Program of Differential Equations, Prof. M. Bardi
Master Degree in Mathematics, a.y. 2022/2023

1. Introduction (references: [E], sect. 1.1, 1.2, 1.3, 2.1)

Generalities on 1st order Partial Differential Equations, linear and nonlinear operators. Examples and motivations: the transport equation, conservation laws, Hamilton-Jacobi (HJ) equations.

2. The method of carachteristics (ref.: [E], sect. 3.2, [L] chapter 1, sect. 1.2)

Derivation of the carachteristic equations for the PDE $F(x, u, Du) = 0$ and inversion of the carachteristic flow. Carachteristics for conservation laws and examples of shocks. Carachteristics for the HJ equation $u_t + H(x, D_x u) = 0$. Local existence theorem of solutions to the Cauchy for the equation $u_t + H(D_x u) = 0$. Examples of global existence of classical solutions, examples of crossing of characteristics and shock of the gradient; links with conservation laws.

3. Links between HJ equations and Calculus of Variations, introduction to Convex Analysis (ref. [E], sect. 3.3 and appendix B.)

Recall of Calculus of Variations and necessary conditions of minimality for action functionals: Euler-Lagrange equations and generalized moment. Connection of Euler-Lagrange equations and Hamiltonian systems. Value function of CoV as a candidate solution to H-J. Recall of convex functions and their properties. Existence of a supporting hyperplane for the graph of a convex function, subgradient. Convex conjugate and the duality theorem for convex and superlinear functions. Jensen inequality. Value function for problems in Calculus of Variations and verification theorem for classical solutions of HJ with H convex.

5. The Hopf-Lax formula (ref. [E], chapter 3, sect. 3.3)

The Hopf-Lax formula for the H-J equation as value function of a problem in Calculus of Variations. Dynamic programming and Lipschitz continuity. The Hopf-Lax formula satisfies the H-J equation a.e. Counterexample to the uniqueness of a.e. solutions. Uniqueness and comparison principle for classical solutions. Semiconcave functions, properties and examples. Uniqueness of solutions to HJ semiconcave in x. The Hopf-Lax formula is semiconcave in x if the initial data are.

6. Viscosity solutions of Hamilton-Jacobi equations (refs. [BCD], ch. II, sect. 1, 2, 3; [E], ch. 10)

Definition of viscosity solution for Hamilton-Jacobi equations and its motivation as the uniform limit of viscous approximations with vanishing viscosity. The Hopf-Lax formula is a viscosity solution. Definition for general equations and consistency of viscosity solutions with classical ones. Sub- and superdifferential of semicontinuous functions, equivalent definition of viscosity solution. Stability with respect to uniform convergence. The example $|u'| - 1 = 0$, $u(-1) = 0 = u(1)$. Comparison Principle for classical and for viscosity solutions in bounded domains; uniqueness of solution to the Dirichlet problem. Change of dependent variable in H-J equations. The distance function from the boundary is the unique solution of the eikonal equation with null boundary condition. Comparison theorem for evolutive equations: behaviour of the solutions at the terminal time. The Hopf-Lax formula is the unique solution of the Cauchy problem.

7. Introduction to optimal control via Dynamic Programming (refs. [BCD], ch. III, sect. 1 and 3 and appendix 5; [E], ch. 10)

Existence and uniqueness of generalized solutions to ordinary differential equations with measurable controls. Value function of an optimal control problem with finite horizon, continuity properties. Dynamic Programming Principle and Hamilton-Jacobi-Bellman equation. The value function is the unique solution of a terminal value problem for the HJB equation.

8. Introduction to Linear-Quadratic control (refs. [FR] cap. IV, sez. 4 and 5; [B])

Feedback controls. Verification theorem and synthesis of an optimal feedback for finite horizon optimal control problems. Linear-Quadratic regulator with finite horizon and Riccati differential equation. Solvability of the Riccati equation.

9. Introduction to game theory (refs.[Bar] ch. 1 and 3; [Bre], [B])

Zero-sum games: value and saddle points. Examples of matrix games. Von Neumann min-max theorem. Mixed strategies and existence of value in matrix games, methods for computing the value. Existence of value in mixed strategies for general 0-sum games.

Two person, non-0-sum games: Nash equilibria and examples (the prisoner's dilemma,...). Pareto optima. Existence of Nash equilibria under convexity assumptions by means of Brouwer fixed point theorem; equilibria in mixed strategies.

10. Introduction to differential games (refs. [Bre], [ES], [BCD] cap. VIII, [B])

Two-person differential games: Nash equilibria in feedback form and verification theorem. Non-0-sum L-Q differential games and Riccati equations. Zero-sum L-Q differential games: Isaacs equations and global existence of solutions of the Riccati equation.

Zero-sum L-Q differential games: nonanticipating strategies and definition of lower and upper value, examples. Dynamic Programming Principle and continuity of the value functions. Characterization of the value functions as unique solutions of the Isaacs equations, existence of the value. Relaxed controls and mixed strategies in differential games, examples.

11. A short introduction to deterministic Mean Field Games (ref. [C], [B])

The continuity equation in weak form for a measure transported by a flow. Motivation and derivation of the Mean Field Games system of 1st order PDEs (MFG). The Monge-Kantorovitch distance on probability measures; continuity of the cost with respect to the Kantorovich-Rubinstein metrics d_1, examples. Uniqueness of a (classical) solution to the MFG system under a monotonicity condition. An existence theorem for solutions of the MFG system (the outline of the proof is optional).

N.B.: Only the proofs presented in class are part of the program, see the pdf files of the lectures available on Moodle.

Bibliography

The items with asterisk * are available on Moodle.

[B] M. Bardi, *Notes of the course "Differential Equations"*, lecture notes 2023.*

[BCD] M. Bardi, I. Capuzzo Dolcetta, **Optimal control and Viscosity solutions of Hamilton-Jacobi-Bellman equations**, Birkhäuser, Boston, 1997; 2nd printing, Modern Birkhäuser Classics, 2008.*

[Bar] N. Barron, **Game Theory**, Wiley, 2008.

[Bre] A. Bressan, *Noncooperative differential games,* , Milan J. Math. 79 (2011), 357–427.*

[C] P. Cardaliaguet, **Notes on Mean Field Games**, 2013, https://www.ceremade.dauphine.fr/ cardaliaguet/.*

[E] L. C. Evans, **Partial Differential Equations**, 2nd edition, American Mathematical Society, 2010.*

[ES] L. C. Evans, P. Souganidis *Differential Games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations*, Indiana Univ. Math. J. 33 (1984), 773–797.*

[FR] W. Fleming, R. Rishel, **Deterministic and stochastic optimal control**, Springer, New York, 1975.

[L] P.-L. Lions, **Generalized solutions of Hamilton-Jacobi equations**, Pitman, Boston, 1982.