LECTURE 26, June 1st, 2023

$$(\mathrm{MFG}')\quad \begin{cases} -u_t + \frac{|D_x u|^2}{2} = F(x,m) & \\ m_t - \mathrm{div}_x(m D_x u) = 0 & \text{in } \mathbb{T}^d \times]0,T[\\ m|_{t=0} = m_0 \ , \ u|_{t=T} = g \end{cases}$$

<u>Theorem</u> (Existence of solutions) Ass. $m_0 \in L^\infty(\mathbb{T}^d)$, $g \in C^2(\mathbb{T}^d)$,

(F) $F : \mathbb{T}^d \times \mathbb{P}(\mathbb{T}^d)$ cont. for $|\cdot|_2 \times d_1$

(F2) $|F(x,\mu)| + |D_x F(x,\mu)| + |D^2_{xx} F(x,\mu)| \le C_1 \quad \forall x \in \mathbb{T}^d, \mu \in \mathbb{P}(\mathbb{T}^d)$.

Then $\exists$ a solution of (MFG')

<u>Proof</u> <u>Step 1</u> $\mathcal{C} = \{ m \in C([0,T], \mathbb{P}(\mathbb{T}^d)), m(0) = m_0 \}$

convex, closed $\subseteq$ Banach space with sup norm.

<u>Step 2</u> Fix $m \in \mathcal{C}$. $\Psi_1(m) = u$ the sol. of

$$-u_t + \frac{|D_x u|^2}{2} = F(x,m), \quad u(x,T) = g(x).$$

"Know that" $u(x,t) \overset{(1)}{=} \inf\limits_{\alpha \in L^2([0,T],\mathbb{R}^d)} \int_t^T \left(\frac{|\alpha(s)|^2}{2} + F(y(s), m(s)) \right) ds$

"$\dot{y} = -\alpha$"

$y(s) = x - \int_t^s \alpha(\tau)\, d\tau$. periodic: $u(x+k,t)$ is $\forall k \in \mathbb{Z}^d$ and hence

From the formula can prove $u \in Lip(\mathbb{T}^d)$ & $\exists C_1$:

u C_1-semiconcave $D^2_{xx} u \le C_1 Id$

Step 3. (the hard one) Given u as above want to solve

$$(CE) \quad \mu_t - \mathrm{div}(\mu D_x u) = 0 \quad , \quad \mu(x,0) = m_0(x).$$

Trouble $D_x u \in L^\infty$ only. Want to build sol.

$$\mu(s) = \Phi(\cdot, s, s) \# m_0$$ for suitable flow ass.

to $\dot{y} = -D_x u(y)$, (ODE)

Idea: traject. of (ODE) should be optimal for probl. (1).

Construction of Φ: $\mathcal{A}(x,t) = \{$opt. controls for (1)$\} \subseteq L^2([0,T], \mathbb{R}^d)$

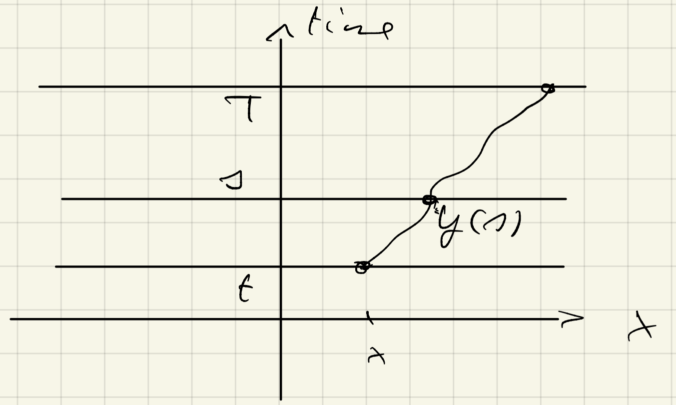
"Standard facts":

- $\mathcal{A}(x,t) \neq \emptyset \quad \forall (x,t)$
- $(x,t) \to \mathcal{A}(x,t)$ has "closed graph": i.e. $(x_n,t_n) \to (x,t)$, $a_n \in \mathcal{A}(t_n,x_n)$, $a_n \to a \Rightarrow a \in \mathcal{A}(x,t)$.
- "Set-valued analysis" $\Rightarrow \exists\ \bar{a}(x,t) \in \mathcal{A}(x,t)$ measurable.

Def $\Phi(x,t,s) := x - \int_t^s \bar{a}(x,\tau)\, d\tau = y(s)$

Properties: • $s > t$

$\bar{a}(x,t)\big|_{[s,T]}$ is optimal for $(y(s), s)$ and it is the unique optimal control.



- $\mathcal{A}(y,s)$ singleton $\Leftarrow u$ diff.le at (y,s) E

$$D_x u(y,s) = \bar{\alpha}(y,s)(s)$$

$\Uparrow$ the unique optimal control.

From these can deduce,

- SEMIGROUP prop. $\forall \ t \le s \le s' \le T$

$$\Phi\left(\Phi(x,t,s), s, s'\right) = \Phi(x, t, s')$$

- Φ solves the (ODE) $\forall x \in \mathbb{R}^d \quad 0 \le t < s \le T$

$$\partial_s \Phi(x,t,s) = -D_x u(\Phi(x,t,s), s)$$

- $$|\Phi(x,t,s) - \Phi(x,t,s')| \le \|D_x u\|_\infty |s - s'|$$

Conclusions of Step. 3. $\mu(s) := \Phi(\cdot, 0, s) \# m_0$

- is a weak solution of (CE)
- it is the UNIQUE WEAK SOLUT. (hard technical fact).

Def. $\Psi_2(u) = \mu$

We defined. $\mathcal{C} \ni m \xrightarrow[\Psi_1]{} u \xrightarrow[\Psi_2]{} \mu \qquad \Psi = \Psi_2 \circ \Psi_1$

is well-defined.

Step. 4 Q 1: $\Psi(m) = \mu \in \mathcal{C}$? Lip estimate in t for μ:

$$\boxed{(S) \quad d_1(\mu(s), \mu(s')) \le \|D_x u\|_\infty |s - s'|.}$$

Pf. of (S). $d_1(\nu, \bar{\nu}) := \inf\left\{ \int_{\mathbb{T}^d \times \mathbb{T}^d} |x-y| \, d\gamma(x,y) : \gamma \text{ with marginals } \nu, \bar{\nu} \right\}$

$\gamma(\mathbb{T}^d \times B) = \nu(B)$, $\gamma(B \times \mathbb{T}^d) = \bar{\nu}(B)$ $\forall B$ Borel.

Def. $G(x) := (\Phi(x,0,s'), \Phi(x,0,s))$

$G: \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{T}^d$ Def. $\gamma := G \# m_0$

Claim: the marginals of γ are $\Phi(x,0,s')\# m_0 = \mu(s')$

& $\Phi(x,0,s)\# m_0 = \mu(s)$

check: $\gamma(\mathbb{T}^d \times B) = m_0(G^{-1}(\mathbb{T}^d \times B)) = m_0(\Phi(\cdot,0,s)^{-1}(B))$

$= \Phi(\cdot,0,s)\# m_0(B) = \mu(s)(B)$ $\square$ claim.

$$\Rightarrow d_1(\mu(s'),\mu(s)) \leq \int_{\mathbb{T}^d \times \mathbb{T}^d} |x-y| \, d\gamma(x,y) =$$

def. of $\#$

$$= \int_{\mathbb{T}^d} |\Phi(x,0,s') - \Phi(x,0,s)| \, dm_0(x)$$

$$\leq \|D_x u\|_\infty |s'-s| \underbrace{\int_{\mathbb{T}^d} m_0(x)}_{=1} = \|D_x u\|_\infty |s-s'|.$$

$\square$

$\Rightarrow \Psi(m) = \mu \in \mathcal{C}$. $\Rightarrow \Psi : \mathcal{C} \to \mathcal{C}$

If $m = \Psi(m)$ is a fixed point of Ψ

$u = \Psi_1(m)$ solves HJ in MFG'.

& m is a weak sol. of CE in MFG'

So if m has a bounded density $(m, \Psi_1(m))$

is a sol. of (MFG'). I need

Lemma $\exists\, C_2$: $\forall\, 0 < s \le T$, $x, \bar{x} \in \mathbb{R}^d$

$$|x - \bar{x}| \le C_2 |\Phi(x, 0, s), - \Phi(\bar{x}, 0, s)|$$

i.e. the inverse $\Theta(\cdot)$ of $\Phi(\cdot, 0, s)$ is Lip with constant C_2.

Pf No, see Notes, use $D^2_{xx} u \le C_1 \, Id$. ∎

Lemma $\forall s$ $\mu(s) = \Psi(m)(s)$ is ABS. CONT. w.r.t. Lebesgue & if $\mu(\cdot, s)$ is its density

(L^∞) $\quad \|\mu(\cdot, s)\|_\infty \le C_2 \|m_0\|_\infty$

Pf. $\mu(s) = \Phi(\cdot, 0, s) \# m_0$. $B \subseteq \mathbb{T}^d$ Borel.

$$\boxed{\mu(s)(B)} = m_0(\Theta(B)) \le \|m_0\|_\infty \mathcal{L}^d(\Theta(B)) \le$$

$\Theta = \Phi(\cdot, 0, s)^{-1}$ $\quad$ Θ C_2 Lip $\quad$ ↑ Lebesgue

$$\boxed{\le \|m_0\|_\infty C_2 \mathcal{L}^d(B)}$$

$\Rightarrow \mu(s)$ A.C. w.r.t. $\mathcal{L}^d$ & (L^∞). ∎

Conclusion of Step 4 : if $m \in \mathcal{C}$ is a fixed point of Ψ then $(m, \Psi_1(m))$ is a sol. of (MFG').

Step 5. Schauder fixed point : $\Psi : \mathcal{C} \to \mathcal{C}$, $\mathcal{C} \subseteq$ Banach Space, convex, closed, Ψ cont. & $\overline{\Psi(\mathcal{C})}$ compact. $\Rightarrow \exists$ a fixed point.

$\mathcal{C} \subseteq \{ m : [0,T] \to \mathcal{M}(\mathbb{T}^d)$ cont. $\}$ with L^∞ norm.

$\uparrow$ signed measures

is convex & closed.

Q1: $\Psi = \Psi_2 \circ \Psi_1$ is continuous?

- $\Psi_1 : m \to u$ is cont because $m \mapsto F(\cdot, m)$ is cont. by (F) $\Rightarrow \Psi_1$ is cont. by the "stability property" of visco. sols.

- Ψ_2 cont. ... more difficult.... [Card.].

$\Rightarrow \Psi$ is cont.

Q2 $\overline{\Psi(\mathcal{C})}$ sequentially compact? Take $\mu_n = \Psi(m_n)$

(S) $\Rightarrow \mu_n$ is equiLipschitz, also equibdd. because $\mu_n \in \mathbb{P}(\mathbb{T}^d)$ which is compact. Ascoli-Arzelà $\Rightarrow \exists\, \mu_{n_k} \xrightarrow[k\to\infty]{} \mu$ unif. in $[0,T]$. $\in \overline{\Psi(\mathcal{C})}$

Concl.: can apply Schauder $\Rightarrow \exists\, m \in \mathcal{C}$: $m = \Psi(m) \overset{\text{Step 4}}{\Longrightarrow} \exists$ sol. of (MFG'). ∎

FINAL REMARKS on MFGs.

- deterministic MFG are HARDER than stochastic MFG.

What is stochastic control?

"$\dot{y} = -a +$ white noise" $\doteq$ "$\dfrac{dw}{dt}$" $\quad w$ = Wiener process or Brownian motion.

give sense to ↗ by Ito integrals.

$$J = \mathbb{E}\left[\int_t^T \dots\right]$$

want to min J

Can do Dyn Prog., find a HJB which is

$$u_t + H(x, D_x u) = \Delta_x u$$

a parabolic equation

has smooth solutions!

- In Stoch. MFG, 2nd eq. is $\qquad$ is smooth.

$$m_t - \mathrm{div}_x(m\, DH(D_x u)) = \Delta_x u$$

Solutions are CLASSICAL!

In stoch MFG the large population limit $N \to \infty$ has been proved rigorously if (M) holds. The general case is open!

Probabilistic approach to stoch MFG: Carmona - Delarue 2 volumes ~ 2018

PLEASE, POINT OUT TO ME ALL TYPOS, MISTAKES or UNCLEAR THINGS in the LECTURE NOTES of the course!