Notes of the course "Differential Equations"

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Contents

1 Verification theorems and synthesis of optimal feedback control

1.1 Finite horizon verification theorems

Consider the control system

$$\begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s > t \\ y(t) = x \end{cases} \tag{1}$$

where $y(s) \in \mathbb{R}^n$, and $\alpha \in \mathcal{A}$, with

$$\mathcal{A} := \{\alpha(\cdot) \colon \mathbb{R} \to A, \text{ measurable and such that the solution of } (1)\ y(\cdot) \text{ exists unique in } [t,T]\}.$$

The variable $y(s)$ represents the state of the system at time s, while α is called *control* and $\mathcal{A}$ is the set of the *admissible controls.* The solution $y(s)$ of (1) will also be denoted by $y_x(s;t,\alpha)$ or by $y(s;x,t,\alpha)$. In this section 1.1 we won't need any hypothesis on the set $A \neq \emptyset$ and on f.

We distinguish two types of controls:

- OPEN LOOP, which depend only on time,
- CLOSED LOOP, or FEEDBACKS or MARKOVIAN, which depend both on time and on the state of the system. Feedback controls are measurable functions $\Phi \colon \mathbb{R}^n \times \mathbb{R} \to A$. Given Φ, the corresponding trajectory, if it exists, is given by $\dot{y}(s) = f(y(s), \Phi(y(s), s))$. Φ is *admissible* if the trajectory exists unique in $[t,T]$, for every initial datum $x \in \mathbb{R}^n$ and if the associated open loop control Φ placing $\alpha_\Phi(s) := \Phi(y(s), s)$ is measurable.

An example of admissible feedbacks ig given by Lipschitz functions of $\mathbb{R}^n$ in A if $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ is Lipschitz (in all its variables). They are admissible since $y \mapsto f(y, \Phi(y))$ is Lipschitz, so we may apply the global existence and uniqueness theorem for ordinary equations.

Definition 1. We define the *cost functional*

$$J(x,t,\alpha) = \int_t^T l(y(s), \alpha(s))\, ds + g(y(T)),$$

where the trajectory is $y(s) = y_x(s;t,\alpha)$.

We call it Lagrange cost if the final cost is $g \equiv 0$, Mayer cost if the current cost is $l \equiv 0$, and Bolza cost if $l \neq 0 \neq g$.

Remark 1. We notice that, adding a variable, the general problem may be reduced to a Mayer problem, i.e. with $l \equiv 0$. Indeed, let's consider a new variable y_{n+1} with dynamics

$$\begin{cases} \dot{y}_{n+1} = l(y, \alpha) \\ y_{n+1}(t) = x_{n+1} \end{cases}$$

Then we have that

$$y_{n+1}(\tau) = x_{n+1} + \int_t^\tau l(y(s), \alpha(s))\, ds.$$

Consider a new problem, with $\tilde{y} = (y, y_{n+1})$ and dynamics $\dot{\tilde{y}} = \tilde{f}(\tilde{y}, \alpha)$, dove $\tilde{f} = (f_1, \dots, f_n, l)$. Let us consider as new cost $\tilde{J}(\tilde{x}, t, \alpha) = y_{n+1}(T) + g(y(T)) := \tilde{G}(\tilde{y}(T))$. We have that $\tilde{J}(x, 0, t, \alpha) = J(x,t,\alpha)$. Hence we may limit ourselves to study problems in which $l \equiv 0$.

Let us now consider the dynamic programming equation of Mayer problem

$$\begin{cases} -W_t + \sup_{a\in A}\{-D_x W \cdot f(x,a)\} = 0, & \text{in } \mathbb{R}^n \times (t_o, T) \\ \\ W(x,T) = g(x). \end{cases} \tag{2}$$

Theorem 1 (verification)**.** *Let $W \in C^1(\mathbb{R}^n \times (t_o, T))$ and continuous at $t = T$ be a solution of* (2)*, and let $t < T$ and $x \in \mathbb{R}^n$ be fixed. Then*

1. *for every $\alpha \in \mathcal{A}$ the function $s \mapsto W(y(s), s)$, where $y(s)$ is the trajectory relative to α, is increasing (in wide sense);*
2. *$W(x,t) \le \inf_{\alpha\in\mathcal{A}} J(x,t,\alpha)$;*
3. *if there exists α^* such that, setting $y^*(s) = y_x(s;t,\alpha^*)$, we have that*
$$W_t(y^*(s), s) + D_x W(y^*(s), s) \cdot f(y^*(s), \alpha^*(s)) = 0, \quad \forall s \in [t,T] \tag{3}$$
then α^* is optimal and $W(x,t) = \inf_{\alpha\in\mathcal{A}} J(x,t,\alpha)$.

Before proving the theorem, let's point out some important remarks.

Remark 2. If $W \in C^1$ we may change sign in the equation (2), which becomes

$$W_t + \inf_{a\in A} D_x W \cdot f(x,a) = 0.$$

Hence (3) is equivalent to state that α^* is such that:

$$D_x W(y^*(s), s) \cdot f(y^*(s), \alpha^*(s)) = \inf_{a\in A} D_x W(y^*(s), s) \cdot f(y^*(s), a)$$

Remark 3. The theorem suggests a method to find the optimal control (Optimal Feedback Synthesis):

1. solve (2);
2. look for an admissible feedback $\Phi(y,s) \in \text{argmin}_{a\in A}(D_x W(y,s) \cdot f(y,a))$;
3. the optimal trajectory is given by $\dot{y}^*(s) = f(y^*(s), \Phi(y^*(s), s))$, $y^*(t) = x$.

Unfortunately, in general (2) has no classical solution, and argmin isn't continuous. Later we'll see an example in which we may apply this procedure.

Let us now prove the theorem.

Proof. We notice that the function $s \mapsto W(y(s), s)$ is absolutely continuous, since $y(s)$ is so. Then it is differentiable for almost every $s \in [t,T]$, and the following holds

$$\frac{d}{ds} W(y(s), s) = W_t + D_x W \cdot f(y(s), \alpha(s)) \ge W_t + \inf_{a\in A}\{D_x W \cdot f(x,a)\} = 0. \tag{4}$$

Then it is increasing. For what is shown in the previous point, we have that

$$W(x,t) \le W(y(T), T) = g(y(T)) = J(x,t,\alpha)$$

Moreover, if hypothesis (3) holds, the inequality (4) becomes

$$\frac{d}{ds} W(y^*(s), s) = 0$$

almost everywhere, hence $W(y^*(\cdot), \cdot)$ is constant. Then, $W(x,t) = g(y^*(T)) = J(x,t,\alpha^*)$. Moreover, we have that, for every $\alpha \in \mathcal{A}$, $J(x,t,\alpha^*) \le J(x,t,\alpha)$, hence

$$W(x,t) = J(x,t,\alpha^*) = \inf_{\alpha\in\mathcal{A}} J(x,t,\alpha)$$

□

Now we give the generalization of the verification theorem to the general case (Bolza problem). Define the pre-Hamiltonian

$$\mathcal{H}(p,x,a) := f(x,a)\cdot p + l(x,a),$$

and the Bellman Hamiltonian

$$H(p,x) := \sup_{a\in A}\{-f(x,a)\cdot p - l(x,a)\} = -\inf_{a\in A}\mathcal{H}(p,x,a).$$

Note that $\mathcal{H}(p,x,a) \geq -H(p,x)$ for all $a \in A$.

Theorem 2 (verification)**.** *Let $W \in C^1(\mathbb{R}^n \times (t_o,T))$ and continuous at $t=T$ be a solution of the Cauchy problem with terminal value*

$$\begin{cases} W_t + \inf\limits_{a\in A}\mathcal{H}(D_xW,x,a) = 0\,, & \text{in } \mathbb{R}^n \times (t_o,T) \\ w(x,T) = g(x)\,. \end{cases} \tag{CT}$$

Then, $\forall x \in \mathbb{R}^n, \forall\, t_0 < t < T$,

1. *the function $s \to W(y_x(s;\alpha,t),s) + \int\limits_t^s l(y(\tau),\alpha(\tau))d\tau$ is increasing (in wide sense) for any admissible control α.*
2. *$W(x,t) \leq v(x,t) = \inf\limits_{\alpha\in\mathcal{A}}(\int\limits_t^T l(y(s),\alpha(s))ds + g(y(T)))$,*
3. *if $\forall s \leq T$ we have*

$$\mathcal{H}(D_xW(y^*(s),s),y^*(s),\alpha^*(s)) = \inf_{a\in A}\mathcal{H}(D_xW(y^*(s),s),y^*(s),a)$$

where y^* is the trajectory controlled by α^*, then α^* is an optimal control, i.e.,

$$J(x,t,\alpha^*) = v(x,t)\,,$$

and $W(x,t) = v(x,t)$.

Proof. Homework. □

Note that the PDE in (CT) is the Hamilton-Jacobi-Bellman equation satisfied by the value function (in viscosity sense)

$$-v_t + H(D_xv,x) = 0\,.$$

1.2 Finite Horizon Linear-Quadratic regulator (LQ control)

Let $A \in \mathcal{M}_{n\times n}$ be an $n\times n$ matrix with real coefficients and let $B \in \mathcal{M}_{n\times m}$. Consider the controlled system

$$\dot{y} = Ay + B\alpha,$$

where $\alpha \in L^2([0,T],\mathbb{R}^m)$, and the cost functional is

$$J(x,t,\alpha) = \int_t^T \left[y(s)^T My(s) + \alpha(s)^T R\alpha(s)\right]\,ds + y(T)^T Qy(T),$$

with $Q, M \in Sym(n)$, $R \in Sym(m)$. Then in this case

$$f(y,a) = Ay + Ba, \qquad l(y,a) = y^T My + a^T Ra, \qquad g(y) = y^T Qy.$$

Notice that the set of controls is $\mathbb{R}^m$, hence non compact, and that all data are unbounded.

Moreover suppose

$$R \quad \text{positive definite.}$$

The dynamic programming equation is

$$W_t + D_xW \cdot Ax + x^tMx + \inf_{a\in\mathbb{R}^m}\{D_xW \cdot Ba + a^TRa\} = 0. \tag{5}$$

Let's compute the Hamiltonian. Let $\tilde{H}(p,a) = p^TBa + a^TRa$. The partial derivative of $\tilde{H}$ is

$$\frac{\partial \tilde{H}}{\partial a}(p,a) = \frac{\partial(p^TBa + a^TRa)}{\partial a} = B^Tp + 2Ra$$

which vanishes in

$$a = -\frac{R^{-1}B^Tp}{2} =: a_{min}.$$

Since $\tilde{H}$ has at most linear growth in p, but quadratic growth in a, we can easily see that

$$\lim_{|a|\to+\infty} \tilde{H}(p,a) = +\infty.$$

Indeed, since R is positive definite, it is enough to estimate from below

$$a^TRa \geq \lambda_{min}|a|^2$$

where λ_{min} is the smallest eigenvalue of R. Hence, for fixed p the function $\tilde{H}(p,a)$ attains its minimum in a_{min}. Then

$$\inf_{a\in\mathbb{R}^m} \tilde{H}(p,a) =$$

$$-\frac{1}{2}p^TBR^{-1}B^Tp + \frac{1}{4}p^TBR^{-1}RR^{-1}B^Tp = -\frac{1}{4}p^TBR^{-1}B^Tp = -\frac{1}{4}p^TSp.$$

where $S := BR^{-1}B^T$. Hence (5) becomes

$$W_t + D_xW \cdot Ax + x^TMx - \frac{1}{4}D_xW^TSD_xW = 0,$$

with final condition $W(x,T) = x^TQx$.
Now let's look for solution of the form $W(x,t) = x^TK(t)x$, with $K(t) \in Sym(n)$, $K \in C^1$ in t. Then

$$W_t(x,t) = x^T\dot{K}(t)x, \qquad D_xW(x,t) = 2K(t)x.$$

Since for any matrix C we have $x^TCx = x^T\frac{C+C^T}{2}x$, the equation can be written as

$$x^T(\dot{K} + KA + A^TK + M - KSK)x = 0, \qquad \forall\, x \in \mathbb{R}^n,$$

that is,

$$\dot{K} + KA + A^TK + M - KSK = 0.$$

We have now obtained a terminal value problem for the Riccati matrix ordinary differential equation:

$$\begin{cases} \dot{K} = -KA - A^TK - M + KSK \\ K(T) = Q. \end{cases} \tag{6}$$

Theorem 3. *If there exists $K(\cdot) \in C^1((t_0,T), Sym(n))$ continuous at $t = T$ solution of (6), then $W(x,t) = x^TK(t)x$ is a C^1 solution of (2), hence $W(x,t) = x^TK(t)x = \min_{\alpha\in\mathcal{A}} J(x,t,\alpha)$, and the feedback control $\Phi(y,s) = -R^{-1}B^TK(s)y$ is admissible and optimal, for every $x \in \mathbb{R}^n$, for every $t_0 < t \leq T$.*

Proof. The first part follows from the previous calculations and the verification theorem, substituting $p = D_x W(y,s) = 2K(s)y$ in a_{min}. It remains to check that $\Phi(y,s) = -R^{-1}B^T K(s)y$ is admissible. The corresponding dynamical system is

$$\dot{y}(s) = (A - SK(s))y(s)\,,$$

which is linear and therefore Lipschitz in the state, so there is existence and uniqueness of the trajectory in $[t_0, T]$. $\square$

Proposition 1. *With the hypotheses stated in this section there exists $t_0 < T$ such that the problem* (6) *has a unique solution $K(\cdot) \in C^1((t_0,T), Sym(n))$.*

Proof. Existence and uniqueness of a matrix which is a solution in (t_0, T) follows from local existence and uniqueness theorem for ordinary differential equations. We are left to check that $K(t) \in Sym(n)$ for all t. Transposing both sides of the two equations in (6) and recalling that M, S, Q are symmetric, we can easily see that $K(\cdot)^T$ solves the same Cauchy problem as K, hence $K(t) = K(t)^T$ for every t. $\square$

Theorem 4. *If we suppose that M, Q are positive semidefinite, and R is positive definite, then* (6) *has a unique solution $K \in C^1((-\infty,T), Sym(n))$ continuous at $t = T$.*

Remark 4. Under the assumptions of this theorem the cost functional is nonnegative, $J(x,t,\alpha) \geq 0$ for all x, t, α.

Proof. Let (t_0, T) be the maximal interval of existence of the solution. If $t_0 > -\infty$ we must have $\lim_{t \to t_0+} \|K(t)\| = +\infty$. If we show that $\forall \tau < T \, \exists \overline{C}_\tau$ such that $\|K(t)\| \leq \overline{C}_\tau, \forall t \in [\tau, T]$, we may conclude. Denote by $\|\cdot\|$ the norm of the trace ($\|X\| := \sum_{i=1}^n |\lambda_i|$ where $\lambda_1, ..., \lambda_n$ are the eigenvalues of $X \in Sym(n)$), with $\|\cdot\|_2$ the Euclidean norm. By Theorem 3 it follows that $x^T K(t)x = \min J(x,t,\alpha) \geq 0$, hence $K(t) \geq 0$ and $\lambda_{min}(K(t)) \geq 0$.

Consider now $\alpha \equiv 0$ and let $y_x^0(\cdot) := y_x(\cdot; t, 0)$ be the corresponding trajectory. Since $y_0^0(\cdot) \equiv 0$, from the estimates on the solution of the controlled system we get that $|y_x^0(s)| \leq C_\tau |x|$, for every $s \in (t, T)$. Thus, for $\tau \leq t \leq T$,

$$\begin{aligned} x^T K(t)x \leq J(x,t,0) = \int_t^T y^0(s)^T M y^0(s)\,ds + y^0(T)^T Q y^0(T) \leq \\ \leq (T-t)\|M\|_2 C_\tau^2 |x|^2 + \|Q\|_2 C_\tau^2 |x|^2 \leq \tilde{C}_\tau |x|^2. \end{aligned}$$

Hence $x^T K(t)x \leq \tilde{C}_\tau |x|^2$ holds, so $K(t) - \tilde{C}_\tau I \leq 0$, i.e., it is a negative semidefinite matrix. Thus we must have $\lambda_{max}(K(t)) - \tilde{C}_\tau \leq 0$. Ultimately, since $\lambda_{min} \geq 0$,

$$\|K(t)\| \leq n \max |\lambda_i| = n \max \lambda_i = n\lambda_{max} \leq n\tilde{C}_\tau =: \overline{C}_\tau.$$

$\square$

An important property of Riccati's ordinary equations is that they can be reduced to linear systems.

Theorem 5. *Let $U, V \in C^1((t_0,T), \mathcal{M}_{n\times n})$ be solutions in (t_0, T) of*

$$\begin{cases} \dot{U} = AU - SV \\ \dot{V} = -MU - A^T V \\ U(T) = I, \quad V(T) = Q, \end{cases}$$

such that $\det U(t) \neq 0$ *for all $t \in (t_0, T)$. Then $K(t) := V(t)U^{-1}(t)$ solves the Cauchy problem for Riccati equation* (6).

Proof. It follows from a direct calculation based on the identity $dU^{-1}/dt = -U^{-1}\dot{U}U^{-1}$. The reader is invited to check it. $\square$

Remark 5. In dimension $n = 1$ Riccati equation is scalar and it may be solved explicitly by separation of variables, or by solving the 2×2 linear system of the previous theorem.

1.3 Teorema di verifica a orizzonte infinito

[Non in programma dal 2019]

Vediamo ora una versione del precedente teorema di verifica nel caso in cui il tempo T di arresto non sia finito. Notiamo che il costo corrente convergerà grazie ad un termine di sconto esponenziale.

Theorem 6 (di verifica)**.** *Consideriamo l'equazione di Bellman stazionaria*

$$\delta W + \sup_{a\in A}\{-DW \cdot f(x,a) - l(x,a)\} = 0, \qquad \text{in } \mathbb{R}^n. \tag{7}$$

Sia $J_\infty(x,\alpha) = \int\limits_0^{+\infty} l(y(s),\alpha(s))e^{-\delta s}\,ds$*. Sia* $W \in C^1$ *soluzione di* (7)*, e sia* $x \in \mathbb{R}^n$ *fissato. Allora,*

1. *per ogni* $\alpha \in \mathcal{A}$ *la funzione* $\varphi(t) = W(y(t))e^{-\delta t} + \int\limits_0^t l(y(s),\alpha(s))e^{-\delta s}\,ds$ *è crescente;*

2. *sia* $\mathcal{A}_x^W = \{\alpha \in \mathcal{A}$ *tali che* $\lim_{t\to\infty} W(y(t))e^{-\delta t} = 0\}$*. Se esiste* $\alpha^* \in \mathcal{A}_x^W$ *tale che, detto* $y^*(s) = y_x(s,\alpha^*)$*, si ha che*

$$\delta W(y^*(s)) - DW(y^*(s)) \cdot f(y^*(s),\alpha^*(s)) - l(y^*(s),\alpha^*(s)) = 0, \quad \text{per q.o. } s > 0 \tag{8}$$

allora α^* *è ottimo in* $\mathcal{A}_x^W$ *e* $W(x,t) = \inf_{\alpha\in\mathcal{A}_x^W} J_\infty(x,\alpha)$*.*

Proof. Come in precedenza calcoliamo la derivata di $\varphi(t)$:

$$\varphi'(t) = e^{-\delta t}(-\delta W + DW \cdot f(y,\alpha) + l(y,\alpha)) \geq e^{-\delta t}(-\delta W + \inf_{a\in A}(DW \cdot f(y,a) + l(y,a))$$

per quasi ogni t, che è 0 per l'ipotesi 7. Quindi, φ è crescente. Se vale (8), la disuguaglianza al punto precedente è in realtà un'uguaglianza. Quindi, φ_* è costante. Allora,

$$J_\infty(x,\alpha^*) = \int_0^{+\infty} l(y^*(s),\alpha^*(s))e^{-\delta s}\,ds = \lim_{t\to+\infty}\varphi_*(t) = \varphi_*(0) = W(x),$$

mentre per un controllo $\alpha \in \mathcal{A}_x^W$ qualsiasi si ha

$$W(x) = \varphi(0) \leq \lim_{t\to+\infty}\varphi(t) = J_\infty(x,\alpha).$$

□

1.4 Problema del regolatore lineare (LQ control) a orizzonte infinito

[Non in programma dal 2019]

Siano $A \in \mathcal{M}_N(\mathbb{R})$, $B \in \mathcal{M}_{N\times m}(\mathbb{R})$, $\alpha \in L^1_{loc}((-\infty,T),\mathbb{R}^m)$. Consideriamo il sistema

$$\dot{y} = Ay + B\alpha,$$

il cui funzionale costo è dato da

$$J_\infty(x,\alpha) = \int_0^{+\infty} \left[y(s)^T M y(s) + \alpha(s)^T R \alpha(s)\right] e^{-\delta s}\,ds,$$

con $M \in Sym(N)$, $R \in Sym(m)$. Quindi, in questo caso $f(y,a) = Ay + Ba$, $l(y,a) = y^T My + a^T Ra$. Supponiamo inoltre che M sia semidefinita positiva, R definita positiva. Si noti che i controlli α ora prendono valori nell'insieme illimitato $\mathbb{R}^n$.

L'equazione di Bellman stazionaria in questo caso è

$$-\delta W + H(DW) + x^T Mx + DW \cdot Ax = 0,$$

dove $H(p) = -\frac{p^T BR^{-1}B^T p}{4}$. Formuliamo il seguente ansatz: cerchiamo la soluzione nella forma $W(x) = x^T Kx$, $K \in Sym(N)$. Allora, l'equazione diventa

$$x^T(-\delta K + KA + A^T K + M - KBR^{-1}B^T K)x = 0.$$

Tale condizione è soddisfatta se e solo se K soddisfa l'equazione matriciale di Riccati

$$-\delta K + KA + A^T K + M - KBR^{-1}B^T K = 0$$

che si può riscrivere ponendo $C := A - \delta I$ e usando l'identità $-K\delta = -\delta\frac{KI+IK}{2}$ nella forma

$$KC + C^T K + M = KBR^{-1}B^T K. \tag{9}$$

Usando il teorema di verifica possiamo provare il seguente risultato:

Theorem 7. *Se esiste una soluzione Kdi (9) allora $W(x) = x^T Kx$ è soluzione C^1 di (7), e se il feedback $\Phi(y) = -R^{-1}B^T K(s)y$ genera un controllo open loop $\alpha_\Phi \in \mathcal{A}_x^W$, allora α_Φ è ottimale, e $W(x) = \min J(x, \alpha)$.*

Example 1 (controllo LQ a orizzonte infinito, $d = 1$)**.** Consideriamo l'equazione

$$\dot{y} = ay + \alpha,$$

con $\alpha(s) \in \mathbb{R}$, $a \in \mathbb{R}$. Supponiamo che il funzionale costo sia dato da

$$J(x, \alpha) = \int_0^{+\infty} e^{-\delta t}(R\alpha(s)^2 + m(y(s) - h)^2)\, ds,$$

con $R > 0$, $m \geq 0$. In questo caso l'equazione di Bellman stazionaria associata è data da

$$-\delta W + axW' - \frac{(W')^2}{4R} + m(x - h)^2 = 0. \tag{10}$$

Formuliamo il seguente ansatz: cerchiamo soluzioni nella forma $W_\delta(x) = k_\delta x^2 + b_\delta x + c_\delta$, cioè che siano polinomi di secondo grado.
Si ottiene un sistema di tre equazioni a tre incognite, che in generale ha due soluzioni. Solo una però è accettabile in questo contesto: infatti, per applicare il teorema di verifica deve essere $\lim_{t\to\infty} e^{-\delta t} W(y^*(t)) = 0$, dove il feedback ottimale è $\Phi(y) = -\frac{k}{R}y - \frac{b}{2R}$ e y^* risolve $\dot{y} = (a - \frac{k}{R})y - \frac{b}{2R}$. Alla fine si ottiene

$$k_\delta = \ldots, \quad b_\delta = \frac{2hm}{a - \delta - \frac{k}{R}}, \quad c_\delta = \frac{1}{\delta}(mh^2 - \frac{b^2}{4R}).$$

Ci poniamo il seguente quesito: cosa succede per $\delta \to 0$? Questo problema prende il nome di *vanishing discount problem* o *problema ergodico.*
Se $m > 0, h \neq 0, a \neq 0$, l'equazione (10) non ha soluzione quadratica per $\delta = 0$. Anche un procedimento di limite per δ che tende a zero nell'espressione di W_δ non dà il risultato sperato, perché $W_\delta \to +\infty$. Invece, la strada giusta è quella di calcolare $\lim_{\delta\to0} \delta W_\delta$. Sostituendo l'espressione esplicita si ottiene

$$\lim_{\delta\to0} \delta W_\delta = \frac{mRh^2a^2}{Ra^2 + m} =: \overline{\lambda}.$$

Inoltre, risulta che l'equazione, detta del controllo ergodico o *equazione critica*,

$$-\lambda + axv' - \frac{(v')^2}{4R} + m(x - h)^2 = 0$$

nelle incognite λ e v ha un'unica soluzione con v polinomio di secondo grado. Tale soluzione è $\lambda = \overline{\lambda}$ e $v(x) = \lim_{\delta\to0}(W_\delta(x) - W_\delta(0)) + cost = k_0x^2 + b_0x + cost.$

2 Game Theory

2.1 Zero-sum games

Let A and B be two sets and $\Phi\colon A \times B \to \mathbb{R}$ be a function, which represents the gain of the first player (player A) and the loss of the second player (player B). The goal of A is to maximize Φ, whereas the goal of B is to minimize it.

2.1.1 First examples

We start by describing some examples of matrix games. Set $A = \{1, \dots, m\}$, $B = \{1, \dots, n\}$. In this case we can represent Φ by a matrix: we set $\Phi(i, j) = \varphi_{ij}$ and define

$$M = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{m1} & \varphi_{m2} & \cdots & \varphi_{mn} \end{pmatrix}$$

During the game player A chooses a row and player B chooses a column.

Example 2 (Head or Tail)**.** Each of the two players chooses head or tail. A wins if both players have made the same choice, otherwise B wins. The associated matrix is found easily:

	T	H
T	1	-1
H	-1	1

Note that this game is equivalent to playing odds and evens with two fingers.

Example 3 (Splitting a cake)**.** Two players must split a cake: A cuts the cake and then B gets to choose what his part is. A can choose between cutting the cake into two almost equal slices, $\frac{1}{2} + \varepsilon$ and $\frac{1}{2} - \varepsilon$, or cutting it into two slices of very different size, say $\frac{2}{3}$ and $\frac{1}{3}$. The corresponding matrix is:

	small	big
equal slices	$\frac{1}{2} + \varepsilon$	$\frac{1}{2} - \varepsilon$
different slices	$\frac{2}{3}$	$\frac{1}{3}$

2.1.2 Definitions and elementary results

From now on we make the following hypothesis:

Hypothesis 1: A and B are compact metric spaces, $\Phi\colon A \times B \to \mathbb{R}$ is continous.

We define the *marginal functions*

$$\Phi^{\max}(b) = \max_{a \in A} \Phi(a, b), \quad \Phi^{\min}(a) = \min_{b \in B} \Phi(a, b),$$

and we call *best response maps* the sets

$$\begin{aligned} R^A(b) &= \{\overline{a} \in A \,:\, \Phi^{\max}(b) = \Phi(\overline{a}, b)\} \\ R^B(a) &= \{\overline{b} \in B \,:\, \Phi^{\min}(a) = \Phi(a, \overline{b})\}. \end{aligned}$$

It holds the following

Lemma 1. *$\Phi^{\max}\colon B \to \mathbb{R}$ and $\Phi^{\min}\colon A \to \mathbb{R}$ are continuous.*

Proof. We prove the continuity of $\Phi^{\max}$ at $b = \bar{b}$. Let $\bar{a}$ be such that $\Phi^{\max}(\bar{b}) = \Phi(\bar{a}, \bar{b})$. Then

$$\Phi^{\max}(\bar{b}) - \Phi^{\max}(b) \le \Phi(\bar{a}, \bar{b}) - \Phi(\bar{a}, b) \le \omega_\Phi(d(b, \bar{b})),$$

where ω_Φ is a modulus of continuity of Φ and $d(b, \bar{b})$ is the distance between b e $\bar{b}$. By exchanging b and $\bar{b}$ we get that

$$|\Phi^{\max}(\bar{b}) - \Phi^{\max}(b)| \to 0, \quad \text{as } d(b, \bar{b}) \to 0.$$

□

Definition 2. We define the *upper value* of the game v^+

$$v^+ := \min_{b \in B} \max_{a \in A} \Phi(a, b),$$

and the *lower value* of the game

$$v^- := \max_{a \in A} \min_{b \in B} \Phi(a, b).$$

A strategy b^* that realizes the minimum in the definition of v^+ is called a *security strategy for B*, i.e.

$$\max_{a \in A} \Phi(a, b^*) = \min_{b \in B} \max_{a \in A} \Phi(a, b),$$

whereas a^* that realizes the maximum in the definition of v^- is called a *security strategy for A*.

Proposition 2. $v^- \le v^+$.

Proof. We have that

$$\min_{b \in B} \Phi(a, b) \le \Phi(a, b')$$

for every $b' \in B$. Then

$$v^- = \max_{a \in A} \min_{b \in B} \Phi(a, b) \le \max_{a \in A} \Phi(a, b')$$

for every $b' \in B$. Thus we have that the inequality also holds for the minimum, i.e.

$$v^- = \max_{a \in A} \min_{b \in B} \Phi(a, b) \le \min_{b' \in B} \max_{a \in A} \Phi(a, b') = v^+.$$

□

Remark 6. v^+ is a "security level" for B: it is the best result that B can obtain if A maximizes Φ for every choice of B.

Definition 3. We say that the game has *value* v if $v^+ = v^- =: v$.

Let's go back to the previous examples.

Example 4 (Head or Tail)**.** It is easy to see that the minimum of each row is -1, whereas the maximum of each column is 1. Thus it immediately follows that

$$v^+ = \min_{b \in B} \max_{a \in A} \Phi(a, b) = \min_{b \in B} \{1, 1\} = 1 > -1 = \max_{a \in A} \{-1, -1\} = \max_{a \in A} \min_{b \in B} \Phi(a, b) = v^-$$

Observe that in this case the value does not exist.

Example 5 (Splitting a cake)**.** Analogously, we can check that in the "splitting a cake" game we have:

$$v^+ = \min_{b \in B} \max_{a \in A} \Phi(a, b) = \min_{b \in B} \{\frac{2}{3}, \frac{1}{2} - \varepsilon\} = \frac{1}{2} - \varepsilon = \max_{a \in A} \{\frac{1}{2} - \varepsilon, \frac{1}{3}\} = \max_{a \in A} \min_{b \in B} \Phi(a, b) = v^-$$

Observe that in this case the game has value $v = \frac{1}{2} - \varepsilon$.

Definition 4. A couple $(a^*, b^*) \in A \times B$ is called a *saddle point* of the game if for every $a \in A$ and for every $b \in B$ it holds that

$$\Phi(a, b^*) \leq \Phi(a^*, b^*) \leq \Phi(a^*, b).$$

Example 6. The name comes from a classical example: if $A = B = [-1, 1]$, $\Phi(a, b) = b^2 - a^2$, then the graph of Φ has the shape of a horse saddle and the saddle point is $(a^*, b^*) = (0, 0)$.

Note that (a^*, b^*) is a saddle point of the game if and only if

$$\max_{a \in A} \Phi(a, b^*) = \Phi(a^*, b^*) = \min_{b \in B} \Phi(a^*, b).$$

Moreover, (a^*, b^*) is a saddle point of the game if and only if $a^* \in R^A(b^*)$ e $b^* \in R^B(a^*)$. If R^A and R^B are functions (i.e. they take values in singletons), then a^* is a fixed point of $R^A \circ R^B$ and b^* is a fixed point of $R^B \circ R^A$.

Theorem 8. *Under Hypothesis 1, the game has value v if and only if it admits a saddle point (a^*, b^*) and in that case $v = \Phi(a^*, b^*)$.*

Proof. Let us assume that (a^*, b^*) is a saddle point of the game. It is enough to prove that $v^- \geq v^+$. This is showed by the following:

$$v^- \geq \min_{b \in B} \Phi(a^*, b) = \Phi(a^*, b^*) = \max_{a \in A} \Phi(a, b^*) \geq v^+.$$

Now we prove the other implication: we assume that the game has a value and we construct a saddle point. Let a^* be a security strategy for A and b^* a security strategy for B. We want to show that (a^*, b^*) is a saddle point of the game. It holds that

$$\forall a \quad \Phi(a, b^*) \leq \max_{a \in A} \Phi(a, b^*) = v^+ = v^- = \min_{b \in B} \Phi(a^*, b) \leq \Phi(a^*, b), \, \forall b.$$

We obtain equality in the previous inequalities by setting $a = a^*$, $b = b^*$ and thus $v = \Phi(a^*, b^*)$ and (a^*, b^*) is a saddle point. ☐

Corollary 1. *If the game has a value, then*
(i) (a^*, b^*) is a saddle point of the game if and only if a^* is a security strategy for A and b^* is a security strategy for B;
(ii) if (a^*, b^*) and (a', b') are saddle point, then also (a^*, b') and (a', b^*) are saddle points (exchangeability property).

Proof. In *(i)* the "if" implication was proved in the previous theorem, whereas the "only if" implication follows from the definition of security strategy for B and

$$\min \max \Phi = \Phi(a^*, b^*) = \max_{a \in A} \Phi(a, b^*)$$

(ii) is implied by *(i)* because a' is a security strategy for A and b' is a security strategy for B. ☐

2.1.3 Other examples

Example 7 (Odds and Evens with three fingers)**.** Two people simultaneously reveal a number of fingers from 1 to 3. Player A wins if the sum of all shown fingers is even, whereas player B wins if it is odd. The associated matrix is

	1	2	3
1	1	-1	1
2	-1	1	-1
3	1	-1	1

This game does not have a value, because $v^+ = 1 > v^- = -1$.

Example 8 (Rock, Scissors, Paper)**.** The associated matrix is:

	S	C	F
S	0	-1	1
C	1	0	-1
F	-1	1	0

This game also does not have a value, because $v^+ = 1 > v^- = -1$.

Example 9 (Nim)**.** There are two boxes, L and R, that contain n and m coins respectively. Two people take turns removing $k \geq 1$ coins from one of the boxes. The player who takes the last coin wins.
We study the case $n = 1, m = 2$. It is easy to represent all possible developments of the game by a tree diagram. This makes it easy to observe that the outcome of the game is completely determined by the first choice of A and by the first choice of B. Let us suppose that A plays first. Then we can construct the matrix of the game, denoting by $1L$ the choice of taking one coin from the box on the left, $1R$ and $2R$ the choice of taking respectively one or two coins from the box on the right. By convention, we also assume that the player that tries to take more coins than the number available in that box loses. Then, supposing as usual that A chooses between the rows of the matrix, we have

	1L	1R	2R
1L	1	1	-1
1R	1	1	1
2R	-1	1	1

In this case the game has value 1 and $(1R, b)$ is a saddle point for every value of b.

Example 10. Set $A = B = [0,1]$ and $\Phi(a,b) = f(a-b)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and periodic of period 1. The game has a value if and only if f is constant. Indeed, $\max_{a\in[0,1]} f(a-b) = \max_{[-b,1-b]} f = \max f$ and $\min_{b\in[0,1]} f(a-b) = \min_{[a,a-1]} f = \min f$ do not depend, respectively, on b and on a, and they coincide if and only if f is constant.

2.1.4 A minmax theorem

Theorem 9 (minmax theorem of Von Neumann)**.** *Let A, B be compact and convex subsets of a vector space and let $\Phi \in C(A \times B)$ be such that, for every b, $a \mapsto \Phi(a,b)$ is concave and, for every a, $b \mapsto \Phi(a,b)$ is convex. Then there exists the value $v = v^+ = v^-$.*

Proof. We prove the theorem in the special case $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$.

1. Let us first suppose that for every a the function $b \mapsto \Phi(a,b)$ is strictly convex. Then it clearly has a unique minimum point, i.e. for every a there exists a unique $r(a)$ such that $\Phi(a, r(a)) = \min_b \Phi(a,b)$. We want to show that $\Phi^{min}(a) = \Phi(a, r(a))$ is continuous. Since Φ is continuous, it suffices to show that r is continuous. Fix $\overline{a}$ and let $\{a_n\}$ be a sequence such that $a_n \to \overline{a}$. Since B is compact, there exists a subsequence a_{n_k} such that $r(a_{n_k})$ converges to some $\overline{b}$. Moreover it holds that $\Phi(a_{n_k}, r(a_{n_k})) \leq \Phi(a_{n_k}, b)$ for every b. Passing to the limit we obtain that $\Phi(\overline{a}, \overline{b}) \leq \Phi(\overline{a}, b)$ for every b, so that it must necessarily be $\overline{b} = r(\overline{a})$, by the uniqueness of the minimum point.

2. Let a^* be a security strategy for A. We know that $v^- = \max \Phi^{min}(a) = \Phi^{min}(a^*) = \Phi(a^*, r(a^*))$. Set $b^* = r(a^*)$ and let us show that (a^*, b^*) is a saddle point. It suffices to prove that $\Phi(a^*, b^*) \geq \Phi(a, b^*)$, $\forall a$.

3. Let a, b be arbitrary, $0 \leq \lambda \leq 1$, $\mu = 1 - \lambda$. It holds

$$\Phi(\lambda a + \mu a^*, b) \geq \lambda\Phi(a,b) + \mu\Phi(a^*, b) \geq \lambda\Phi(a,b) + \mu\Phi^{min}(a^*),\ \forall b,$$

where the first inequality is true by the concavity hypothesis and the second by definition of marginal. Set $a_\lambda = \lambda a + \mu a^*$. We have that

$$\Phi^{min}(a^*) \geq \Phi^{min}(a_\lambda) = \Phi(a_\lambda, r(a_\lambda)) \geq \lambda\Phi(a, r(a_\lambda)) + \mu\Phi^{min}(a^*).$$

This implies that $\Phi^{min}(a^*) \geq \Phi(a, r(a_\lambda))$ and by the continuity of r, passing to the limit as $\lambda \to 0$, we obtain

$$\Phi^{min}(a^*) = \Phi(a^*, b^*) \geq \Phi(a, r(a^*)) = \Phi(a, b^*), \; \forall a$$

which proves the theorem under the initial additional assumption of strict convexity in the b variable.

4. Now we are left with the general case. Let us consider $\Phi_\varepsilon(a, b) = \Phi(a, b) + \varepsilon|b|^2$, which is strictly convex in the b variable. This implies that there exists a saddle point $(a_\varepsilon, b_\varepsilon)$ for Φ_ε, namely

$$\forall a \quad \Phi_\varepsilon(a, b_\varepsilon) \leq \Phi_\varepsilon(a_\varepsilon, b_\varepsilon) \leq \Phi_\varepsilon(a_\varepsilon, b), \; \forall b.$$

By compactness there exists a subsequence ε_n such that $a_{\varepsilon_n} \to a^*$, $b_{\varepsilon_n} \to b^*$ and by continuity $\Phi_\varepsilon(a_{\varepsilon_n}, b_{\varepsilon_n}) \to \Phi(a^*, b^*)$. Moreover, $\Phi(a, b_{\varepsilon_n}) \leq \Phi_\varepsilon(a_{\varepsilon_n}, b_{\varepsilon_n}) \leq \Phi(a_{\varepsilon_n}, b) + \varepsilon|b|^2$. Passing to the limit, we obtain the desired inequality: $\Phi(a, b^*) \leq \Phi(a^*, b^*) \leq \Phi(a^*, b)$.

$\square$

Example 11. Let M be a $n \times m$ matrix and consider $\Phi(a, b) = a^T M b$, where $a \in A$, $b \in B$ with $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ compact and convex. Then Φ is bilinear and satisfies the hypotheses of the previous theorem.

2.1.5 Mixed strategies for matrix games

If A and B are finite sets, then obviously they are compact but not convex and so the hypothesis of the previous theorem do not hold. In particular, this implies that the previous theorem does not apply to matrix games and indeed in Example 4 the game does not have a value. However, given two sets A and B, we can define the sets of *mixed strategies* for player A and B, respectively, as

$$\mathcal{P}(A) := \{\mu \text{ probability measure on } A\}, \qquad \mathcal{P}(B) := \{\nu \text{ probability measure on } B\}.$$

To every $\mu \in \mathcal{P}(A)$ we can associate a random variable X defined by $P(X \in S) = \int_S d\mu$, for every $S \subseteq A$. Analogously, to every $\nu \in \mathcal{P}(B)$ we can associate a random variable Y. Assuming that X, Y are indipendent, we define

$$\tilde{\Phi}(\mu, \nu) := E[\Phi(X, Y)] := \iint_{A \times B} \Phi(a, b) \, d\mu(a) d\nu(b).$$

Definition 5. If the game defined by $(\tilde{\Phi}, \mathcal{P}(A), \mathcal{P}(B))$ has value v, we say that the game (Φ, A, B) has *value v in mixed strategies.*

We observe that there is "copy" of A in the set of mixed strategies for player A (and analogously there is a "copy" of B in $\mathcal{P}(B)$) and that $\tilde{\Phi}$ extends Φ. Indeed, it suffices to consider the application from A into $\mathcal{P}(A)$ that maps a to δ_a, the Dirac delta concentrated at a (and analogously for B). Moreover, $\tilde{\Phi}(\delta_a, \delta_b) = \Phi(a, b)$. From now on these strategies will be called *pure strategies.*

Now let us consider an important example of mixed strategies.

Example 12 (matrix games)**.** If $A = \{1, \ldots, m\}$, the probability measures on A can be identified with the m-tuples that belong to the m-dimensional simplex Δ_m:

$$\mathcal{P}(A) \leftrightarrow \Delta_m = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m \; : \; x \in [0, 1]^m, \sum_{i=1}^m x_i = 1\},$$

and the analogous holds for $B = \{1, \dots, n\}$. Obviously Δ_m is compact and convex. Moreover, we observe that

$$\tilde{\Phi}(x, y) = \sum_i \sum_j \varphi_{ij} x_i y_j = x^T M y$$

is bilinear. Thus we can apply Von Neumann's theorem to the game in mixed strategies and we obtain the following

Corollary 2 (Von Neumann)**.** *Every matrix game has a value (and at least a saddle point) in mixed strategies.*

Now we ask ourselves how to find saddle points in mixed strategies of matrix games.

We recall that (x^*, y^*) is a saddle point if and only if x^* is a security strategy for the first player and y^* is a security strategy for the second player, namely if and only if

$$\tilde{\Phi}^{\min}(x^*) = \max_x \tilde{\Phi}^{\min}(x) = \max_x \min_y \tilde{\Phi}(x, y) = \max_x \min_y x^T M y,$$

and analogously $\tilde{\Phi}^{\max}(y^*) = \min_y \max_x x^T M y$, where $x \in \Delta_m$ and $y \in \Delta_n$. It is known that the minimum of a linear function defined on a polyhedron is reached at the vertices (if not, check it as an exercise). The function $y \mapsto x^T M y$ is linear and so it reaches its minimum for y of the kind $(0, \dots, 1, 0, \dots, 0)$, vertex of Δ_m. It follows that

$$\tilde{\Phi}^{\min}(x) = \min_j \sum_i \varphi_{ij} x_i = \min_j (x^T M)_j.$$

and analogously

$$\tilde{\Phi}^{\max}(y) = \max_i (My)_i.$$

Let us try to apply these last results to some of the matrix games that we have introduced before.

Example 13 (Head or Tail)**.** We recall that the matrix of the game is given by

	T	C
T	1	-1
C	-1	1

Let us compute $\tilde{\Phi}^{\min}(x)$. Since

$$x^T M = (x_1 - x_2, -x_1 + x_2),$$

$\tilde{\Phi}^{\min}(x) = -|x_1 - x_2|$. The maximum of $\tilde{\Phi}^{\min}(x)$ on Δ_2 is 0 and it is obtained at $x_1 = x_2 = 1/2$. By symmetry, the minimum of $\tilde{\Phi}^{\max}$ is obtained at $y_1 = y_2 = 1/2$. We conclude that the pair $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is the unique saddle point in mixed strategies of the game, with value 0.

Example 14. Now we study a general 2×2 game, with matrix

$$M = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$$

In order to determine $\tilde{\Phi}^{\max}(y)$, first we compute My, then we take the maximum of the two components and finally we minimize it in Δ_2. However it is easy to reduce this problem to a minimum problem of a single variable: it sufficient to observe that $y_2 = 1 - y_1$. In this setting the security strategies for the second player correspond to the minimum points in $[0, 1]$ of the maximum between two affine functions. Such a maximum is reached either at the endpoints of the interval $[0, 1]$ or at the point in which the two affine functions coincide. We conclude that, in 2×2 matrix games, if there is no saddle point in pure strategies, then there exists a unique saddle point in mixed strategies (and it is possible to compute it explicitly solving two linear equations: try it as an exercise).

Example 15 (Rock, Paper, Scissors)**.** This is a 3×3 matrix game with the following associated matrix

$$M = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

We have that

$$x^T M = (x_2 - x_3, -x_1 + x_3, x_1 - x_2).$$

We want to compute $\max_{x \in \Delta_3} \min(x_2 - x_3, -x_1 + x_3, x_1 - x_2)$. We observe that the minimum can not be positive, so it suffices to find $x^* \in \Delta_3$ such that $\min(x_2 - x_3, -x_1 + x_3, x_1 - x_2) = 0$. For example, set $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. By symmetry, we have that $y^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a security strategy for the second player and (x^*, y^*) is a saddle point in mixed strategies.

2.1.6 Mixed strategies in more general cases

Let $(\mathcal{P}(A), \mathcal{P}(B), \tilde{\Phi})$ be a game in mixed strategies with A or B not finite. We would like $\mathcal{P}(A)$ and $\mathcal{P}(B)$ to be compact metric spaces and $\tilde{\Phi}$ to be continuous. On $\mathcal{P}(A)$ and $\mathcal{P}(B)$ we consider the weak-star topology that comes from thinking of them as the dual spaces of $C(A)$ and $C(B)$ endowed with the uniform convergence topology.

Definition 6. A sequence (μ_k) in $\mathcal{P}(A)$ converges weakly* to μ, and we write $\mu_k \overset{*}{\rightharpoonup} \mu$, if

$$\int_A f(x)\, d\mu_k(x) \to \int_A f(x)\, d\mu \quad \text{as } k \to \infty, \quad \forall\, f \in C(A).$$

An analogous definition is given for $\mathcal{P}(B)$. The following result comes from functional analysis.

Theorem 10. *If A is compact, $\mathcal{P}(A)$ endowed with the weak* convergence is metrizable and (sequentially) compact, i.e. from every sequence (μ_k) in $\mathcal{P}(A)$ it is possible to extract a subsequence (μ_{k_j}) such that $\mu_{k_j} \overset{*}{\rightharpoonup} \mu \in \mathcal{P}(A)$ as $j \to \infty$.*

Remark 7. It is possible to prove that $\tilde{\Phi}$ is continuous on $\mathcal{P}(A) \times \mathcal{P}(B)$ with respect to the product topology of the weak* topologies, i.e. that for every $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{P}(A)$ and $\nu_k \overset{*}{\rightharpoonup} \nu$ in $\mathcal{P}(B)$ it holds that

$$\lim_{k \to \infty} \iint_{A \times B} \Phi(a, b)\, d\mu_k(a) d\nu_k(b) = \iint_{A \times B} \Phi(a, b)\, d\mu(a) d\nu(b) = \tilde{\Phi}(\mu, \nu).$$

Theorem 11 (generalization of Von Neumann's theorem)**.** *Under hypothesis 1, there exists the value (and at least a saddle point) in mixed strategies.*

Proof. Convexity of $\mathcal{P}(A)$ and $\mathcal{P}(B)$ is immediate: if $\mu, \tilde{\mu} \in \mathcal{P}(A)$, then $\lambda\mu + (1-\lambda)\tilde{\mu}$ is a measure such that $(\lambda\mu + (1-\lambda)\tilde{\mu})(A) = 1$ and hence it is an element of $\mathcal{P}(A)$. The function $\tilde{\Phi}$ is concave-convex because it is bilinear, as it is easily verified. All other hypotheses of Von Neumann's minmax theorem are satistied for the game in mixed strategies by Theorem 10 and Remark 7.

The tutorial [Bre] presents a more constructive proof, which only uses the minmax theorem for subsets of Euclidean spaces (which is the one that we have proved). We approximate the infinite dimensional game with matrix games. Let $\{a_n\}_n$ be dense in A and $\{b_n\}_n$ be dense in B. Let us consider the sets $A_N = \{a_1, \ldots, a_N\}$ and $B_N = \{b_1, \ldots, b_N\}$. By Corollary 2 the game has a saddle point (μ_N, ν_N) in mixed strategies and by compactness there exists a subsequence converging to (μ^*, ν^*). Finally, it can be shown that (μ^*, ν^*) is a saddle point of the game. See [Bre] for a complete proof of the more general case of Non-Zero Sum Games, which is the topic of the next section. □

2.2 Non-Zero Sum Games

2.2.1 Notions of equilibrium

In general, Non-Zero Sum Games involve n players, each of them trying to maximize its gain Φ^i. For simplicity, we only deal with the case $n = 2$. Let A, B be sets and $\Phi^A, \Phi^B : A \times B \to \mathbb{R}$ be the payoff that the first and the second player respectively try to maximize. Observe that if $\Phi^A = \Phi^B$, then we have an optimization problem, whereas if $\Phi^A = -\Phi^B$ we have a Zero Sum Game. We make the following hypothesis:

Hypothesis $1'$**:** A, B compact sets; Φ^A, Φ^B continous functions.

Example 16 (Bimatrix games)**.** Set $A = \{1, \ldots, m\}$, $B = \{1, \ldots, n\}$. The game is represented by the matrix

φ^B_{11} φ^A_{11}	φ^B_{12} φ^A_{12}	$\cdots$ $\cdots$
φ^B_{21} φ^A_{21}	φ^B_{22} φ^A_{22}	$\cdots$ $\cdots$
$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$

Definition 7. A pair (a^*, b^*) is a *Nash equilibrium* if

$$\Phi^A(a, b^*) \leq \Phi^A(a^*, b^*) \quad \forall\, a \in A \quad \text{and } \Phi^B(a^*, b) \leq \Phi^B(a^*, b^*) \quad \forall\, b \in B.$$

In other words, (a^*, b^*) is a Nash equilibrium if it is not convenient for any player to deviate from its strategy if the other player does not.

Observe that in the case of Zero Sum Games, i.e. $\Phi^A + \Phi^B = 0$, Nash equilibria are exactly the saddle points.

Definition 8. (a^*, b^*) is a *Pareto optimum* if there does not exist any pair (a, b) such that either

$$\Phi^A(a, b) > \Phi^A(a^*, b^*) \quad \text{and} \quad \Phi^B(a, b) \geq \Phi^B(a^*, b^*),$$

or

$$\Phi^A(a, b) \geq \Phi^A(a^*, b^*) \quad \text{and} \quad \Phi^B(a, b) > \Phi^B(a^*, b^*).$$

Observe that in the case of Zero Sum Games, every pair is a Pareto optimum. Moreover, Pareto optima always exist: fix $\lambda \in]0, 1[$ and let (a^*, b^*) be a maximum point for $F(a, b) := \lambda\Phi^A(a, b) + (1-\lambda)\Phi^B(a, b)$ (the existence of such a point is guaranteed thanks to hypothesis $1'$). Then (a^*, b^*) is a Pareto optimum, because otherwise, since $\lambda, 1 - \lambda > 0$, the point would not be a maximum for F.

Let us study some examples of Non Zero Sum Games.

Example 17 (The Prisoner's Dilemma)**.** Two suspects of theft are offered a bargain: if they both confess, each of them serves six years in prison, if one remains silent and the other confesses, the first one serves eight years and the second one is free, if both remain silent they both serve one year. The matrix that represent the game is:

	C	S
C	-6 -6	-8 0
S	0 -8	-1 -1

where C indicates the choice to confess, S the one to remain silent. The only Nash equilibrium is (C, C), whereas (C, S), (S, S), (S, C) are Pareto optima. Observe that (C, C) is not a Pareto optimum and it produces the maximum total number of years of jail for the two supects.

Example 18 (Arms race)**.** We analyze a really simplified model of the arms race happened during the Cold War. In this model, USA and URSS only have two choices: to continue to own only conventional arms (C) or to build a nuclear arsenal (N). The matrix of the game is:

	N		C	
N		-1		-5
	-1		10	
C		10		0
	-5		0	

Again we have a unique Nash equilibrium (N, N), whereas the other pairs are Pareto optima.

Example 19 (Chicken run)**.** This game was sometimes played by youngsters in the US some decades ago[1]. Two cars are driven straight against each other. Both drivers can choose whether to turn (T) or continue to go straight (S). If no one turns, they collide and lose (-10). If both turn, then they are even ($+1$). If one turns and the other does not, the one who turned loses (chicken!). The matrix of the game is:

	T		S	
T		1		3
	1		-2	
S		-2		-10
	3		-10	

The pairs (T, S) and (S, T) are both Nash equilibria and Pareto optima, whereas (T, T) is only a Pareto optima and (S, S) is not an equilibrium. There are two important differences with respect to Zero Sum Games. First of all, the two Nash equilibria produce two completely different results of the game and so it is not possible to define a unique "value of the game". Moreover, the exchangeability property for saddle points of Corollary 1 *(ii)* does not hold anymore.

Example 20 (of a game with a continuum of strategies)**.** Let A, B be closed and bounded intervals and Φ^A and Φ^B differentiable. Suppose that there exists a Nash equilibrium (a^*, b^*) in the interior of $A \times B$. Then

$$\frac{\partial \Phi^A}{\partial a}(a^*, b^*) = 0 \qquad \text{and} \qquad \frac{\partial \Phi^B}{\partial b}(a^*, b^*) = 0$$

Let $(x(t), y(t))$ be a regular parametrization of the level set of Φ^A corresponding to the value $\Phi^A(a^*, b^*)$ and let us suppose that the implicit function theorem holds (i.e., $\nabla \Phi^A \neq 0$); then

$$\frac{\partial \Phi^A}{\partial a} x' + \frac{\partial \Phi^A}{\partial b} y' = 0$$

implies $y' = 0$, i.e. the level set of Φ^A has horizontal tangent at (a^*, b^*). Analogously, we obtain that the level set of Φ^B has a vertical tangent at (a^*, b^*). Hence we have a necessary condition for a point in the interior of $A \times B$ to be a Nash equilibrium: it must lie in the intersection of two curves, that are level sets of Φ^A and Φ^B respectively, and that satisfy these geometric conditions at that point.

Example 21 (Cournot's duopoly model 1838)**.** We consider a market consisting of only two firms, which produce the same good. The quantity produced by firm A is $a \in [0, M_1]$, whereas firm B produces $b \in [0, M_2]$ (in quintals q). The selling price p is determined by the law of demand: $p = P - k(a + b)$ (in dollars \$), where $k > 0$ (the unit of measurement of k is $\frac{\$}{q}$). The returns of the two firms are

$$\Phi^A(a, b) = pa = Pa - ka^2 - abk, \qquad \Phi^B(a, b) = pb = Pb - kb^2 - abk.$$

[1] see, e.g., the movies Rebels without a cause (Gioventù bruciata, 1955) and American graffiti (1973). The game was also used by Bertrand Russell in a book of 1962 as a metaphor of the ongoing arms race.

From the previous example, we have that a Nash equilibrium in the interior of the constraint must satisfy the following equations

$$P - 2ka - bk = 0, \qquad P - 2kb - ak = 0,$$

which have $(\frac{P}{3k}, \frac{P}{3k})$ as the only solution. Supposing $\frac{P}{3k} < M_1$ and $\frac{P}{3k} < M_2$, it is possible to show that this point is really a Nash equilibrium of the game.

2.2.2 Nash theorem

Theorem 12 (Nash, 1951). *Let A, B be compact and convex, Φ^A, Φ^B continuous. If for every b the function $a \mapsto \Phi^A(a, b)$ is concave and for every a the function $b \mapsto \Phi^B(a, b)$ is concave, then there exists at least one Nash equilibrium.*

Before proving the theorem, let us make some remarks. First of all, observe that the minmax theorem is a particular case of this one. Then, let us extend some of the definitions given for Zero Sum Games. The best response maps become $R^B(a) = \operatorname{argmax}_b \Phi^B(a, b)$ and $R^A(b) = \operatorname{argmax}_a \Phi^A(a, b)$. We have that (a^*, b^*) is a Nash equilibrium if and only if $a^* \in R^A(b^*)$ and $b^* \in R^B(a^*)$.

Definition 9. A *multifunction* $F : X \rightsquigarrow X$ is a function with domain X and values in $\mathcal{P}(X)$.

Definition 10. If $F : X \rightsquigarrow X$ is a multifunction, x^* is a *fixed point* of F if $x^* \in F(x^*)$.

With these definitions, (a^*, b^*) is a Nash equilibrium if and only if it is a fixed point of the multifunction $(a, b) \rightsquigarrow R^A(b) \times R^B(a)$. Thus the proof of the above theorem can be obtained using Kakutani's theorem, a general fixed point theorem for multifunctions. This was indeed the first proof, proposed by Nash in [Na50] (see [Bre] for more details). We will use instead a classical fixed point theorem for single-valued functions, namely:

Theorem 13 (Brouwer). *Let $K \subseteq \mathbb{R}^n$ be convex and compact, $f : K \to K$ continuous. Then there exists a fixed point of f.*

Remark 8. If $K \subseteq \mathbb{R}$, the proof is an easy consequence of the Intermediate Zero Theorem: draw a picture!

Let us go back to the proof of Nash's theorem.

Proof. For simplicity, we restrict ourselves to the case $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$.

1. Suppose that Φ^A, Φ^B are strictly concave in a and b respectively. Then they have a unique maximum point and the best reply map is single valued: $R^A(b) = \{r^A(b)\}$ and $R^B(a) = \{r^B(a)\}$. Moreover, r^A and r^B are continuous, as shown in the proof of Von Neumann's theorem. Since $K := A \times B$ is compact and convex and $f(a, b) := (r^A(b), r^B(a))$ is continuous, Brouwer's theorem asserts that f has a fixed point and hence there exists a Nash equilibrium (a^*, b^*).

2. If Φ^A, Φ^B are concave in a and b respectively, $\Phi^A_\varepsilon(a, b) := \Phi^A(a, b) - \varepsilon|a|^2$ and $\Phi^B_\varepsilon(a, b) := \Phi^B(a, b) - \varepsilon|b|^2$ are strictly concave in a and b respectively. Hence, by step 1, there exists a Nash equilibrium $(a^*_\varepsilon, b^*_\varepsilon)$. By compactness, there exists a subsequence $\varepsilon_k \to 0$ such that $(a^*_{\varepsilon_k}, b^*_{\varepsilon_k})$ converges to some (a^*, b^*). Finally, it is easily checked that (a^*, b^*) is a Nash equilibrium for Φ^A, Φ^B.

$\square$

Corollary 3. *All bimatrix games (i.e. games with A, B finite) have at least one Nash equilibrium in mixed strategies.*

Proof. Recall that $\mathcal{P}(A) \leftrightarrow \Delta_m$ and $\mathcal{P}(B) \leftrightarrow \Delta_n$, compact and convex. Moreover, $\tilde{\Phi}^A(x,y) = x^T M_A y$ and $\tilde{\Phi}^B(x,y) = x^T M_B y$ are bilinear maps and so they clearly satisfy the hypotheses of theorem. □

We also state the analogous of Theorem 11 for Non-Zero Sum Games:

Theorem 14. *If A, B are compact and Φ^A, Φ^B are continuous, then there exists a Nash equilibrium in mixed strategies.*

Proof. We check that the hypotheses of Nash's theorem are satisfied by reasoning as in the proof of Theorem 11. □

Example 22 (Welfare game)**.** In order to encourage job seekers, a government prefers to give financial support to unemployed who are actively looking for a job, rather than to help unemployed who do not. On the other hand, if the welfare is too generous, unemployed who get the financial support prefer not to look for a job.

	seeking job	not seeking job
welfare	2 3	3 -1
no welfare	1 -1	0 0

This game does not have Nash equilibria in pure strategies (check), but the previous theorem guarantees that it has Nash equilibria in mixed strategies.

3 Differential Games

Let us consider the controlled system

$$\begin{cases} \dot{y} = f(y(s), a(s), b(s)), \quad s > t, \\ y(t) = x, \end{cases} \tag{11}$$

where $f : \mathbb{R}^n \times A \times B \to \mathbb{R}^n$ is continuous, and the two associated cost functionals:

$$J^A(x,t,a,b) = \int_t^T l_A(y(s), a(s), b(s))\, ds + g_A(y(T)),$$

$$J^B(x,t,a,b) = \int_t^T l_B(y(s), a(s), b(s))\, ds + g_B(y(T)).$$

Player A wants to maximize J^A, while player B wants to maximize J^B. Other possible gain functionals may be infinite horizon functionals, i.e. of type

$$J^A(x,a,b) = \int_0^{+\infty} l_A(y(s), a(s), b(s)) e^{-\delta s}\, ds,$$

or, if $\mathcal{T}$ is a target, functionals of type *minimal time*, i.e.

$$t_x = \inf\{t \,|\, y_x(t; a, b) \in \mathcal{T}\}.$$

Example 23 (pursuit-evasion)**.** In pursuit-evasion games player A runs, while player B runs after him. The associated system is

$$\begin{cases} y'_A = f_A(y, a) \\ y'_B = f_B(y, b) \end{cases}$$

where $y_A, y_B \in \mathbb{R}^{\frac{n}{2}}$, n even, and the target is $\mathcal{T} = \{(y_A, y_B) \,|\, y_A = y_B\}$.
We ask ourselves which kind of control functions we may choose. At first we try with open-lopp controls, e.g. $a(\cdot) \in L^1(I, A), b(\cdot) \in L^1(I, B)$, where I is an interval. We notice that this choice doesn't fit for zero-sum games; indeed, back in our example, let's suppose that the run takes place around a table. Then, if $J^P = \inf\{t \,|\, y_P(t) = y_E(t)\}$, we have that $J^E = -J^P = J$ and it is a zero-sum game. In this case, it is clear that it is completely irrealistic to suppose that the two players choose at the initial time their strategies for every following time. In reality the strategy is chosen moment by moment, looking at one's own position and at the opponent's position.

3.1 Verification theorems

Given $T > 0$, with u_A, u_B we'll denote the feedback controls

$$u_A \colon [0, T] \times \mathbb{R}^n \to A, \quad u_B \colon [0, T] \times \mathbb{R}^n \to B,$$

whose associated dynamics is $f(y(s), u_A(s, y(s)), u_B(s, y(s)))$.

Definition 11. A measurable couple (u_A, u_B) is admissible if for all $x \in \mathbb{R}^n$ and for all $t \in [0, T]$ there exists an unique solution defined in $[t, T]$ to the system:

$$\begin{cases} y'(s) = f(y(s), u_A(s, y(s)), u_B(s, y(s))) \\ y(t) = x \end{cases}$$

Definition 12. An admissible couple (u_A^*, u_B^*) is a Nash equilibrium in feedback strategies for initial time t_0 and initial position x_0 if

- u_A^* is optimal for the first player, i.e. it maximizes $J^A(x_0, t_0, u_A, u_B^*)$ among feedbacks u_A such that (u_A, u_B^*) is admissible, and
- u_B^* is optimal for the first player, i.e. it maximizes $J^B(x_0, t_0, u_A^*, u_B)$among feedbacks u_B such that (u_A^*, u_B) is ammissible.

Let us now state the following hypothesis:
Hypothesis 2: There exists a couple of continuous functions $(u_1^\sharp, u_2^\sharp) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to A \times B$ such that $\forall\, x \in \mathbb{R}^n, \forall\, p_1, p_2 \in \mathbb{R}^n$, $(u_1^\sharp, u_2^\sharp)(x, p_1, p_2)$ is a Nash equilibrium of the static game with gain functions

$$\Phi^A(a, b; x, p_1) = p_1 \cdot f(x, a, b) + l_A(x, a, b), \qquad \Phi^B(a, b; x, p_2) = p_2 \cdot f(x, a, b) + l_B(x, a, b),$$

namely,

$$u_1^\sharp(x, p_1, p_2) \in \operatorname{argmax}_{a \in A} \Phi^A(a, u_2^\sharp; x, p_1), \quad u_2^\sharp(x, p_1, p_2) \in \operatorname{argmax}_{b \in B} \Phi^B(u_1^\sharp, b; x, p_2). \tag{12}$$

Sufficient conditions for the existence of at least one equilibrium $(u_1^\sharp, u_2^\sharp)$ are given by Nash's theorem, while the possibility to choose it in a continuous way should be checked case by case. The following lemma gives some more explicit conditions. We will use it in our main application, namely linear-quadratic games.

Lemma 2. *Let's suppose that the dynamic and the current gain functions have the decoupled form*

$$f(x, a, b) = f_0(x) + B_1(x)a + B_2(x)b,$$

with $B_1(x) \in \mathcal{M}_{n \times m_1}$*,* $B_2(x) \in \mathcal{M}_{n \times m_2}$*,*

$$l_j(x, a, b) = l_{j1}(x, a) + l_{j2}(x, b), \qquad j = A, B,$$

and that

1. $A \subseteq \mathbb{R}^{m1}, B \subseteq \mathbb{R}^{m2}$ *are closed (possibly unbounded),*

2. either A is compact or $\lim_{|a|\to\infty} l_{A1}(x,a)/|a| = -\infty$,

3. either B is compact or $\lim_{|b|\to\infty} l_{B2}(x,b)/|b| = -\infty$.

Then there exists $(u_1^\sharp, u_2^\sharp)$ with property (12). If moreover A and B are convex and the functions $a \mapsto l_{A1}(x,a)$ and $b \mapsto l_{B2}(x,b)$ are strictly concave, then the Nash equilibrium is unique.

Proof. By the decoupling hypothesis, for every x, p_1, p_2, $(u_1^\sharp, u_2^\sharp)$ are determined by

$$u_1^\sharp \in \operatorname{argmax}_{a\in A} \{p_1 \cdot B_1(x)a + l_{A1}(x,a)\}, \qquad u_2^\sharp \in \operatorname{argmax}_{b\in B} \{p_2 \cdot B_2(x)b + l_{B2}(x,b)\}.$$

Then the existence of $u_1^\sharp$ and $u_2^\sharp$ comes from Weierstrass' theorem, while uniqueness follows immediately from the strict concavity hypothesis. □

Theorem 15 (Verification theorem for Nash equilibria)**.** *Besides Hypothesis 2 let's suppose there exist $W_1, W_2 \in C^1(\mathbb{R}^n \times (t_0,T))$ continuous in t_0 and in T solving the system*

$$\begin{cases} \partial_t W_1 + DW_1 \cdot f(x, u_1^\sharp, u_2^\sharp) + l_A(x, u_1^\sharp, u_2^\sharp) = 0 \\ \partial_t W_2 + DW_2 \cdot f(x, u_1^\sharp, u_2^\sharp) + l_B(x, u_1^\sharp, u_2^\sharp) = 0 \\ u_1^\sharp = u_1^\sharp(x, DW_1, DW_2), \quad u_2^\sharp = u_2^\sharp(x, DW_1, DW_2) \\ W_1(x,T) = g_A(x) \\ W_2(x,T) = g_B(x), \end{cases} \tag{SHJ}$$

and let's suppose that the feedbacks couple

$$\Big(u_1^\sharp(x, DW_1(x,t), DW_2(x,t)), u_2^\sharp(x, DW_1(x,t), DW_2(x,t))\Big)$$

is admissible. Then such couple is a Nash equilibrium for t_0 and for all $x_0 \in \mathbb{R}^n$.

Notice that the first equation in (SHJ) may be rewritten, thanks to the definition of $u_1^\sharp$,

$$\partial_t W_1 + \max_{a\in A}\{DW_1 \cdot f(x, a, u_2^\sharp) + l_A(x, a, u_2^\sharp)\} = 0, \tag{13}$$

and in the same way for the second equation, referring to the definition of $u_2^\sharp$,

$$\partial_t W_2 + \max_{b\in B}\{DW_2 \cdot f(x, u_1^\sharp, b) + l_B(x, u_1^\sharp, b)\} = 0. \tag{14}$$

Hence (SHJ) is a system of two H-J-B equations coupled via $u_1^\sharp(x, DW_1, DW_2)$ and $u_2^\sharp(x, DW_1, DW_2)$.

The theorem we just stated is a direct consequence of the verification theorem for non-autonomous equations (i.e. with time-depending data) which is an easy variation of the previously presented verification theorems (the proof is left as an exercise for the reader).

Theorem 16 (Verification theorem for non-autonomous HJB equations)**.** *Let $\tilde f : [t_0,T] \times \mathbb{R}^n \times A \to \mathbb{R}^n$ and $l : [t_0,T] \times \mathbb{R}^n \times A \to \mathbb{R}$ be measurable in t and continuous in (x,a) and let $W \in C^1(\mathbb{R}^n \times (t_0,T))$ be continuous in t_0 and in T, solving*

$$\begin{cases} \partial_t W + \max_{a\in A}\{DW \cdot \tilde f(t,x,a) + l(t,x,a)\} = 0 \\ W(x,T) = g(x). \end{cases} \tag{15}$$

If α^* is admissible and such that, defining $y^*(s) := y_x(s; t, \alpha^*)$, we have

$$(W_t + DW \cdot f + l)|_{(s, y^*(s), \alpha^*(s))} = 0,$$

then α^* is optimal for the cost functional $J(x,t,\alpha) := \int_t^T l(s, y(s), \alpha(s))ds + g(y(T))$ on the trajectories of $\dot y(s) = \tilde f(s, y(s), \alpha(s))$ with initial conditions $y(t) = x$. Moreover $W(x,t) = J(x,t,\alpha^*)$.

Proof ot the theorem 15. By (13), W_1 satisfies (15) with

$$\tilde{f}(t,x,a) := f\left(x, a, u_2^\sharp\left(x, DW_1(x,t), DW_2(x,t)\right)\right),$$

$$l(t,x,a) := l_A\left(x, a, u_2^\sharp(x, DW_1(x,t), DW_2(x,t))\right),$$

which are continuous, and moreover the max in (15) is reached for $a = u_1^\sharp(x, DW_1(x,t), DW_2(x,t))$. Then Theorem 16 implies that $u_1^\sharp(x, DW_1(x,t), DW_2(x,t))$ is optimal for the first player if the second player uses $u_2^\sharp$. A symmetric argument about equation (14) completes the proof that the couple $(u_1^\sharp, u_2^\sharp)$ is a Nash equilibrium. □

Let us look now at some application of the theorem.

3.1.1 LQ differential Games

Consider the equation

$$\dot{y} = Ay + B_1 a + B_2 b,$$

where $A \in \mathcal{M}_{n\times m}$, $B_1 \in \mathcal{M}_{n\times m_1}$, $B_2 \in \mathcal{M}_{n\times m_2}$, and the controls a, b are respectively in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$. The cost functionals are

$$J^A(x,t,\alpha,\beta) = -\int_t^T \left(y(s)^T \frac{M_1}{2} y(s) + \frac{|\alpha(s)|^2}{2}\right) ds + y(T)^T \frac{Q_1}{2} y(T),$$
$$J^B(x,t,\alpha,\beta) = -\int_t^T \left(y(s)^T \frac{M_2}{2} y(s) + \frac{|\beta(s)|^2}{2}\right) ds + y(T)^T \frac{Q_2}{2} y(T),$$

with $M_1, M_2, Q_1, Q_2 \in Sym(n)$.

Let's check Hypothesis 2 holds: consider the static game whose cost functionals are given by

$$\Phi^A(a,b) = p_1 \cdot (Ax + B_1 a + B_2 b) - x^T \frac{M_1}{2} x - \frac{|a|^2}{2},$$
$$\Phi^B(a,b) = p_2 \cdot (Ax + B_1 a + B_2 b) - x^T \frac{M_2}{2} x - \frac{|b|^2}{2}.$$

We are in the same hypotheses of Lemma 2 and it may be easily computed that the maximum of $\Phi^A(\cdot, b)$ is reached only by $a = B_1^T p_1$, while the maximum of $\Phi^B(a, \cdot)$ is reached only by $b = B_2^T p_2$. Hence the only possible Nash equilibrium is $(B_1^T p_1, B_2^T p_2)$, and

$$u_1^\sharp(p_1) = B_1^T p_1, \qquad u_2^\sharp(p_2) = B_2^T p_2$$

are continuous, so Hypothesis 2 holds. The system becomes

$$\begin{cases} \partial_t W_1 + DW_1 \cdot (Ax + B_1 B_1^T DW_1 + B_2 B_2^T DW_2) = x^T \frac{M_1}{2} x + \frac{|B_1^T DW_1|^2}{2} \\ \partial_t W_2 + DW_2 \cdot (Ax + B_1 B_1^T DW_1 + B_2 B_2^T DW_2) = x^T \frac{M_2}{2} x + \frac{|B_2^T DW_2|^2}{2} \\ W_1(x,T) = \frac{x^T Q_1 x}{2}, \quad W_2(x,T) = \frac{x^T Q_2 x}{2} \end{cases} \tag{16}$$

As we did for optimal control, let's state the following *ansatz*: let's look for solutions of the problem of the form

$$W_i(x,t) = \frac{x^T K_i(t) x}{2}, \quad i \in \{1,2\}$$

with K_i symmetric matrix $n \times n$. Let's substitute in (16) and set

$$S_i := B_i B_i^T, \quad i = 1, 2,$$

noticing that they are symmetric matrices; moreover in each quadratic form which appears in the expression we write the matrix in symmetrized form, namely

$$K_iS_jK_j = \frac{K_iS_jK_j + K_j^T S_j K_i}{2}, i \neq j.$$

So we get an equality bewtween two quadratic forms which holds for all x if the following system of Riccati ordinary matricial equations is verified

$$\begin{cases} \dot{K_1} = M_1 - K_1S_1K_1 - K_1(A + S_2K_2) - (A + S_2K_2)^T K_1 \\ \dot{K_2} = M_2 - K_2S_2K_2 - K_2(A + S_1K_1) - (A + S_1K_1)^T K_2 \\ K_1(T) = Q_1, \quad K_2(T) = Q_2. \end{cases} \tag{17}$$

Theorem 17. *If Riccati system (17) has solutions $K_1(\cdot), K_2(\cdot) \in C^1((t_0, T), Sym(n))$, K_i continuous in T, then $W_i(x,t) = x^T K_i(t)x$ are solutions of the system (16) in $(t_0, T] \times \mathbb{R}^n$, hence thet are the value corresponging to a feedback Nash equilibrium given by*

$$u_1^*(x,t) = u_1^\sharp(DW_1(x,t)) = B_1^T K_1(t)x$$
$$u_2^*(x,t) = u_2^\sharp(DW_2(x,t)) = B_2^T K_2(t)x$$

Moreover, there exists $t_0 < T$ such that (17) has solution in $C^1([t_0, T], Sym(n))$ and such a solution is unique.

Proof. The first statement comes from verification theorem and previous calculations.

Existence and uniqueness for a local solution for the system (17) comes from the general theory of ordinary differential equations thanks to the local Lipschitz continuity of the right hand side of (17). The fact that the matricesK_i are symmetric can be shown as in Proposition 1, namely checking that the couple $K_1(\cdot)^T, K_2(\cdot)^T$ is a solution of (17), thanks to the symmetry of M_i and Q_i, and then using the uniqueness of the solution to get $K_i = K_i^T$. □

3.1.2 Zero-sum LQ differential games

Let's consider the special case of 0-sum differential games, where $l_B = -l_A = -l$, $g_B = -g_A = -g$, $p_1 = -p_2 = p$. Hypothesis 2 can be simplified and replaced by the following.

Hypothesis 2′: There exists a couple of continuous functions $(u_1^\sharp, u_2^\sharp) : \mathbb{R}^n \times \mathbb{R}^n \to A \times B$ such that $\forall\, x \in \mathbb{R}^n, \forall p \in \mathbb{R}^n$, $(u_1^\sharp, u_2^\sharp)(x,p)$ is a saddle point of the 0-sum static game with gain-cost functional $\Phi(a, b; x, p) = p \cdot f(x, a, b) + l(x, a, b)$, i.e.

$$u_1^\sharp(x,p) \in \operatorname{argmax}_{a \in A} \Phi(a, u_2^\sharp; x, p), \quad u_2^\sharp(x,p) \in \operatorname{argmin}_{b \in B} \Phi(u_1^\sharp, b; x, p).$$

Definition 13. (u_1^*, u_2^*) admissible is a saddle point of the 0-sum differential game in feedback strategies if

$$J(t, x, u_1, u_2^*) \leq J(x, t, u_1^*, u_2^*) \leq J(x, t, u_1^*, u_2),$$

for every u_1 feedback such that (u_1, u_2^*) is admissible, for every u_2 feedback such that (u_1^*, u_2) is admissible.

Corollary 4. *Let's suppose Hypothesis 2' holds and that $W \in C^1$ solves*

$$\begin{cases} \partial_t W + DW \cdot f(x, u_1^\sharp, u_2^\sharp) + l(x, u_1^\sharp, u_2^\sharp) = 0 \\ u_i^\sharp = u_i^\sharp(x, DW) \\ W(x,T) = g(x) \end{cases} \tag{18}$$

If $(u_1^*, u_2^*) = (u_1^\sharp(x, DW(x,t)), u_2^\sharp(x, DW(x,t)))$ is admissible, then it is a saddle point of the 0-sum differential game in feedback strategies, and W is the value of the game, i.e., $W(x,t) = J(x, t, u_1^*, u_2^*)$.

Proof. Note that $W_1 = W$ e $W_2 = -W$ satisfy the first two equations of the system (SHJ). By Theorem 15, the control (u_1^*, u_2^*) is a Nash equilibrium for the differential game in feedback strategies. Then it is also a saddle point. By Theorem 16 we get also $W(x,t) = J(x,t,u_1^*,u_2^*)$. $\square$

Remark 9. If the static game of Hypothesis $2'$ has a saddle point, then it has value:

$$\Phi(u_1^\sharp, u_2^\sharp; x, p) = \min_b \max_a \Phi(a, b; x, p) = \max_a \min_b \Phi(a, b; x, p).$$

Such request is called *Isaacs condition*; it is weaker than Hypothesis $2'$ since it doesn't require the continuity of the saddle point with respect to (x,p). In this case (18) becomes

$$\partial_t W + \min_{b\in B} \max_{a\in A} \{DW \cdot f(x,a,b) + l(x,a,b)\} = 0,$$

or also

$$\partial_t W + \max_{a\in A} \min_{b\in B} \{DW \cdot f(x,a,b) + l(x,a,b)\} = 0,$$

which go by the name of *Isaacs equations*.

Application 1 (0-sum LQ games)**.** Consider the linear system

$$\dot{y} = Ay + B_1 a + B_2 b,$$

where $A \in \mathcal{M}_{n\times m}$, $B_1 \in \mathcal{M}_{n\times m_1}$, $B_2 \in \mathcal{M}_{n\times m_2}$, and the controls a, b are respectively in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$. Let the functional J, which represents the gain of the first player (hence the cost for the second player) be given by:

$$J = -\int_t^T \left(y^T \frac{M}{2} y + \frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} \right) ds + y^T \frac{Q}{2} y.$$

We get

$$u_1^\sharp = B_1^T p, \quad u_2^\sharp = -B_2^T p,$$

and looking for solutions of the form $W(x,t) = \frac{1}{2} x^T K(t) x$, we get a single ordinary matrix Riccati equation

$$\begin{cases} \dot{K} = M - KA - A^T K + K(B_2 B_2^T - B_1 B_1^T) K\,, \\ K(T) = Q\,. \end{cases} \tag{19}$$

Besides the existence for small times of the feedback Nash equilibrium, which comes from Theorem 17, under additional hypotheses we may prove the existence also for larger times.

Corollary 5. *If $Q \geq 0$, $M \leq 0$, and $B_2 B_2^T - B_1 B_1^T \geq 0$, then there exists a unique solution of* (19) *in* $(-\infty, T]$.

Proof. Since $B_2 B_2^T - B_1 B_1^T \geq 0$, by a well-know fact of linear algebra there exists a matrix B such that $BB^T = B_2 B_2^T - B_1 B_1^T$. Hence the ordinary matrix Riccati equation has the same form of the optimal control one, and by the hypotheses of semi-definiteness of Q and M we can use Theorem 4 to get the existence of a global solution $K \in C^1((-\infty, T), Sym(n))$. $\square$

3.1.3 An example: advertising in a duopoly

Let y_1, y_2 be the market percentages of the two firms, with $y_1 + y_2 = 1$. The controls α_i are the amount of money invested in advertising ($\alpha_2 = \beta$). The dynamics, called Lanchester dynamics, is

$$\dot{y}_i = \alpha_i (1 - y_i) - \alpha_j y_i, \quad i \neq j,\ \alpha_i \geq 0.$$

If the unitary cost of the advertising is equal for the two firms, the gain may be modeled with

$$J^i(x, t, \alpha_1, \alpha_2) = \int_t^T \left(r_i y_i - \frac{\alpha_i^2}{2} \right) ds + R_i y_i(T),$$

for suitable constants $r_i > 0$, $R_i \geq 0$. We want to phrase the problem as a 0-sum game. Using the equality $y_2 = 1 - y_1$ we get

$$J^1 - J^2 = \int_t^T \left((r_1 + r_2)y(s) - \frac{\alpha^2}{2} + \frac{\beta^2}{2} \right) ds + (R_1 + R_2)y(T) - r_2(T-t) - R_2.$$

Define $r = r_1 + r_2$ and $R = R_1 + R_2$; the 0-sum game with cost functional

$$J(x,t,\alpha,\beta) = \int_t^T \left(ry - \frac{\alpha^2}{2} + \frac{\beta^2}{2} \right) ds + Ry(T).$$

is equivalent to the previous game. Set $f(x,a,b) = (1-x)a - xb$, $l(x,a,b) = rx - \frac{a^2}{2} + \frac{b^2}{2}$, $g(x) = Rx$. Isaacs equation is

$$W_t + W_x f(x, u_1^\sharp, u_2^\sharp) + l(x, u_1^\sharp, u_2^\sharp) = 0,$$

where $(u_1^\sharp, u_2^\sharp)$ is a saddle point for the static game $\Phi(a,b,x,p) = pf(x,a,b) + l(x,a,b)$. We easily get that $u_1^\sharp(x,p) = p(1-x)$ if $p \geq 0$, 0 if $p < 0$, and that $u_2^\sharp(x,p) = px$ if $p \geq 0$, 0 if $p < 0$. Hence, if $W_x \geq 0$ the equation becomes

$$\begin{cases} W_t + \frac{W_x^2}{2}(1-2x) + rx = 0 \\ W(T,x) = Rx \end{cases}$$

Let's state the following *ansatz*: we look for solutions of the form $W(t,x) = c(t) + K(t)x$, with K and c of class C^1, $K \geq 0$. Hence we get the equation

$$\dot{c} + \dot{K}x + \frac{1-2x}{2}K^2 + rx = 0, \forall x \in \mathbb{R},$$

which leads to the system

$$\begin{cases} \dot{K} - K^2 + r = 0 \\ \dot{c} + \frac{K^2}{2} = 0 \\ K(T) = R \\ c(T) = 0 \end{cases}$$

We immediatly get that $c(t) = \int_t^T K^2(s)/2\, ds$. The equation for K may be solved explicitly, for example with variables separation, and we see that the solution exists for all $t \leq T$ and moreover $K(t) \geq 0$. This may also be shown by a simple qualitative analysis. Indeed $\sqrt{r}$ and $-\sqrt{r}$ are stationary solutions (equilibria) of the equation, hence if $R > \sqrt{r}$ also $K(t) > \sqrt{r}$ for all t, and if $0 \leq R < \sqrt{r}$ we get $-\sqrt{r} < K(t) < \sqrt{r}$. Moreover, in the firts case $\dot{K}(t) > 0$, hence the solution exists unique in $(-\infty, T)$ and it is increasing. In the second case $\dot{K}(t) < 0$, hence K is decreasing and so $K(t) \geq R \geq 0$, and then also in this case the solution exists in $(-\infty, T)$. In conclusion, there exists a unique solution in this form, $W(t,x) = c(t) + K(t)x$, for every $R \geq 0$, with K and c of class C^1 and defined for every $t \leq T$.

Moreover, the feedbacks $u_1^*(x,t) = u_1^\sharp(x, W_x(x,t)) = (1-x)K(t)$ and $u_2^*(x,t) = u_2^\sharp(x, W_x(x,t)) = xK(t)$ are the saddle point of the differential game.

3.2 Zero-sum differential games

3.2.1 Value functions and Dynamic Programming

In this section we want to give a definition of value of a 0-sum differential game such that it satisfies a suitable equation of Hamilton-Jacobi type. First we need some definitions.

Definition 14. Let $\mathcal{A}_t = \{a\colon [t,T] \to A \text{ measurable}\}$, and $\mathcal{B}_t = \{b\colon [t,T] \to B \text{ measurable}\}$. A strategy for the 1st player is a map $\alpha : \mathcal{B}_t \to \mathcal{A}_t$; it is non-anticipating if for every $t \leq s \leq T$, and for every $b, \hat{b} \in \mathcal{B}_t$ such that $b(u) = \hat{b}(u)$ q.o. $u \leq s$, we have that $\alpha[b](u) = \alpha[\hat{b}](u)$, q.o. $u \leq s$.

Let $\Gamma_t := \{\alpha : \text{ non-anticipating strategy of A}\}$ and $\Delta_t := \{\beta : \text{ non-anticipating strategy of B}\}$. Note that $\forall b \in \mathcal{B}_t$, $\forall \alpha \in \Gamma_t$ there exists a unique solution of

$$\begin{cases} \dot{y}(s) = f(y(s), \alpha[b](s), b(s)), \quad t < s \\ y(t) = x \end{cases} \tag{20}$$

Recall that the gain of player A and the cost of player B are given by the single functional

$$J(t, x, a, b) = \int_t^T l(y(s), a(s), b(s))\, ds + g(y(T)).$$

Definition 15. The lower value of the game is

$$V(t, x) = \inf_{\beta \in \Delta_t} \sup_{a \in \mathcal{A}} J(t, x, a, \beta[a]);$$

The upper value of the game is

$$U(t, x) = \sup_{\alpha \in \Gamma_t} \inf_{b \in \mathcal{B}} J(t, x, \alpha[b], b).$$

The game has a value if $V = U$.

Example 24. Examples of strategies:

- Constant strategies. Let $\overline{a} \in \mathcal{A}_t$ and set $\alpha[b] = \overline{a}$ for all $b \in \mathcal{B}_t$.
- Let $\Psi : B \to A$ be given and set $\alpha[b] = \Psi \circ b$. If $\Psi(b(\cdot))$ is measurable for any $b \in \mathcal{B}_t$, then $\alpha \in \Gamma_t$.
- Feedbacks: let $\Phi : \mathbb{R}^n \times [t, T] \to A$ be such that for all $b \in \mathcal{B}_t$ there exists a unique solution of
$$\begin{cases} \dot{q}(s) = f(q(s), \Phi(q(s), s), b(s)) \\ q(t) = x \end{cases}$$
and such that $s \to \Phi(q(s), s)$ be measurable. Then set $\alpha[b](s) = \Phi(q(s), s)$.

Example 25 (Berkovitz - the lower value and the upper values can be different)**.** Consider

$$\begin{cases} \dot{y} = (a - b)^2 \\ y(0) = x \in \mathbb{R} \end{cases}$$

where $A = B = \{0, 1\}$, $l = 0$, g strictly increasing. If B wants to minimize the cost functional we expect he plays the strategy $\beta^*[a](s) = a(s)$. For such choice we get

$$V(t, x) \leq \sup_a g(y_x(T, t, a, \beta^*[a])) = g(x).$$

On the other hand we expect that A plays the strategy

$$\alpha^*[b](s) = \begin{cases} 1 & \text{if } b(s) = 0 \\ 0 & \text{if } b(s) = 1 \end{cases}$$

For such choice we have $\dot{y}(s) = (\alpha^*[b]s) - bs))^2 = 1$ for all $s \geq t$, and therefore $y(s) = x + s - t$ for all $b \in \mathcal{B}_t$. Then

$$U(t, x) \geq g(x + T - t) > g(x) \geq V(t, x).$$

Remark 10. From the example we understand that there is an information advantage for the player choosing non-anticipating strategies. However, any more realistic notion of value lies between V and U, and so, if $V = U$ all reasonable notions of value coincide because in such a case the information advantage is irrelevant.

Here are the standing assumptions in this section. We suppose that A, B are compact and

- $f\colon \mathbb{R}^n \times A \times B \to \mathbb{R}^n$ continuous, bounded, and globally Lipschitz in x, uniformly in (a, b), i.e., there exists $C_1 > 0$ such that $|f| \le C_1$ and $|f(x, a, b) - f(\hat{x}, a, b)| \le C_1|x - \hat{x}|$, for all $x, \tilde{x}$ and all $(a, b) \in A \times B$;
- $l\colon \mathbb{R}^n \times A \times B \to \mathbb{R}^n$ continuous, bounded, and globally Lipschitz in x, uniformly in (a, b), i.e., there exists $C_2 > 0$ tsuch that $|l| \le C_2$ and $|l(x, a, b) - l(\hat{x}, a, b)| \le C_2|x - \hat{x}|$, for all $x, \tilde{x}$ and all $(a, b) \in A \times B$;
- $g\colon \mathbb{R}^n \to \mathbb{R}$ bounded and globally Lipschitz , i.e., there exists $C_3 > 0$ such that $|g| \le C_3$, $|g(x) - g(\hat{x})| \le C_3|x - \hat{x}|$, for all $x, \hat{x}$.

Theorem 18 (Dynamic Programming Principle)**.** *Under the previous assumptions we have the following:* $\forall\, 0 \le t \le t + \sigma \le T$, $\forall x \in \mathbb{R}^n$

$$V(t, x) = \inf_{\beta \in \Delta_t} \sup_{a \in \mathcal{A}_t} \left\{ \int_t^{t+\sigma} l(y_x, a, \beta[a])\, ds + V(t + \sigma, y_x(t + \sigma)) \right\}$$

where $y_x(s)$ *solves* $\dot{y} = f(y, a, \beta[a])$, *for* $t < s$, $y(t)=x$,

$$U(t, x) = \sup_{\alpha \in \Gamma_t} \inf_{b \in \mathcal{B}_t} \left\{ \int_t^{t+\sigma} l(y_x, \alpha[b], b)\, ds + U(t + \sigma, y_x(t + \sigma)) \right\}$$

where $y_x(s)$ *solves* (20).

Proof. For simplicity we suppose $l = 0$; we know from the previous study of the Bolza problem that the case $l \ne 0$ can be reduced to a Mayer problem, i.e., with $l = 0$. We define

$$W(t, x) := \inf_{\beta \in \Delta_t} \sup_{a \in \mathcal{A}_t} V(t + \sigma, y_x(t + \sigma))$$

where y_x is the solution of the system given the controls β e a. We want to prove the inequalities $V(t, x) \le W(t, x) \le V(t, x)$.

For the first inequality we fix $\varepsilon > 0$. Then, by definition of inf there exists $\delta \in \Delta_t$ such that

$$W(t, x) \ge \sup_{a \in \mathcal{A}_t} V(t + \sigma, y_x(t + \sigma)) - \varepsilon.$$

Moreover, for any $z \in \mathbb{R}^n$ consider the lower value starting from z at the initial time $t + \sigma$:

$$V(t + \sigma, z) = \inf_{\beta \in \Delta_{t+\sigma}} \sup_{a \in \mathcal{A}_{t+\sigma}} g(y_z(T))$$

with y_z solution of

$$\begin{cases} \dot{y} = f(y, a, \beta[a]) \\ y(t + \sigma) = z \end{cases}$$

Then there is a $\delta_z \in \Delta_{t+\sigma}$ such that

$$V(t + \sigma, z) \ge \sup_{a \in \mathcal{A}_{t+\sigma}} g(y_z(T)) - \varepsilon.$$

We define a strategy for the 2nd player based on the choices above:

$$\bar{\beta}[a](s) = \begin{cases} \delta[a](s) & t \le s \le t + \sigma \\ \delta_z[a](s) & t + \sigma \le s \le T, \quad z = y_x(t + \sigma; t, a, \delta[a]) \end{cases}$$

Note that $\bar{\beta} \in \Delta_t$ and $y_z(T; t + \sigma, a, \delta_z[a]) = y_x(T; t, a, \bar{\beta}[a])$. Then

$$W(t, x) \ge \sup_{a \in \mathcal{A}_t} g(y_x(T; t, a, \beta[a])) - 2\varepsilon \ge V(t, x) - 2\varepsilon.$$

Letting ε tend to 0 we get the 1st inequality.
[Non in programma dal 2019]

Vediamo ora come ottenere la seconda. Fissiamo ancora una volta $\varepsilon > 0$: esiste $\beta_1 \in \Delta_t$ tale che

$$V(t,x) \geq \sup_{a\in\mathcal{A}_t} g(y_x(T)) - \varepsilon$$

Per definizione $W(t,x) \leq \sup_{a\in\mathcal{A}_t} V(t+\sigma, y_x(t+\sigma))$. Quindi, esiste $a_1 \in \mathcal{A}_t$ tale che

$$W(t,x) \leq V(t+\sigma, y_x^1(t+\sigma)) + \varepsilon$$

dove y_x^1 è la soluzione del sistema con controllo a_1. Come fatto in precedenza definiamo una strategia, questa volta per il primo giocatore, a partire dalla strategia a. Per ogni $a \in \mathcal{A}_{t+\sigma}$ definisco un controllo $\tilde{a} \in \mathcal{A}_t$ con

$$\tilde{a}(s) = \begin{cases} a_1(s) & t \leq s \leq t+\sigma \\ a(s) & t+\sigma \leq s \leq T \end{cases}$$

e definisco $\beta_2 \in \Delta_{t+\sigma}$ come $\beta_2[a](s) = \beta_1[\tilde{a}](s)$, per $s \in [t+\sigma, T]$. Allora,

$$V(t+\sigma, z) \leq \sup_{a\in\mathcal{A}_t} g(y_z(T; t+\sigma, a, \beta_2[a]))$$

quindi esiste $a_2 \in \mathcal{A}_{t+\sigma}$ tale che

$$V(t+\sigma, z) \leq g(y_z(T; t+\sigma, a_2, \beta_2[a_2])) + \varepsilon.$$

Definiamo

$$a_3(s) = \begin{cases} a_1(s) & t \leq s \leq t+\sigma \\ a_2(s) & t+\sigma \leq s \leq T \end{cases}$$

Notiamo che per come abbiamo costruito i controlli, $y_z(T; t+\sigma, a_2, \beta_2[a_2]) = y_x(T; t, a_3, \beta_1[a_3])$. Mettendo assieme le disuguaglianze precedenti si ottiene che

$$\begin{aligned} W(t,x) \leq V(t+\sigma, y_x^1(t+\sigma)) + \varepsilon \leq g(y_z(T; t+\sigma, a_2, \beta_2[a_])) + 2\varepsilon = \\ = g(y_x(T; t, a_3, \beta_1[a_3])) \leq \sup_{a\in\mathcal{A}_t} g(y_x(T)) \leq V(t,x) + 3\varepsilon \end{aligned} \tag{21}$$

Facendo tendere ε a 0 si ottiene la seconda disuguaglianza, da cui $W = V$.

The second statement of the theorem, concerning U, can be obtained in a similar way by defining $\tilde{W}(t,x) = \sup_{\alpha\in\Gamma_t} \inf_{b\in\mathcal{B}_t} U(t+\sigma, y_x(t+\sigma))$ and proving the inequalities $U \leq \tilde{W} \leq U$. ☐

Theorem 19. *There exists $C_4 > 0$ dipending on T such that*

1. $|V(t,x)|, |U(t,x)| \leq C_4$, *per ogni* $0 \leq t \leq T$, *per ogni* x;
2. $|V(t_1,x_1) - V(t_2,x_2)| \leq C_4(|t_1 - t_2| + |x_1 - x_2|)$, *per ogni* $0 \leq t_1, t_2 \leq T$, *per ogni* x_1, x_2;
3. $|U(t_1,x_1) - U(t_2,x_2)| \leq C_4(|t_1 - t_2| + |x_1 - x_2|)$, *per ogni* $0 \leq t_1, t_2 \leq T$, *per ogni* x_1, x_2.

Proof. To prove *1* it is enough to note that by the assumptions on f and l we have

$$|J(t,x,a,b)| \leq C_2(T-t) + C_3 \leq C_2 T + C_3.$$

[Non in programma dal 2019]

Dimostriamo il punto 3 (la dimostrazione di 2 è analoga): siano x_1, x_2 fissati, e $0 \leq t_1 \leq t_2 \leq T$. Vogliamo stimare $U(t_1,x_1) - U(t_2,x_2)$. Esiste una strategia α_1 tale che $U(t_1,x_1) \leq \inf_{b\in\mathcal{B}_{t_1}} J(t_1, x_2, \alpha, b) + \varepsilon$. Dato $b_0 \in B$ costante e $b \in \mathcal{B}_{t_2}$, definiamo il controllo $\tilde{b} \in \mathcal{B}_{t_1}$ così:

$$\tilde{b}(s) = \begin{cases} b_0 & t_1 \leq s \leq t_2 \\ b(s) & t_2 \leq t \leq T \end{cases}$$

Sia $\underline{\alpha} \in \Gamma_{t_2}$ definita da $\underline{\alpha}[b](s) = \alpha[\tilde{b}](s)$, per $s \in [t_2, T]$. Allora vale:

$$U(t_2, x_2) \geq \inf_{b \in \mathcal{B}_{t_2}} J(t_2, x_2, \underline{\alpha}, b).$$

Quindi, esiste $b \in \mathcal{B}_{t_2}$ tale che $U(t_2, x_2) \geq J(t_2, x_2, \underline{\alpha}, b) - \varepsilon$. Confrontiamo le traiettorie $y_{x_1}(T; t_1, \alpha, \tilde{b})$ e $y_{x_2}(T; t_2, \underline{\alpha}, b)$: si ha che

$$|y_{x_1}(t_2) - x_1| \leq C|t_1 - t_2|,$$
$$|y_{x_1}(s) - y_{x_2}(s)| \leq e^{C_1 T}|y_{x_1}(t_2) - x_2|$$

per $t_2 \leq s \leq T$, dove la seconda stima si ottiene dalla prima con la disuguaglianza di Gronwall perché $b(s) = \tilde{b}(s)$ e $\underline{\alpha}[b](s) = \alpha[\tilde{b}](s)$ per $t_2 \leq s \leq T$. Sommando e sottraendo x_1 si ottiene

$$|y_{x_1}(s) - y_{x_2}(s)| \leq C_T(|t_1 - t_2| + |x_1 - x_2|).$$

Infine supponiamo per semplicità $l \equiv 0$ e osserviamo che

$$\begin{aligned} J(t_1, x_1, \alpha, \tilde{b}) - J(t_2, x_2, \underline{\alpha}, b) = g(y_{x_1}(T)) - g(y_{x_2}(T)) \leq C_3|y_{x_1}(T) - y_{x_2}(T)| \leq \\ \leq C_3 C_T(|t_1 - t_2| + |x_1 - x_2|). \end{aligned}$$

□

3.2.2 Isaacs Equations

We define the upper and lower Hamiltonians by

$$H^+(x, p) := \min_b \max_a \{p \cdot f(x, a, b) + l(x, a, b)\}$$

$$H^-(x, p) := \max_a \min_b \{p \cdot f(x, a, b) + l(x, a, b)\}.$$

Proposition 3. *The Hamiltonians $H^+, H^- : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are continuous and there exists $K > 0$ such that, for $F = H^+, H^-$, we have*

$$|F(x, p) - F(\hat{x}, p)| \leq K|x - \hat{x}|(1 + |p|) \tag{Lx}$$

$$|F(x, \hat{p}) - F(x, p)| \leq C_1|p - \hat{p}|. \tag{Lp}$$

Proof. We show the estimate for H^+. There exists $b' \in B$ such that

$$H^+(x, p) = \max_a \{p \cdot f(x, a, b') + l(x, a, b')\};$$

moreover, there exists a' such that

$$H^+(\hat{x}, p) \leq p \cdot f(x, a', b') + l(x, a', b').$$

Then

$$H^+(\hat{x}, p) - H^+(x, p) \leq |p||f(x, a', b') - f(\hat{x}, a', b')| + C_2|x - \hat{x}| \leq K|x - \hat{x}|(1 + |p|).$$

We conclude by exchanging x and $\hat{x}$.

Next we prove the continuity in p. Take b' as above; there exists $\hat{a}$ such that $H^+(x, \hat{p}) \leq \hat{p} \cdot f(x, \hat{a}, b') + l(x, \hat{a}, b')$. Moreover,

$$H^+(x, p) = \max_a \{p \cdot f(x, a, b') + l(x, a, b')\} \geq p \cdot f(x, \hat{a}, b') + l(x, \hat{a}, b'),$$

and then

$$H^+(x, \hat{p}) - H^+(x, p) \leq |f(x, \hat{a}, b')||p - \hat{p}| \leq C_1|p - \hat{p}|.$$

By exchanging p and $\hat{p}$ we get (Lp), which combined with (Lx) implies the continuity of H^+. □

Theorem 20. *Under the current assumptions the upper value U is the unique bounded and continuous viscosity solution in $[0,T]\times\mathbb{R}^n$ of*

$$\begin{cases} -(U_t + H^+(x, DU)) = 0 & \text{in }]0,T[\times\mathbb{R}^n, \\ U(T,x) = g(x), \end{cases}$$

and the lower value V is the unique bounded and continuous viscosity solution of

$$\begin{cases} -(V_t + H^-(x, DV)) = 0 & \text{in }]0,T[\times\mathbb{R}^n, \\ V(T,x) = g(x). \end{cases}$$

To prove Theorem 20 we need the following technical lemma.

Lemma 3. *Suppose $\Lambda \in C(A\times B)$. Then $\forall\,\sigma > 0, \forall\, t_0 \in \mathbb{R}$*

$$\max_{a\in A}\min_{b\in B}\Lambda(a,b) = \inf_{\beta\in\Delta_{t_0}}\sup_{a\in\mathcal{A}_{t_0}}\frac{1}{\sigma}\int_{t_0}^{t_0+\sigma}\Lambda(a(s),\beta[a](s))\,ds.$$

Proof. Call λ the left hand side of the equation, θ the right hand side.We show first that $\lambda \le \theta$. Take a^* such that $\lambda = \min_{b\in B}\Lambda(a^*,b)$. Then

$$\theta \ge \inf_{\beta\in\Delta_{t_0}}\frac{1}{\sigma}\int_{t_0}^{t_0+\sigma}\Lambda(a^*,\beta[a](s))\,ds \ge \lambda,$$

because the integrand is $\ge \lambda$ at each point.

Next we prove the other inequality. First we show that, for all $\varepsilon > 0$, there exists $\varphi\colon A \to B$ measurable such that

$$\Lambda(a,\varphi(a)) \le \min_{b\in B}\Lambda(a,b) + \varepsilon.$$

For all a there is $b(a)$ such that $\Lambda(a,b(a)) = \min_b \Lambda(a,b) := \lambda(a)$, but we do not know if the map b is measurable. By continuity, for all a there is $r > 0$ such that

$$\Lambda(a',b(a)) \le \lambda(a') + \varepsilon, \qquad \forall\, a' \in B(a,r).$$

By compactness there exist $a_1,\dots,a_n$ and $r_1,\dots,r_n$ such that $A \subseteq \bigcup_{i=1}^n B(a_i,r_i)$, and setting $b_i := b(a_i)$ we get

$$\Lambda(a',b_i) \le \lambda(a') + \varepsilon,$$

for all $a' \in B(a_i,r_i)$. Then we can define $\varphi(a) = b_i$ if $a \in B(a_i,r_i)$ and $a \notin B(a_k,r_k)$, $k < i$. This function is measurable and it satisfies the desired inequality.

Now we define the strategy $\beta^* \in \Delta_{t_0}$ as $\beta^*[a](s) = \varphi(a(s))$. Then

$$\begin{aligned}\theta &\le \sup_{a\in\mathcal{A}_{t_0}}\frac{1}{\sigma}\int_{t_0}^{t_0+\sigma}\Lambda(a(s),\beta^*[a](s))\,ds \\ &\le \sup_{a\in\mathcal{A}_{t_0}}\frac{1}{\sigma}\int_{t_0}^{t_0+\sigma}\min_b\Lambda(a(s),b)\,ds + \varepsilon \le \lambda + \varepsilon.\end{aligned}$$

□

Proof of Theorem 20. We will show only the statement about V, the other is analogous.
1. By definition $V(T,x) = g(x)$.
2. We prove that V is a subsolution. Let (t_0,x_0) be a local maximum point of $V - \varphi$, with $\varphi \in C^1$. Assume by contradiction thta $-(\varphi_t(t_0,x_0) + H^-(x_0, D\varphi(t_0,x_0))) > \theta > 0$, i.e.,

$$\varphi_t(t_0,x_0) + H^-(x_0, D\varphi(t_0,x_0)) < -\theta < 0. \tag{22}$$

For simplicity we assume $l = 0$, and then

$$H^-(x, p) = \max_a \min_b p \cdot f(x, a, b).$$

Set $\Lambda(a, b) := f(x_0, a, b) \cdot D\varphi(t_0, x_0)$. Then, by the previous Lemma,

$$H^-(x_0, D\varphi(t_0, x_0)) = \inf_{\beta \in \Delta_{t_0}} \sup_{a \in \mathcal{A}_{t_0}} \frac{1}{\sigma} \int_{t_0}^{t_0+\sigma} f(x_0, a(s), \beta[a](s)) \cdot D\varphi(t_0, x_0)\, ds.$$

Set $y(s) = y_{x_0}(s; t_0, a, \beta[a]))$. For σ sufficiently small, from (22), the estimate $|y(s) - x_0| \le C_1 s$ for all $a \in \mathcal{A}_{t_0}, \beta \in \Delta_{t_0}$, and the continuity of $f, \varphi_t, D\varphi$, we get

$$\inf_{\beta \in \Delta_{t_0}} \sup_{a \in \mathcal{A}_{t_0}} \frac{1}{\sigma} \int_{t_0}^{t_0+\sigma} [\varphi_t(s, y(s)) + f(y(s), a(s), \beta[a](s)) \cdot D\varphi(s, y(s))]\, ds \le -\frac{\theta}{2}.$$

We can integrate and obtain

$$\inf_{\beta \in \Delta_{t_0}} \sup_{a \in \mathcal{A}_{t_0}} [\varphi(t_0 + \sigma, y(t_0 + \sigma) - \varphi(t_0, x_0)] \le -\frac{\sigma\theta}{2}.$$

Recalling that (t_0, x_0) is a local maximum point of $V - \varphi$, we arrive at

$$\inf_{\beta \in \Delta_{t_0}} \sup_{a \in \mathcal{A}_{t_0}} [V(t_0 + \sigma, y(t_0 + \sigma)) - V(t_0, x_0)] \le -\frac{\sigma\theta}{2},$$

a contradiction with the Dynamic Programming Principle, which says that the right hand side is 0.
3. The proof that V is a supersolution follows a similar argument: the reader is invited to work out the details.
4. The uniqueness of U and V as solutions in $BUC([0, T] \times \mathbb{R}^n$ of the terminal value problem for the corresponding Isaacs equation follows from the Comparison Principle for viscosity solutions, which holds by the properties (Lx) and (Lp) of the Hamiltonians: see, e.g., Theorem 1, sect. 10.2, p. 547 of [E]. □

The following corollary of the previous theorem guarantees the existence of the value the Isaacs conditions is satisfied.

Corollary 6. *1. $V(t, x) \le U(t, x)$, for all $(t, x) \in [0, T] \times \mathbb{R}^n$;*

2. if $H^+(x, p) = H^-(x, p)$ for all (x, p), then $U = V$.

Proof. It is an immediate consequence of the inequality $H^+ \ge H^-$, following from the definitions, and of the Comparison Principle among viscosity sub- and supersolutions, which holds because the Hamiltonians satisfy the properties (Lx) and (Lp). □

3.2.3 Existence of the value for differential games in mixed strategies

Let $\mathbb{P}(A), \mathbb{P}(B)$ be the sets of probability measures on A e B, respectively. Define

$$\tilde{f}(x, \mu, \nu) := \iint_{A \times B} f(x, a, b)\, d\mu(a)\, d\nu(b)$$

$$\tilde{l}(x, \mu, \nu) := \iint_{a \times B} l(x, a, b)\, d\mu(a)\, d\nu(b).$$

Lemma 4. *$\tilde{f}$ and $\tilde{l}$ satisfy the same hypotheses as l and f, if we endow $\mathbb{P}(A), \mathbb{P}(B)$ with the topology induced by the weak-star convergence.*

Proof. $\mathbb{P}(A), \mathbb{P}(B)$ are compact with the weak-star topology by Theorem 10. Let us prove that $\tilde{f}$ and $\tilde{l}$ are continuous. Let $(x_n, \mu_n, \nu_n) \to (x, \mu, \nu)$, then:

$$\begin{aligned} |\tilde{f}(x_n, \mu_n, \nu_n) - \tilde{f}(x, \mu, \nu)| \leq \\ |\tilde{f}(x_n, \mu_n, \nu_n) - \tilde{f}(x, \mu_n, \nu_n)| + |\tilde{f}(x, \mu_n, \nu_n) - \tilde{f}(x, \mu, \nu)| \leq \\ \leq \iint_{A\times B} |f(x_n, a, b) - f(x, a, b)|\, d\mu_n(a) d\nu_n(b) + \\ + \left| \iint_{A\times B} f(x, a, b)\, d\mu_n(a) d\nu_n(b) - \iint_{A\times B} f(x, a, b)\, d\mu(a) d\nu(b) \right| \leq \\ \leq C_1 |x_n - x| \cdot 1 + o(1) \end{aligned} \tag{23}$$

for $n \to \infty$, where we used that μ_n, ν_n are probability measures and Remark 7. Moreover,

$$|\tilde{f}(x, \mu, \nu)| \leq \iint_{A\times B} |fx, a, b)|\, d\mu(a) d\nu(b) \leq C_1,$$

e

$$|\tilde{f}(\hat{x}, \mu, \nu) - \tilde{f}(x, \mu, \nu)| \leq \iint_{A\times B} |f(\hat{x}, a, b) - f(x, a, b)|\, d\mu(a) d\nu(b) \leq C_1 |\hat{x} - x|.$$

□

We define the values in *mixed strategies* as

$$\tilde{V}(t, x) = \inf_{\beta\in\tilde{\Delta}_t} \sup_{a\in\tilde{\mathcal{A}}_t} \tilde{J}(t, x, a, \beta[a]), \qquad \tilde{U}(t, x) = \inf_{\alpha\in\tilde{\Gamma}_t} \sup_{b\in\tilde{\mathcal{B}}_t} \tilde{J}(t, x, \alpha[b], b)$$

where $\tilde{\mathcal{A}}_t$ and $\tilde{\mathcal{B}}_t$ are the mixed strategies for the two players (in control theory they are called *relaxed controls*), $\tilde{\Delta}_t$ is the set of non-anticipating mixed strategies $\tilde{\mathcal{A}}_t \to \tilde{\mathcal{B}}_t$ for the second player, whereas $\tilde{\Gamma}_t$ is the analogue for the first.

Application 1. We first apply relaxed controls to optimal control problems with a single player. The interest of this result is that the existence of optimal controls can be proved within relaxed controls (see the references in [FR] or [BCD]).

Proposition 4. *Let $\tilde{J}(t, x, b)$ be the cost functional for a control system (with a single player B who wishes to minimize a cost). The value in mixed strategies defined above coincides with the value previuosly defined, i.e., $\tilde{V}(t, x) = V(t, x)$.*

Proof. We compare

$$\tilde{H}(x, p) = \min_{\nu\in\mathcal{P}(B)} \{\tilde{f}(x, \nu) \cdot p + \tilde{l}(x, \nu)\}$$

and

$$H(x, p) = \min_{b\in B} \{f(x, b) \cdot p + l(x, b)\},$$

that are the Hamiltonians associated to the values $\tilde{V}$ and V, respectively. We have

$$\tilde{H}(x, p) \leq \min_{b\in B} \{\tilde{f}(x, \delta_b) \cdot p + \tilde{l}(x, \delta_b)\} = H(x, p).$$

Moreover, $\forall\nu$:

$$\begin{aligned} \tilde{f}(x, \nu) \cdot p + \tilde{l}(x, \nu) = \int_B (f(x, b) \cdot p + l(x, b))\, d\nu(b) \geq \\ \geq \int_B \min_{b\in B} \{f(x, b) \cdot p + l(x, b)\}\, d\nu(b) = \int_B H(x, p) d\nu(b) = H(x, p) \cdot 1 \end{aligned} \tag{24}$$

because ν is a probability. Then $\tilde{H}(x, p) \geq H(x, p)$, which is the other inequality. Then the Hamilton-Jacobi-Isaacs equations for $\tilde{v}$ and v coincide. By the uniqueness of the solution of such equation with prescribed terminal data we get the conclusion. □

Application 2. Now we apply mixed strategies to 2-person 0-sum differential games. First we show that the value in mixed strategies exists.

Corollary 7. *The game A, B, f, l, g has a value in mixed strategies.*

Proof. Let $\tilde{H}^+$ and $\tilde{H}^-$ be the Hamiltonians of the differential game in mixed strategies:

$$\tilde{H}^+(x,p) := \min_{\mu\in\mathcal{P}(B)} \max_{\nu\in\mathcal{P}(A)} \{p\cdot f(x,\nu,\mu) + l(x,\nu,\mu)\} = \min_{\mu\in\mathcal{P}(B)} \max_{\nu\in\mathcal{P}(A)} F(\nu,\mu)$$

$$F(\nu,\mu) = \iint_{A\times B} (f(x,a,b)\cdot p + l(x,a,b))\, d\mu(b) d\nu(a)$$

The function F is bilinear. In particular it is concave in ν and convex in μ. Therefore we can apply Von Neumann Theorem (recall that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are compact and convex) and we get $\tilde{H}^+ = \tilde{H}^- =: \tilde{H}$.

As in the previous proof, the uniqueness of solutions to Hamilton-Jacobi-Isaacs equations with prescribed terminal data implies $\tilde{V} = \tilde{U}$, and therefore the existence of the value. □

Finally we show that the value in pure strategies, if it exists, coincides with the one in mixed strategies.

Corollary 8. *Under the previous assumptions*

$$V(t,x) \le \tilde{V}(t,x) = \tilde{U}(t,x) \le U(t,x).$$

Proof. We show that $V(t,x) \le \tilde{V}(t,x)$. The proof of $\tilde{U} \le U$ is analogous. We introduce the lower value function $\tilde{V}_A(t,x)$ of the game where the first player uses relaxed controls $\tilde{\mathcal{A}}_t$ while the second uses nonanticipating strategies with values pure strategies, i.e., $\tilde{\mathcal{A}}_t \to \mathcal{B}_t$, that we denote with $\bar{\Delta}_t$. Then the definitions give

$$V(t,x) \le \tilde{V}_A(t,x) := \inf_{\beta\in\bar{\Delta}_t} \sup_{a\in\tilde{\mathcal{A}}_t} \tilde{J}(t,x,a,\beta[a]).$$

Note that $\tilde{V}_A$ solves an Isaacs equation with Hamiltonian

$$H_A^-(x,p) := \max_{\mu\in\mathcal{P}(A)} \min_{b\in B}\{\tilde{f}(x,\mu,\delta_b)\cdot p + \tilde{l}(x,\mu,\delta_b)\} =$$
$$= \max_{\mu\in\mathcal{P}(A)} \min_{\nu\in\mathcal{P}(B)} \{\tilde{f}(x,\mu,\delta_b)\cdot p + \tilde{l}(x,\mu,\delta_b)\}, \quad (25)$$

as we saw in Proposition 4. Then $H_A^- = \tilde{H}^-$ and by uniqueness $\tilde{V}_A = \tilde{V}$, and thus $V \le \tilde{V}$. □

We conclude the section by computing the value function in mixed strategies for an example of game without value in pure strategies.

Example 26. Given the sistem $\dot{y} = (a-b)^2$ in $\mathbb{R}$, with controls $a, b \in \{0,1\}$, $l \equiv 0$ and $g \in C^1(\mathbb{R})$ with $g' > 0$, we compute H^-, H^+, V, U. This is the Berkovitz example and we know that the value does not exist. We have

$$H^-(x,p) = \max_a \min_b [(a-b)^2 p] = \begin{cases} 0 & \text{se } p \ge 0 \\ p & \text{se } p < 0, \end{cases}$$

$$H^+(x,p) = \min_b \max_a [(a-b)^2 p] = \begin{cases} p & \text{se } p \ge 0 \\ 0 & \text{se } p < 0. \end{cases}$$

We guess that the upper and lower value are

$$U(t,x) = g(x+T-t), \qquad V(x,t) = g(x).$$

In order to prove it we first note that they satisfy the terminal conditions of the Cauchy problem. Moreover, $\frac{\partial V}{\partial t} = 0$ and $\frac{\partial V}{\partial x} = g'(x) > 0$. Since H^- is null on positive numbers, $H^-(V_x) = 0$, and then $V_t + H^-(V_x) = 0$ is satisfied. By uniqueness, $V(t,x) = g(x)$ is the solution of the Hamilton-Jacobi-Isaacs equations.

In the same way $\frac{\partial U}{\partial t} = -g'(x + T - t)$ and $\frac{\partial U}{\partial x} = g'(x + T - t) > 0$. Since H^+ is the identity on positive numbers, $H^+(U_x) = g'(x + T - t)$, and then $U_t + H^+(U_x) = 0$.

Finally we observe that $V(t,x) < U(t,x)$, because $g' > 0$.

Next we look for the value function $\tilde{V}$ in mixed strategies. To this purpose we compute the Hamiltonian $\tilde{H}$, which corresponds to the value in mixed strategies of the static game with $\Phi(a,b) = (a-b)^2 p$. The associated matrix is

$$(\Phi_{i,j}) = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$$

We look for a mixed strategy $x = (x_1, x_2)$, with $x_1 + x_2 = 1$ and $x_1, x_2 \geq 0$ that maximizes the expected value of Φ. By the method of Section 2.1.5 we compute

$$\max_{(x_1,x_2)} \min_{j=0,1} \sum_{i=0}^{1} \Phi_{i,j} x_i = \max_{(x_1,x_2)} \min\{p x_2, p x_1\} = \max_{0 \leq x \leq 1} \min\{p(1-x), px\} = \frac{p}{2},$$

as it can be seen by analysing separately the cases $p > 0$ and $p < 0$. Then we solve the Cauchy problem

$$\begin{cases} \tilde{V}_t + \frac{1}{2}\tilde{V}_x = 0 \\ \tilde{V}(T,x) = g(x). \end{cases}$$

This is a linear transport equation, and the solution is

$$\tilde{V}(t,x) = g\left(x + \frac{T-t}{2}\right).$$

Then we have

$$V(t,x) < \tilde{V}(t,x) = \tilde{U}(t,x) < U(t,x) \quad \forall\, t < T.$$

Observe also that the three value functions have all the same profile as the final cost g, but V is stationary, whereas the velocity of propagation of $\tilde{V}$ is a half of the one of the upper value U, and then it is the mean between the velocities of propagation of the two value functions in pure streategies.

4 An introduction to deterministic Mean Field Games

Nash equilibria are very hard to analyse in dynamic games when the number of players N is not small. Suppose that the state x of each player lives in $\mathbb{R}^d$, so the state space is R^{Nd}. Then the system of Hamilton-Jacobi equations in the verification theorem has N PDEs, each in dimension Nd. The numerical solution of such a system in hard if $Nd > 3$ and intractable if $Nd > 10$. The problems with multi-agent system that are interesting for applications have a much larger scale. For instance, the agents of the finantial markets are of the order $N \sim 10^4$, models of crowd motion involve at least $N \sim 10^2$ pedestrians, the consumers of the energy market in a big country like the US are of the order $N \sim 10^5$.

However, in these fields the players can be assumed to be very similar among them and to influence only the cost functional via their *empyrical mean*

$$\frac{1}{N-1} \sum_{i \neq j} \delta_{x_i}, \qquad \delta_x = \text{ Dirac mass in } x.$$

Then one can try to carry ideas from the Mean-field theories in Physics into these games. This was done independently by Lasry and Lions in France and Caines, Huang and Malhame in Canada, starting around 2005-6. The theory took the name of Mean Field Games, usually abbreviated MFG. It aims at approximating, for large population size N, the system of N H-J PDEs (or matrix Riccati ODEs in the L-Q case) by a simpler model, possibly in dimension d, that describes macroscopically the behavior of the population of players.

In MFG the population of agents is modelled by a probability measure μ_t on the state space of a single player $\mathbb{R}^d$, such that $\mu_t(B) = \int_B d\mu_t(x)$ represents the percentage of agents that are in the Borel set $B \subseteq \mathbb{R}^d$ at time t if the population is at an equilibrium (to be properly defined in the spirit of Nash). Often μ_t has a density $m(x,t)$ and in that case

$$\mu_t(B) = \int_B d\mu_t(x) = \int_B m(x,t)dx.$$

The other unknown of the MFG is the value of the game for a generic agent at the equilibrium. Before deriving an MFG model we need some preliminaries.

4.1 The continuity equation.

We need a preliminary notion of measure theory. Given two metric spaces X and Y, a measurable map $\Psi : X \to Y$, and $\mu \in \mathcal{P}(X)$, the *push-forward of μ by Ψ, $\Psi\#\mu$*, is defined by

$$\Psi\#\mu(B) := \mu(\Psi^{-1}(B)), \quad \forall \text{ Borel } B \subseteq Y.$$

If $\mu(\Psi^{-1}(Y)) = 1$, e.g., Ψ is surjective, then $\Psi\#\mu \in \mathcal{P}(Y)$. Observe that, if χ_B is the characteristic function of the set B, the previous definition reads

$$\int_Y \chi_B(y)d(\Psi\#\mu)(y) = \int_X \chi_{\Psi^{-1}(B)}(x)d\mu(x) = \int_X \chi_B(\Psi(x))d\mu(x).$$

Then, for any bounded Borel-measurable function g, by approximation with simple functions we can get

$$\int_Y g(y)d(\Psi\#\mu)(y) = \int_X g(\Psi(x))d\mu(x). \tag{26}$$

Now we want to describe the evolution of a population with distribution in space at time t described by $\mu_t \in \mathcal{P}(\mathbb{R}^d)$, if each agent moves with dynamics driven by a vector field $f : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ satisfying the global existence and uniqueness of trajectories.

Definition 1. *$\Phi : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ is the flow associated to f if $\Phi(x,s) = y(s)$ where $y(\cdot)$ solves*

$$\begin{cases} \dot{y}(s) = f(y(s),s), & s > 0, \\ y(0) = x. \end{cases}$$

We know that $x \mapsto \Phi(x,s)$ is invertible and denote the inverse with $\Phi^{-1}(\cdot,s)$.

Definition 2. *If μ_o is a Borel probability measure on $\mathbb{R}^d$, we denote with $\mu_s := \Phi(\cdot,s)\#\mu_o \in \mathcal{P}(\mathbb{R}^d)$ its push-forward by the flow Φ, i.e.*

$$\mu_s(B) := \mu_o\left(\Phi^{-1}(\cdot,s)(B)\right), \qquad \forall \text{ Borel set } B \subseteq \mathbb{R}^d.$$

Now we look for an equation satisfied by μ_s. By (26), for all bounded and measurable $\psi : \mathbb{R}^d \times [0,T]$ we have

$$\int_{\mathbb{R}^d} \psi(y,s)d\mu_s(y) = \int_{\mathbb{R}^d} \psi(\Phi(x,s),s)d\mu_o(x). \tag{27}$$

If $\psi \in C^1$ we can use (27), differentiate under the integral sign, and then apply (27) again to get

$$\frac{d}{ds}\int_{\mathbb{R}^d} \psi(y,s)d\mu_s(y) = \int_{\mathbb{R}^d} \left(\psi_s + D_x\psi \cdot \Phi_s\right)(\Phi(x,s),s)d\mu_o(x)$$

$$= \int_{\mathbb{R}^d} \left(\psi_s + D_x\psi \cdot f\right)(y,s)d\mu_s(y).$$

Now we take ψ with compact support in $\mathbb{R}^d \times [0,T)$ and integrate both sides in time from 0 to T to get

$$\int_{\mathbb{R}^d} \psi(y,0)d\mu_o(y) = -\int_0^T \int_{\mathbb{R}^d} (\psi_s + D_x\psi \cdot f)\,(y,s)d\mu_s(y)\,ds, \quad \forall\, \psi \in C_c^1(\mathbb{R}^d \times [0,T)). \tag{WCE}$$

Definition 3. *If $\mu_\cdot : [0,T) \to \mathcal{P}(\mathbb{R}^d)$ satisfies (WCE), we say it is a weak or distributional solution of the initial-value problem for the* continuity equation

$$\frac{\partial m}{\partial s} + \mathrm{div}_x(mf) = 0 \tag{CE}$$

with initial condition μ_0.

So we have proved the following

Lemma 5. *The push-forward $\mu_s = \Phi(\cdot,s)\#\mu_o$ of $\mu_o \in \mathcal{P}(\mathbb{R}^d)$ by the flow Φ associated to f satisfies* (WCE)*, i.e., it is a weak solution of* (CE) *in $\mathbb{R}^d \times (0,T)$ with the initial condition μ_o.*

The motivation of the last definition is the following. Assume

$$d\mu_o(x) = m_o(x)dx, \qquad d\mu_s = m(x,s)ds, \tag{H1}$$

i.e., μ_o and μ_s are absolutely continuous with respect to the Lebesgue measure, and therefore have a (locally integrable) density. Suppose in addition that f is C^1 in space and $m \in C(\mathbb{R}^d \times [0,T)) \cap C^1(\mathbb{R}^d \times (0,T))$. In this case, if μ solves (WCE), then the density m solves (CE). To prove this claim we take $\psi \in C_c^1(\mathbb{R}^d \times [0,T))$ and rewrite (WCE) as

$$\int_{\mathbb{R}^d} \psi(y,0)m_o(y)dy = -\int_0^T \int_{\mathbb{R}^d} (\psi_s + D_x\psi \cdot f)(y,s)\,m(y,s)\,dy\,ds\,. \tag{28}$$

We integrate first by parts in time to get

$$-\int_{\mathbb{R}^d}\int_0^T (\psi_s m)(y,s)\,dsdy = \int_{\mathbb{R}^d} \psi(y,0)m_o(y)dy + \int_{\mathbb{R}^d}\int_0^T (\psi m_s)(y,s)\,ds$$

and note that the first term on the r.h.s. equals the l.h.s. of (28). Next we will use the product rule

$$\mathrm{div}_x(mf\psi) = \mathrm{div}_x(mf)\psi + mD_x\psi \cdot f$$

and the divergence theorem, which gives

$$\int_{\mathbb{R}^d} \mathrm{div}_x(m\psi f)(y,s)dy = \int_{\partial \mathrm{supp}\psi} m\psi f \cdot \nu d\sigma = 0\,,$$

where ν is the exterior normal to suppψ. Plugging these identities in (28) we obtain

$$0 = \int_0^T \int_{\mathbb{R}^d} (m_s + \mathrm{div}_x(mf))\,(y,s)\psi(y,s)dy\,ds.$$

This implies (CE) by the arbitrariness of ψ: if (CE) fails at some point (x,t) we get a contradiction by choosing $\psi > 0$ with sufficiently small support centered at (x,t).

Viceversa, it is easy to see in the same way that (CE) and the initial condition $m(y,0) = m_o(y)$ (attained continuously) imply that μ satisfies (WCE).

4.2 A heuristic derivation of the MFG system

Consider a problem in Calculus of Variations, i.e., a control problem with dynamics

$$\dot{y}(s) = -a(s) \qquad a(s) \in \mathbb{R}^d,$$

initial condition $y(t) = x$, and cost functional to minimize

$$J(t,x,a) := \int_t^T (L(a(s)) + F(y(s),s))\, ds + g(y(T)).$$

We assume L is convex, $\lim_{|a|\to\infty} L(a)/a = +\infty$, and define the Hamiltonian

$$H(p) := L^*(p) := \max_{a\in\mathbb{R}^d}\{a \cdot p - L(a)\} \tag{29}$$

i.e., the convex conjugate of L. The HJB equation associated to the problem is

$$\begin{cases} -u_t + H(D_x u) = F(x,t)\,, & \text{in } \mathbb{R}^d \times]0,T[, \\ u(x,T) = g(x). \end{cases} \tag{HJB}$$

Lemma 6. *Assume $L \in C^1$, $DL : \mathbb{R}^d \to \mathbb{R}^d$ is invertible and $(DL)^{-1} \in C^1$. Then $H \in C^1$ and $DH = (DL)^{-1}$. Moreover, if $u \in C^1$ is a solution of* (HJB) *then $DH(D_x u(x,t))$ is an optimal feedback, i.e., the solutions of*

$$\begin{cases} \dot{y}(s) = -DH(D_x u(y(s),s))\,, & s > t, \\ y(t) = x. \end{cases}$$

are optimal trajectories for the problem of minimizing $J(t,x,\cdot)$.

Proof. The argmax in the definition of H (29) is attained at critical points of the pre-Hamiltonian, i.e., at a such that $p - DL(a) = 0$. Since DL is invertible this has a unique solution $Q(p) := (DL)^{-1}(p)$. Then $H(p) = Q(p) \cdot p - L(Q(p))$, so $H \in C^1$ and

$$DH(p) = Q(p) + DQ(p) \cdot p - DL(Q(p)) \cdot DQ(p) = Q(p)$$

because $p = DL(Q(p))$. This proves the first statement.

The optimality of the feedback control $a = Q(D_x u) = DH(D_x u)$ follows from the verification theorems of Section 1.1. $\square$

Then Lemma 5 tells us that, under the assumptions of Lemma 6, a population of agents with the same individual cost F and all behaving optimally, with initial density m_o, evolves as a weak solution of the continuity equation (of Fokker-Planck type)

$$\begin{cases} \frac{\partial m}{\partial t} - \operatorname{div}_x(mDH(D_x u)) = 0\,, & \text{in } \mathbb{R}^d \times]0,T[, \\ \\ m(x,0) = m_o(x). \end{cases} \tag{FP}$$

Now suppose that we have N agents, with position $y_i(s)$ at time s, and the cost F of the $N-th$ agent depends on its own position in the state space and on the empirical mean of the others

$$\mu_s^N := \frac{1}{N-1} \sum_{i=1}^{N-1} \delta_{y_i(s)}.$$

So from now on $F : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, and for measures μ with a density m we will also write $F(x,m) = F(x,\mu)$ (with a slight abuse of notation). The value function v^N of the N-th agent satisfies

$$\begin{cases} \frac{\partial v^N}{\partial t} + H(D_x v^N) = F(x,\mu^N(t))\,, & \text{in } \mathbb{R}^d \times]0,T[, \\ \\ v^N(x,T) = g(x). \end{cases}$$

The measure μ^N describing a population of agents all equal to the N-th evolves according to the weak form of the continuity equation

$$\begin{cases} \frac{\partial \mu^N}{\partial t} - \mathrm{div}_x(\mu^N DH(D_x v^N)) = 0\,, \qquad \text{in } \mathbb{R}^d \times]0,T[, \\ \mu^N(x,0) = \mu_o^N(x). \end{cases}$$

Now let us make the strong assumption that, as $N \to \infty$, $\mu_N \overset{*}{\rightharpoonup} \mu$ and $v^N \to u$, with its derivatives, in a suitable sense. If the limit μ has a density m, we may expect that m and u satisfy the system of PDEs

$$\begin{cases} -u_t + H(D_x u) = F(x,m)\,, \qquad \text{in } \mathbb{R}^d \times]0,T[, \\ m_t - \mathrm{div}_x(mDH(D_x u)) = 0\,, \qquad \text{in } \mathbb{R}^d \times]0,T[, \\ m(x,0) = m_o(x)\,, \quad u(x,T) = g(x)\,. \end{cases} \tag{MFG}$$

This is called the first order MFG system of PDEs. Note that the first equation is backward in time with a terminal condition, and the second is forward in time with an initial condition, a very unusual feature in the theory of PDEs.

The rigorous justification of the heuristic derivation given above is a very hard and largely open problem. However, the system (MFG) can also be easily interpreted in terms of Nash-type equilibria in the following sense.

Definition 4. *A pair (μ, u) with $\mu : [0,T] \to \mathcal{P}(\mathbb{R}^d)$, $u : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ is a* MFG equilibrium *if*

- *u is the value function of the optimal control problem with running cost $F(y(s), \mu_s) + L(a(s))$,*
- *μ is the distribution of a population of agents following the optimal feedback $DH(D_x u)$.*

In other words, as long as the population evolves with distribution μ, it is not convenient for any player to deviate from the feedback $DH(D_x u)$ associated to μ itself. Note that, if μ has a density m, the system (MFG) expresses exactly this equilibrium situation, with initial distribution of agents m_o and terminal cost g.

4.3 Distances in the space of measures

Let Q be a compact metric space with distance d. We want to define a metric on $\mathcal{P}(Q)$. For $p \geq 1$ the Monge-Kantorovich distance on $\mathcal{P}(Q)$ is

$$\mathbf{d}_p(\mu, \nu) := \inf \left\{ \left(\int_{Q^2} d(x,y)^p d\gamma(x,y) \right)^{1/p} \, : \, \gamma \in \mathcal{P}(Q^2) \text{ with marginals } \mu \text{ and } \nu \right\}$$

where the definition of marginals is

$$\gamma(B \times Q) = \mu(B),\ \gamma(Q \times B) = \nu(B) \quad \forall \text{ Borel } B \subseteq Q.$$

This definition is strongly related to the theory of optimal transportation, originated in the work of Gaspard Monge (1781) on how to move in an optimal way a mass of soil from one configuration to another. Fundamental contributions were given by Leonid Kantorovich (1945) using linear programming for infinite-dimensional problems in spaces of measure: he proved that the inf in the above definition is attained as a min (see, e.g., Lemma 5.1 in [C]). Kantorovich won the Nobel prize in Economics in 1975. The mathematical theory of optimal transport had an enormous growth in the last 20 years and is a current subject of intensive research: we refer the reader to the introductory book [S] and to the two monographs of the Fields Medalist C. Villani.

It can be proved that $\mathbf{d}_p$ is indeed a distance on $\mathcal{P}(Q)$ (see, e.g., Lemma 5.2 in [C]). The importance of these distances is that they metricize $\mathcal{P}(Q)$ with the weak-$*$ convergence of measures.

Theorem 21. *(Prop. 5.3 in [C]) For a sequence μ_n in $\mathcal{P}(Q)$, $\mu_n \overset{*}{\rightharpoonup} \mu$ if and only if $\mathbf{d}_p(\mu_n, \mu) \to 0$ as $n \to \infty$.*

The most popular distance in optimal transport is $\mathbf{d}_2$, called the Wassertstein distance. Here we will use instead the metric $\mathbf{d}_1$. It is also called Kantorovich-Rubinstein distance and it admits the following very useful dual formulation.

Theorem 22. *(Kantorovich-Rubinstein Thm. 5.5 in [C]) For all $\mu, \nu \in \mathcal{P}(Q)$*

$$\mathbf{d}_1(\mu, \nu) = \sup \left\{ \int_Q \varphi(x) d(\mu - \nu)(x) \,|\, \varphi : Q \to \mathbb{R} \text{ Lipschitz with } Lip(\varphi) = 1 \right\}.$$

In the sequel, in order to work in a compact set but avoid the difficulties coming from boundary conditions, we will work in the d-dimensional torus, i.e., $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. In other words, we assume that all data and solutions are 1-periodic in each direction:

$$F(x+k, \mu) = F(x, \mu), \quad g(x+k) = g(x), \quad m_o(x+k) = m_o(x), \quad \forall\, k \in \mathbb{Z}^d, x \in \mathbb{R}^d, \mu \in \mathcal{P}(\mathbb{T}^d).$$

The basic assumption on the cost F of the interactions among players is

$$F : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R} \text{ continuous in the product metric } |\cdot|_2 \times \mathbf{d}_1, \tag{F}$$

where $|\cdot|_2$ is the usual Euclidean distance in $\mathbb{R}^d$.

Example 27. Consider $F(x, \mu) = \int_{\mathbb{T}^d} K(x-y) d\mu(y)$ with a kernel $K : \mathbb{T}^d \to \mathbb{R}$ Lipschitz with constant L_K. Then

$$\begin{aligned} &|F(x,\mu) - F(z,\nu)| \le \\ &\quad \left| \int_{\mathbb{T}^d} K(x-y) d\mu(y) - \int_{\mathbb{T}^d} K(z-y) d\mu(y) \right| + \left| \int_{\mathbb{T}^d} K(z-y) d\mu(y) - \int_{\mathbb{T}^d} K(z-y) d\nu(y) \right| \\ &\qquad \le L_K |x-z| \int_{\mathbb{T}^d} d\mu(y) + L_K \left| \int_{\mathbb{T}^d} \frac{K(z-y)}{L_K} d(\mu - \nu)(y) \right| \le L_K(|x-z| + \mathbf{d}_1(\mu, \nu)). \end{aligned}$$

Then F is Lipschitz for the metric of $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$ with constant L_K.

Since we do not expect the system (MFG) to have classical solutions, in the next sections we will consider solutions in the following generalised sense.

Definition 5. *A pair (u, m) is a solution of* (MFG) *if*

- $u \in \mathrm{Lip}_{loc}(\mathbb{T}^d \times [0,T])$,
- *$m(\cdot, t)$ is the density of μ_t, where $\mu \in C([0,T]; \mathcal{P}(\mathbb{T}^d))$ (μ_t is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$ for all $t \in [0,T]$), and $m(\cdot,t)$ is bounded uniformly in t;*
- *u solves the 1st equation of* (MFG) *in viscosity sense,*
- *m solves the 2nd equation in the weak sense of distributions,*
- *$m(x, 0) = m_o(x)$ and $u(x,T) = g(x)$ for all $x \in \mathbb{T}^d$.*

4.4 A uniqueness result for the MFG system

The crucial assumptions to get uniqueness of solutions to (MFG) is the following *monotonicity condition* due to Lasry and Lions [LL]:

$$\int_{\mathbb{T}^d} (F(x,\mu_1) - F(x,\mu_2))\, d(\mu_1 - \mu_2)(x) > 0\,, \quad \forall \mu_1, \mu_2 \in \mathcal{P}(\mathbb{T}^d), \quad \mu_1 \ne \mu_2. \tag{M}$$

With a slight abuse of notation, for measures $\mu_i = m_i dx$ having densities, we will write

$$\int_{\mathbb{T}^d} (F(x,m_1) - F(x,m_2))\, (m_1 - m_2)\, dx > 0\,, \quad \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d), \quad m_1 \ne m_2.$$

Theorem 23. *Let* $(u_1, m_1), (u_2, m_2)$ *be solutions of* (MFG) *in* $\mathbb{T}^d \times [0, T]$ *(i.e., 1-periodic in all space directions). If* $H \in C^1$ *is convex and* F *satisfies* (F) *and* (M), *then* $u_1 = u_2$ *and* $m_1 = m_2$.

Proof. We assume for simplicity that u_i and m_i are smooth, so that we can use the classical derivation of a product and integration by parts for

$$u := u_1 - u_2 \quad \text{and} \quad m := m_1 - m_2.$$

The general case can be achieved by approximating u with smooth functions and multiplying it by a cutoff function ξ_ε (i.e., a smooth function with compact support which is $\equiv 1$ in $[2\varepsilon, T - 2\varepsilon] \times \mathbb{T}^d$ and $\equiv 0$ in $([0, \varepsilon] \cup [T - \varepsilon, T] \times \mathbb{T}^d)$ and use $u\xi_\varepsilon$ as a test function in the equations for m_i. Then the boundedness of the derivatives of u_i and of m_i are used to pass to the limit in the integrals using (F) (see the proof of Thm. 8 in Sect. 1.3.3 of [CP]).

Observe that m solves the equation

$$m_t - \text{div}G = 0, \qquad G := m_1 DH(Du_1) - m_2 DH(Du_2).$$

We multiply this equation by u and integrate in space and time to get

$$\int_0^T \int_{\mathbb{T}^d} m_t u dx dt = \int_0^T \int_{\mathbb{T}^d} u \ \text{div}(G) dx dt$$

and then

$$\int_0^T \int_{\mathbb{T}^d} \left(\frac{d}{dt}(mu) - mu_t \right) dx dt = \int_0^T \int_{\mathbb{T}^d} (\text{div}(uG) - G \cdot Du) \, dx dt.$$

We use at the left hand side the fundamental theorem of calculus and the initial and terminal conditions

$$m(x, 0) = 0\,, \qquad u(x, T) = 0$$

and apply the divergence theorem to the right hand side, to get

$$0 = \int_0^T \int_{\mathbb{T}^d} (mu_t - G \cdot Du) \, dx dt\,, \tag{30}$$

because $\int_{\partial \mathbb{T}^d} uG \cdot n \, d\sigma = 0$ by the periodicity of uG (n is the unit normal at $\partial \mathbb{T}^d$). Next we use that

$$u_t = H(Du_1) - H(Du_2) + F(m_2) - F(m_1)$$

to rewrite (30) as

$$0 = \int_0^T \int_{\mathbb{T}^d} (F(m_2) - F(m_1)) \, (m_1 - m_2) dx dt +$$
$$\int_0^T \int_{\mathbb{T}^d} ((H(Du_1) - H(Du_2)) \, (m_1 - m_2) - (Du_1 - Du_2) \cdot (m_1 DH(Du_1) - m_2 DH(Du_2))) \, dx dt.$$

We claim that the last integral is non-positive as a consequence of the convexity of H. In fact this implies, for all $p_1, p_2 \in \mathbb{R}^d$,

$$H(p_1) - H(p_2) \geq DH(p_2) \cdot (p_1 - p_2)\,, \quad H(p_1) - H(p_2) \leq DH(p_1) \cdot (p_1 - p_2)\,.$$

We multiply the first inequality by m_2, the second by m_1, then subtract the 1st from the 2nd, and get

$$(m_1 - m_2)(H(p_1) - H(p_2)) \leq (p_1 - p_2) \cdot (m_1 DH(p_1) - m_2 DH(p_2))\,, \quad \forall \, p_i \in \mathbb{R}^d, m_i > 0,$$

which proves the claim. Then we obtain

$$0 \leq \int_0^T \int_{\mathbb{T}^d} (F(m_2) - F(m_1)) \, (m_1 - m_2) dx dt.$$

On the other hand, the monotonicity condition (M) says that

$$\int_{\mathbb{T}^d} (F(m_2) - F(m_1))\,(m_1 - m_2)\,dx \leq 0 \quad \forall\, t \text{ and } < 0 \text{ if } \exists \bar{t} \,:\, m_1(\bar{t}) \neq m_2(\bar{t}).$$

By the continuity in time of the integrand we get that $m_1 = m_2$. Now we conclude that $u_1 = u_2$ because they are viscosity solutions of the same Hamilton-Jacobi equation and therefore we can use the comparison principle for such equations. □

Example 28. If $F(x, \mu) = f(x, m(x))$ for $\mu = mdx$, with $f : \mathbb{T}^d \times [0, +\infty) \to \mathbb{R}$ and $\partial f / \partial m > 0$, then the monotonicity condition (M) is satisfied. This example does not satisfy the continuity assumption (F) of F on $\mathcal{P}(\mathbb{T}^d)$, but the proof of uniqueness of solutions works with no changes if f is continuous and $m_i : \mathbb{T}^d \times [0, T] \to \mathbb{R}$ are continuous.

Example 29. Take $F(x, \mu) = f(\cdot, m*\xi(\cdot))*\xi(x)$ for $\mu = mdx$, with f as in the preceding example and $\xi \in C^1(\mathbb{T}^d)$ even. The reader can check that

$$\int_{\mathbb{T}^d} (F(x, m_1) - F(x, m_2))\,(m_1 - m_2)dx = \int_{\mathbb{T}^d} (f(x, m_1 * \xi) - f(x, m_2 * \xi))\,(m_1 * \xi - m_2 * \xi) > 0,$$

and then the condition (M) holds true.

Remark 1. There are examples of existence of multiple solutions of (MFG) when the monotonicity condition (M) does not hold. The meaning of this condition is that it is costly to imitate the behaviour of the other players, so it describes games where the population tends to spread around rather than aggregate.

Remark 2. Condition (M) requires a strict monotonicity. It can be relaxed to monotonicity in wide sense if the Hamiltonian H is strictly convex. The proof begins with the same argument, but then it requires a uniqueness theorem for the continuity equation driven by a discontinuous vector field. This is a deep and difficult result, see, e.g., [C] or Thm. 8 in Sect. 1.3.3 of [CP].

4.5 An existence theorem for the MFG system

Here the main assumption is the following.

$$\exists\, C \text{ such that } \sup_{x \in \mathbb{T}^d} \left(|F(x, \mu)| + |D_x F(x, \mu)| + |D^2_{xx} F(x, \mu)| \right) \leq C \quad \forall\, \mu \in \mathcal{P}(\mathbb{T}^d). \tag{F2}$$

It is easy to check that it is satisfied by the Example 27, $F(x, \mu) = K * \mu(x)$, if $K \in C^2(\mathbb{T}^d)$ (the periodicity implies the boundedness of the derivatives of the kernel, and therefore of F).

We will also restrict to the model case of $H(p) = |p|^2/2$, although the result can be extended to more general uniformly convex Hamiltonians with bounded second derivatives.

Theorem 24. *(Thm. 4.1 in [C], simplified to the periodic setting). Assume* (F), (F2), *$m_o \in L^\infty(\mathbb{T}^d)$, and $g \in C^2(\mathbb{T}^d)$ with bounded 1st and 2nd derivatives. Then there exists a solution of*

$$\begin{cases} -u_t + \frac{|D_x u|^2}{2} = F(x, m)\,, & \text{in } \mathbb{T}^d \times]0, T[\\ m_t - \operatorname{div}_x(m D_x u) = 0\,, & \text{in } \mathbb{T}^d \times]0, T[\\ m(x, 0) = m_o(x)\,, \quad u(x, T) = g(x)\,. \end{cases} \tag{MFG'}$$

Outline of the proof. The proof is obtained by a fixed point argument based on Schauder's Theorem.

Step 1. Define the space

$$\mathcal{C} := \{ m \in C([0, T]; \mathcal{P}(\mathbb{T}^d)) : m(0) = m_o \}$$

and note that it is a subset of the Banach space of continuous functions from $[0, T]$ to the signed measures on $\mathbb{T}^d$, endowed with the uniform convergence. Note also that $\mathcal{C}$ is convex.

Step 2. For each $m \in \mathcal{C}$ consider the terminal value problem for the Hamilton-Jacobi equation

$$-u_t + \frac{|D_x u|^2}{2} = F(x, m), \quad \text{in } \mathbb{T}^d \times]0, T[, \quad u(x, T) = g(x). \tag{HJ}$$

It has a unique bounded and uniformly continuous viscosity solution, which is periodic in space (by uniqueness and the periodicity of the data). We call it $u := \Psi_1(m)$. Moreover it is the value function of the problem in Calculus of Variations associated to (HJ), and form this representation and assumption (F2) one can prove that u is Lipschitz continuous in $\mathbb{T}^d \times [0, T]$ and semiconcave in the space variables, i.e., $D^2_{xx} u \le C_1 Id$: this is Lemma 4.7 in [C].

Step 3. We want to associate to $u := \Psi_1(m)$ a $\mu \in \mathcal{C}$ that solves in the weak sense the continuity equation

$$\mu_t - \operatorname{div}_x(\mu D_x u) = 0, \quad \text{in } \mathbb{T}^d \times]0, T[, \quad \mu(x, 0) = m_o(x). \tag{CE}$$

From Section 4.1 we know that a natural candidate is $\mu(s) := \Phi(\cdot, 0, s)\# m_o$, the push-forward of the initial measure m_o by the flow $\Phi(x, t, s)$ associated to the ODE

$$\dot y(s) = -D_x u(y(s), s), \quad s > t, \quad y(t) = x. \tag{ODE}$$

However, the vector field $-D_x u$ is not Lipschitz, in fact it is merely defined almost everywhere and can be discontinuous. The definition of a flow solving in a generalized sense (ODE) requires a very careful analysis of the regularity properties of the optimal controls and trajectories of the underlying problem in C. of V.: see Lemmata 4.8, 4.9, 4.11 and 4.12 in [C]. Once this is done it can be proved that $\mu(s) := \Phi(\cdot, 0, s)\# m_o$ is a weak solution of (CE): Lemma 4.15 in [C].

In order to define $\Psi_2(u) := \mu$, the solution of (CE), we also need to know that it is unique. This is standard if the vector field is locally Lipschitz in space (Lemma 4.16 in [C]), but we only have $-D_x u \in L^\infty$. The uniqueness result for (CE), Theorem 4.18 in [C], requires very subtle arguments of measure theory applied to transport equations with discontinuous vector fields.

Step 4. Now we define $\Psi := \Psi_2 \circ \Psi_1$ and check that it takes values in $\mathcal{C}$, i.e., μ built in Step 3 is continuous in time. Indeed we can prove the estimate (Lemma 4.14 in [C])

$$\mathbf{d}_1(\mu(s), \mu(s')) \le \|D_x u\|_\infty |s - s'|, \quad \forall\, t \le s \le s' \le T. \tag{S}$$

Then $\Psi : \mathcal{C} \to \mathcal{C}$ is well-defined. Moreover, for all s, $\mu(s)$ is absolutely continuous with respect to the Lebesgue measure, and its density $\mu(\cdot, s)$ satisfies, for some constant C_2,

$$\|\mu(\cdot, s)\|_\infty \le C_2 \|m_o\|_\infty \quad \forall\, s \in [0, T]. \tag{L^∞}$$

From the construction of Ψ and these properties, we can conclude that a fixed point m of Ψ, if it exists, is such that $(m, \Psi_1(m))$ is a solution of (MFG').

Step 5. Now we want to apply the Schauder fixed point theorem (see, e.g., Chapt. 3 of [BCD]). The conditions to be checked are that $\mathcal{C}$ is a closed convex subset of a Banach space, $\Psi : \mathcal{C} \to \mathcal{C}$ is continuous and $\overline{\Psi(\mathcal{C})}$ is compact. The properties of $\mathcal{C}$ are trivial. The continuity of Ψ follows from the continuity of both Ψ_1 and Ψ_2. The map $\Psi_1(m) = u$ is continuous by the stability property of viscosity solutions to (HJ) with respect to the uniform convergence of the data. The continuity of $\Psi_2(u) = \mu$ is proved in Lemma 4.19 of [C]. Finally, the compactness of $\overline{\Psi(\mathcal{C})}$ follows from the Ascoli-Arzelà theorem: the equicontinuity is due to (S), and the equiboundedness is true because $\mathcal{P}(\mathbb{T}^d)$ is compact for the metric $\mathbf{d}_1$.

Some more details on Steps 3 and 4. The flow Φ in Step 3 is built as follows. From Step 2 we have that

$$u(x, t) = \inf_{a \in L^2([t,T], \mathbb{R}^d)} \int_t^T \frac{|a(s)|^2}{2} + F(y(s), m(s)) ds + g(y(T)),$$

where $y(s) = x - \int_t^s a(\tau)d\tau$. By a standard result in C. of V. the inf is attained, so we denote with $\mathcal{A}(x,t) \neq \emptyset$ the set of optimal controls. It can be proved that the graph of the multivalued map $(x,t) \mapsto \mathcal{A}(x,t) \subseteq L^2([t,T],\mathbb{R}^d)$ has a closed graph, i.e., if $(x_n,t_n) \to (x,t)$, $a_n \in \mathcal{A}(x_n,t_n)$, and $a_n \to a$ weakly, then $a \in \mathcal{A}(x,t)$. A classical theorem in set-valued analysis then implies the existence of a Borel-measurable selection $\bar{a}$ such that $\bar{a}(x,t) \in \mathcal{A}(x,t)$ for all (x,t). Now we define

$$\Phi(x,t,s) := x - \int_t^s \bar{a}(x,t)(\tau)d\tau.$$

The properties of the flow Φ depend on the connection between the regularity of the value function u and the set $\mathcal{A}(x,t)$, Lemma 4.9 of [C]. In particular

- for any $s \in (t,T]$ the restriction of $\bar{a}$ to the time interval $[s,T]$ is optimal and the unique element of $\mathcal{A}(y(s),s)$;
- u is differentiable at (x,t) if and only if $\mathcal{A}(x,t)$ is a singleton, and in this case $D_x u(x,t) = \bar{a}(x,t)(t)$.

Then it is easy to show that Φ has the semi-group property

$$\Phi(x,t,s') = \Phi(\Phi(x,t,s),s,s') \quad \forall\, t \le s \le s' \le T,$$

and it solves the system (ODE)

$$\partial_s \Phi(x,t,s) = -D_x u(\Phi(x,t,s),s) \quad \forall x \in \mathbb{R}^d, s \in (t,T). \tag{31}$$

This also give the estimate

$$|\Phi(x,t,s) - \Phi(x,t,s')| \le \|D_x u\|_\infty |s-s'|. \tag{32}$$

Now we turn to the estimates of Step 4. The proof of (S) is a nice combination of the definition of $\mathbf{d}_1$ and the definition of μ as push-forward of m_o by Φ.

Proof of (S). Consider the map $G : \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{T}^d$ defined by $G(x) := (\Phi(x,0,s'), \Phi(x,0,s))$ and the measure $\gamma \in \mathcal{P}(\mathbb{T}^d \times \mathbb{T}^d)$ defined as $\gamma = G\sharp m_o$. It is easy to check that the marginals of γ are $\Phi(\cdot,0,s')\sharp m_o = \mu(s')$ and $\Phi(\cdot,0,s)\sharp m_o = \mu(s)$. Then the definition of $\mathbf{d}_1$ gives

$$\begin{aligned}\mathbf{d}_1(\mu(s'),\mu(s)) \le \int_{R^{2d}} &|x-y|d\gamma(x,y) \\ &= \int_{\mathbb{R}^d} |\Phi(x,0,s') - \Phi(x,0,s)|dm_o(x) \le \|D_x u\|_\infty |s-s'|,\end{aligned}$$

where the last inequality comes from (32). □

In order to prove the absolute continuity of $\mu(s)$ with bounded density (L^∞) we need the following.

Lemma 7. *For some constant C_2 the flow Φ constructed above satisfies*

$$|x-\bar{x}| \le C_2|\Phi(x,0,s) - \Phi(\bar{x},0,s)|, \quad \forall\, 0 < s \le T,\ x,\bar{x} \in \mathbb{R}^d, \tag{33}$$

and so the map $x \mapsto \Phi(x,0,s)$ has a Lipschitz inverse on its image $\Phi(\mathbb{R}^d,0,s)$.

Proof. We will use the property of a semiconcave function w with constant C_1 on $\mathbb{R}^d$ that

$$(Dw(x) - Dw(\bar{x})) \cdot (x-\bar{x}) \le C_1|x-y|^2, \quad \forall\, x,\bar{x} \in \mathbb{R}^d \text{ where } w \text{ is differentiable,} \tag{34}$$

which generalizes the classical monotonicity of the gradient of concave functions.

For $s > 0$ define

$$y(\tau) := \Phi(x, 0, s - \tau), \qquad z(\tau) := \Phi(\bar{x}, 0, s - \tau), \quad \tau \in [0, s].$$

Then (31) implies

$$\dot{y}(\tau) = D_x u(y(\tau), s - \tau), \quad \dot{z}(\tau) = D_x u(z(\tau), s - \tau), \quad \tau \in [0, s],$$

with initial conditions $y(0) = \Phi(x, 0, s)$, $z(0) = \Phi(\bar{x}, 0, s)$. Now for a.e. $\tau \in [0, s]$

$$\frac{1}{2}\frac{d}{d\tau}|y - z(\tau)|^2 = (\dot{y} - \dot{z})(\tau) \cdot (y - z)(\tau) \leq |(y - z)(\tau)|^2,$$

where the inequality comes from (34) and the semiconcavity estimate $D^2_{xx} u \leq C_1 Id$ stated in Step 2. A standard application of Gronwall inequality gives

$$|x - \bar{x}| = |(y - z)(\tau)| \leq e^{C_1(s-\tau)}|(y - z)(0)|, \quad \tau \in [0, s],$$

which implies (33) with $C_2 := e^{C_1 T}$.

□

Proof of (L^∞). Call $\Theta : \Phi(\mathbb{R}^d, 0, s) \to \mathbb{R}^d$ the inverse of $x \mapsto \Phi(x, 0, s)$ and recall that $\mu(s) := \Phi(\cdot, 0, s)\#m_o$. Then, for any Borel set $B \subseteq \mathbb{R}^d$ and denoting with $\mathcal{L}^d$ the Lebesgue measure,

$$\mu(s)(B) = m_o(\Theta(B)) \leq \|m_o\|_\infty \mathcal{L}^d(\Theta(B)) \leq \|m_o\|_\infty C_2 \mathcal{L}^d(B),$$

where the first inequality uses the existence of a density in $L^\infty(\mathbb{T}^d)$ for the initial measure, and the last inequality follows from Lemma 7. This proves the absolute continuity of $\mu(s)$ with respect to the Lebesgue measure and the estimate (L^∞) for its density $\mu(\cdot, s)$. □

For the complete proof and references to the literature we refer to [C] (sect. 4, pages 18-31). □

5 References

Section 1 on verification theorems and LQ control is based on [FR] and [En].

The introduction to game theory in section 2 follows [Bar] and partially [Bre]. The original proofs by Nash of Corollary 3 are in [Na50, Na-thesis, Na51].

Section 3.1 is taken from [Bre], the example in 3.1.3 comes from [JZ]. Section 3.2 on 0-sum differential games follows [ES], with the addition of examples and applications from Chapter VIII of [BCD] (for infinite horizon games).

Section 4 is taken from [C], see also [CP] and [S].

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6 Some historical notes and perspectives on Differential Games.

6.1 History

The first trace of game theory seems to go back to the Talmud, 500 b.C., where a mysterious solution to a problem of division of a heritage was proposed without explanation and resisted all attempts of interpretation until 1985, when Aumann and Maschler studied it within the theory of coalitions in cooperative games and gave a rational motivation to it.

In the 19th century some economists started the analysis of equilibria and introduced ideas that are nowadays considered part of Game Theory. The most important were Antoine Cournot, who studied in 1838 a model of duopoly and found an equilibrium that today is called a Nash equilibrium (in pure strategies), and Vilfredo Pareto, who introduced in 1896 the notion of optima in non-cooperative games that took his name.

In the 20th century the paper [Bo] by Emile Borel proposed the name "théorie du jeu" but did not have much impact. Even less impact had the paper [Ro1] by C.F. Roos who introduced in the context of Calculus of Variations a notion of solution coinciding again with Nash equilibrium (in pure strategies): in the data base MathSciNet of the A.M.S. the paper has no citations at all so far, although published on a prestigious journal.

In 1928 John Von Neumann published the first of a series of papers that marked the official beginning of Game Theory. His work culminated in the book [VNM] with Oskar Morgenstern, dealing with non-cooperative and cooperative games with economic motivations and applications. In 1934 the economist Von Stackelberg introduced the notion of equilibrium that took his name for problems where a player has the role of Leader and the others the role of Followers (e.g., the government and the enterprises, or the central bank of a state and the banks). (The reader can find this notion in [Bre].)

After World War 2 the Rand Corporation was founded in Santa Monica to study the strategic and military scenarios arising in the Cold War. Several mathematicians worked there for some periods, including Von Neumann, Nash, Fleming, Isaacs, and almost all the main american game theorists of the time. A very nice account of the developments of Game Theory and its interplay with decision making in the US foreign policy is the book [P]. The work of John Nash started in 1951 with his thesis at Princeton [N-thesis], where he proved the existence of equilibria for non-cooperative N-person games in mixed strategies.

Two person, 0-sum, differential games started with a series of classified Rand Corporation reports by Rufus Isaacs in 1954-56. In the Soviet Union the group of Pontryagin was working on the same problems and with the same motivations, mostly pursuit-evasion problems arising in the defense from ballistic missiles. The book [I] marked the public official birth of the theory of Differential Games (DG in the following).

The approach of Isaacs was based on the method of characteristics for the H-J-I equation to produce piecewise smooth solution, and in a careful study of singularities where the characteristics meet, in low-dimensional state space. He initiated a theory of singular surfaces associated to 0-sum DG that allowed to solve explicitly some model problems.

The first definition of value function leading to a rigorous connection with the Isaacs equation is due to Wendell Fleming via a limiting procedure on discrete-time approximations [F61]. Other definitions of value were proposed and studied by A. Friedman [Fri] and in Russia by N.N. Krassovski and A.I. Subbotin [KS], all based on the convergence of suitable approximate concepts. The notion of value based on non-anticipating strategies was introduced by Varaiya and Roxin, and mostly studied by Elliott and Kalton [EK] in a series of papers in the 70s.

The proof that the value function of a 0-sum DG is a viscosity solution was obtained in [ES] for the Elliott-Kalton value, and by various authors for Friedman, Fleming and other notions of value. The comparison principle for viscosity solutions of Crandall and Lions [CEL], and the consequent uniqueness results for the Cauchy and Dirichlet problems for H-J-I equations, therefore give a sort of meta-theorem stating that all reasonable notions of value of a DG must coincide. The notion of value of Krassovski-Subbotin was proved by Subbotin to be the unique "minimax solution" of the Isaacs equation [S84], a notion developed independently from the viscosity theory and later recognized to be equivalent, see the book [S95].

A comprehensive survey of dynamic and differential games and their applications until 1995 is in the book of T. Basar and G.J. Olsder [BO]. The ideas of Isaacs on singular surfaces were developed by authors such as J. Breakwell, P. Bernhard, Lewin, and especially Arik Melikyan, whose book [Mel] combines the classical differential geometric methods with the theory of viscosity solutions. L.C. Evans revisited and made rigorous some results by Isaacs and by Melikyan [Ev14] and pointed out some challenging open problems.

Although DG were presented from the very beginning as the first step towards a mathematical theory of conflicts, it was recognized as early as in 1961 by Fleming [F61] that they could model different problems, where a controller wants to achieve some goal but the system governing the state dynamics is affected by an unknown, uncertain disturbance. If one knows the statistics of the noise, this problem can be modeled within stochastic control by minimizing the expectation of the cost functional. If, instead, one wants or must perform a worst-case analysis, then the disturbance can be modeled as an opposing player, therefore leading to a 0-sum deterministic DG. In stochastic control this point of view is called risk-aversion. Two classical applications are the landing of an airplane in a windshear, deeply studied in 80s-90s, and collision avoidance between two vehicles: this was studied at length in aerospace and naval engineering and it is a current topic of research in the automotive industry. A further important development of the connection between deterministic DG and stochastic control was introduced in the book of T. Basar and P. Bernhard [BB] for linear systems and developed by Soravia for nonlinear systems (using viscosity solutions, see App. B of [BCD]): it is called H^∞-control and is related to other relevant subjects in system theory such as disturbance attenuation, risk-sensitive optimal control and robust control.

Dynamic and differential games found a very large number of applications, especially in Economics and Management. Two fields where DG had an important impact in the last decades are population genetics, where the theory of Evolutionary Games was started in the 70s by John Maynard Smith and was continued by many mathematicians and biologists (for an introduction see the book [HS]), and the control of pollution, see the survey [JMZ].

6.1.1 Nobel Prizes.

Nobel Prizes in Economics were awarded four times to groups of researchers in Game Theory. Some of the recipients were mathematicians, others theoretical economists with a strong background in Mathematics. They were

1994: Nash, Harsany, and Selten, for the theory of equilibria and its applications;

2005: Robert Aumann and Thomas Schelling, for the theory and applications of repeated games;

2007: Myerson, Hurvicz, and Maskin for the theory of "mechanism design";

2012: L.S. Shapley and Roth for the theory of cooperative games and coalitions.

6.1.2 Scientific societies.

There are two scientific societies in the field. They organize periodic meetings and publish journals and series of books on the subject. They are the following.

- The International Society of Dynamic Games was founded in 1990 and joins different groups, mostly of control engineers and applied mathematicians, with broad interests in games involving dynamical system and not necessarily fitting in the classical axiomatic framework.

- The Game Theory Society was founded in 1999 and Aumann was its first president, it gathers the community of "classical" game theorists, mostly economists and mathematicians.

6.2 Some future directions of Differential Games.

Besides the numerous potential applications, DG present also several challenging mathematical problems. In all models presented in this course each player has full information at least on the state of the whole system. This is not realistic in many instances, and modelling and handling partial information, although studied in some examples, remains a major open problem.

Deterministic N-person DG give raise to strongly coupled systems of H-J equations. There is no general theory for such PDEs. In fact, some negative results on the well-posedness were proven by Bressan and co-workers (see [Bre]). So there seems to be no hope for a theory somehow parallel to the one for the 0-sum case. If the control system is affected by white noise, the H-J-B equations associated to the DG by the methods of stochastic control are of 2nd order uniformly elliptic or parabolic, and a deep theory about them was developed, especially by Bensoussan and Frehse. It leads to many existence results of smooth solutions but not to uniqueness, that is in general false for such system of PDEs as well as for Nash equilibria. Therefore N-person DG remain a challenging open field of research, both in the deterministic and in the stochastic setting, especially if the number of player is not small.

The main emerging theory in the area, Mean Field Games (briefly, MFG), aims at modelling precisely the hard case of a large population of players. This is possible if the players are very similar and their influence on each other is small: what influences the costs and the dynamics of an agent is only the distribution of the other agents (and white noise). Some ideas come from mean-field theories in physics and statistical mechanics. The pioneers were Caines, Huang and Malhame in Canada, using mostly methods of stochastic control, and, indipendently, Lasry and Lions in France [LL] with an approach based on PDEs. The system of N HJB equation, each in $\mathbb{R}^{Nd}$, is replaced in the limit as $N \to \infty$ by a single HJB equation for the value of the representative agent coupled with a Kolmogorov-Fokker-Planck equation for the distribution of the players, both in $\mathbb{R}^d$. There was an explosion of interest for MFG in the last decade, on the theoretical side and especially on the side of applications, which are potentially a large number, e.g. financial market, issues in social sciences such as opinion dynamics and crowd motion, decentralized communications, energy markets, etc. For an account of the recent advances in the analytical aspects of MFG see the lecture notes [CP] and the references therein. The probabilistic approach to MFG was developed in the monograph by Carmona and Delarue [CD].

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