(b,b) \wedge \Phi(a,b)) && = 9\\
n_{01} &= \#\text{SAT}(\neg R(a,a) \wedge \neg R(b,b) \wedge \neg P(a,a) \wedge P(b,b) \wedge \Phi(a,b)) && = 9\\
n_{02} &= \#\text{SAT}(\neg R(a,a) \wedge \neg R(b,b) \wedge P(a,a) \wedge \neg P(b,b) \wedge \Phi(a,b)) && = 9\\
n_{03} &= \#\text{SAT}(\neg R(a,a) \wedge \neg R(b,b) \wedge P(a,a) \wedge P(b,b) \wedge \Phi(a,b)) && = 9\\
n_{11} &= \#\text{SAT}(\neg R(a,a) \wedge R(b,b) \wedge \neg P(a,a) \wedge P(b,b) \wedge \Phi(a,b)) && = 9\\
n_{12} &= \#\text{SAT}(\neg R(a,a) \wedge R(b,b) \wedge P(a,a) \wedge \neg P(b,b) \wedge \Phi(a,b)) && = 0\\
n_{13} &= \#\text{SAT}(\neg R(a,a) \wedge R(b,b) \wedge P(a,a) \wedge P(b,b) \wedge \Phi(a,b)) && = 9\\
n_{22} &= \#\text{SAT}(R(a,a) \wedge \neg R(b,b) \wedge P(a,a) \wedge \neg P(b,b) \wedge \Phi(a,b)) && = 9\\
n_{23} &= \#\text{SAT}(R(a,a) \wedge \neg R(b,b) \wedge P(a,a) \wedge P(b,b) \wedge \Phi(a,b)) && = 9\\
n_{33} &= \#\text{SAT}(R(a,a) \wedge R(b,b) \wedge P(a,a) \wedge P(b,b) \wedge \Phi(a,b)) && = 9
\end{aligned}$$

We can now replace the elements in the general formula obtaining

$$\begin{aligned}
\sum_{\boldsymbol{k}} \binom{4}{\boldsymbol{k}} \prod_{\substack{0 \le i \le j \le 3\\ (i,j) \ne (1,2)}} 9^{k(i,j)} &= \sum_{\boldsymbol{k}} \binom{4}{\boldsymbol{k}} 9^{6-k_1 k_2}\\
&= \sum_{\substack{\boldsymbol{k}\\ k_1 k_2 = 0}} \binom{4}{\boldsymbol{k}} 9^6 + \sum_{\substack{\boldsymbol{k}\\ k_1 k_2 = 1}} \binom{4}{\boldsymbol{k}} 9^5 + \sum_{\substack{\boldsymbol{k}\\ k_1 k_2 = 2}} \binom{4}{\boldsymbol{k}} 9^4 + \sum_{\substack{\boldsymbol{k}\\ k_1 k_2 = 3}} \binom{4}{\boldsymbol{k}} 9^3 + \sum_{\substack{\boldsymbol{k}\\ k_1 k_2 = 4}} \binom{4}{\boldsymbol{k}} 9^2
\end{aligned}$$

□

Exercise 30:

Using the formula in the slides compute the first order model counting for $\forall xy(R(x,y) \to \neg R(x,x) \wedge \neg R(y,y))$ in the domain of 3 elmenets. **Solution**

We have 1 unary predicate which is $R(x,x)$ (it is binary but applied to the same variable it becomes like a unary predicate). Therefore we have to compute n_{00} n_{01} and n_{11}. Each n_{ij} indicates the number of assignments to unary and binary atoms that makes true the grounding of the formula with two elements a, b.

$R(a,a)$	$R(b,b)$	$R(a,b)$	$R(b,a)$	$R(a,a) \to \neg R(a,a) \wedge \neg R(a,a) \wedge$ $R(a,b) \to \neg R(a,a) \wedge \neg R(b,b) \wedge$ $R(b,a) \to \neg R(b,b) \wedge \neg R(a,a) \wedge$ $R(b,b) \to \neg R(b,b) \wedge \neg R(b,b)$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

Therefore $n_{00} = 4$ and $n_{01} = n_{11} = 0$. The formula for first order model counting is

$$\sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{i \leq j} n_{ij}^{k(i,j)}$$

where n is the size of the domain and $\boldsymbol{k}$ is a vector of 2^u positive integers that sum to n, where u is the number of unary predicates. In our case $n = 3$, $u = 1$, and therefore $\boldsymbol{k}$ is a vector containing two elements that sum to 3. We can therefore use the binomial coefficient $\binom{n}{k_0}$ instead of the multinomial $\binom{n}{k_0, k_1}$, as they are equivalent. The expansion of the fomula is as follows:

$$\sum_{k_0=0}^{3} n_{00}^{\frac{k_0(k_0-1)}{2}} n_{01}^{k_0(3-k_0)} n_{11}^{\frac{(3-k_0)(3-k_0)}{2}}$$
$$= \sum_{k_0=0}^{3} 4^{\frac{k_0(k_0-1)}{2}} 0^{k_0(3-k_0)} 0^{\frac{(3-k_0)(3-k_0)}{2}}$$

Notice that if $k \neq 3$ then the product is equal to 0 and therefore it does not contributes to the sum. We have only to consider the case in which $k_0 = 2$, obtaining the following expression

$$4^{\frac{3 \cdot 2}{2}} = 4^3 = 2^6$$

□

Exercise 31:

Using the formula for first order model counting compu¯te the number of models of the FO^2-formula

$$\forall xy(R(x,x) \to (R(x,y) \to R(y,x)))$$

Solution Let us first determine which are the 1-types and the 2-tables. The

1-types are

$$1(x) \triangleq R(x,x),$$
$$2(x) \triangleq \neg R(x,x),$$

and the following 2-tables

$$1(x,y) \triangleq R(x,y) \land R(y,x) \land x \neq y$$
$$2(x,y) \triangleq R(x,y) \land \neg R(y,x) \land x \neq y$$
$$3(x,y) \triangleq \neg R(x,y) \land R(y,x) \land x \neq y$$
$$4(x,y) \triangleq \neg R(x,y) \land \neg R(y,x) \land x \neq y$$

Now let us compute n_{11}, n_{12} and n_{22}. To do so we have to do the grounding of the formula obtaining:

$$\begin{aligned}
&(R(c,c) \rightarrow (R(c,c) \rightarrow R(c,c))) \land \\
&(R(c,c) \rightarrow (R(c,d) \rightarrow R(d,c))) \land \\
&(R(d,d) \rightarrow (R(d,c) \rightarrow R(c,d))) \land \\
&(R(d,d) \rightarrow (R(d,d) \rightarrow R(d,d)))
\end{aligned}$$

which can be simplified in

$$(R(c,d) \rightarrow (R(c,d) \rightarrow R(d,c))) \ \land (R(d,d) \rightarrow (R(d,d) \rightarrow R(c,d))) \tag{15}$$

Let us now construct the truth table

2-type	$R(c,c)$	$R(d,d)$	$R(c,d)$	$R(d,c)$	(15)
$111(c,d)$	T	T	T	T	T
$112(c,d)$	T	T	T	F	F
$113(c,d)$	T	T	F	T	F
$114(c,d)$	T	T	F	F	T
$121(c,d)$	T	F	T	T	T
$122(c,d)$	T	F	T	F	T
$123(c,d)$	T	F	F	T	T
$124(c,d)$	T	F	F	F	T
$221(c,d)$	F	F	T	T	T
$222(c,d)$	F	F	T	F	T
$223(c,d)$	F	F	F	T	T
$224(c,d)$	F	F	F	F	T

from which we have that $n_{11} = 2$, $n_{12} = 3$ and $n_{22} = 4$ We can then replace in the formula for FOMC

$$\sum_{k=1}^{4} \binom{4}{k} n_{11}^{\frac{k(k-1)}{2}} n_{12}^{k(4-k} n_{22}^{\frac{(4-k)(3-k)}{2}} = 4^6 + 4(3^3 4^3) + 6(2^1 3^4 4^1) + 4(2^3 3^3) + 2^6$$

□

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