LECTURE 23, May 31, 2023

Need a notion of continuity w.r.t. measures.

Let Q be compact metric space with distance $d(\cdot,\cdot)$

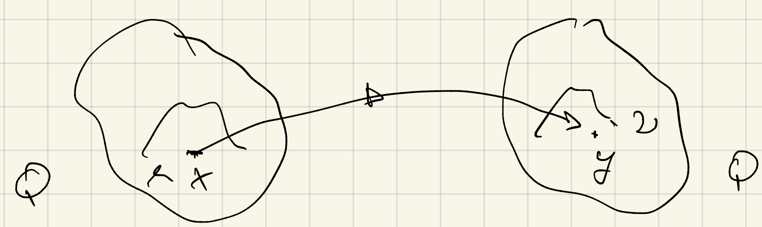
Look for a metric on $\mathbb{P}(Q) \longleftrightarrow$ weak $*$ convergence.

<u>Def.</u> $p \geq 1$, $\mu, \nu \in \mathbb{P}(Q)$, Monge-Kantorovich distance
(transport ..)

$$d_p(\mu,\nu) := \inf\left\{ \left(\int_Q d(x,y)^p \, d\gamma(x,y) \right)^{1/p}, \ \gamma \in \mathbb{P}(Q\times Q) \text{ with marginals } \mu \ \& \ \nu \right\}$$

Marginals means $\gamma(B\times Q) = \mu(B)$, $\gamma(Q\times B) = \nu(B)$

$\forall B \subseteq Q$ Borel.



G. Monge 1781

Kantorovich ~1945

<u>Thm.</u> $\forall p \geq 1$ d_p is a metric on $\mathbb{P}(Q)$ &

$$d_p(\mu_n, \mu) \to 0 \text{ as } n\to\infty \iff \mu_n \xrightarrow{*} \mu$$

<u>Ref.</u> [Cand.] [F. Santambrogio ... Opti transport for applied math.]

$p=2$ Wasserstein distance (see Villani's books).

We use $p=1$, d_1 = Kantorovich - Rubinstein dist.

Dual formulation:

Thm. (K-R) $\forall\ \mu, \nu \in \mathbb{P}(Q)$

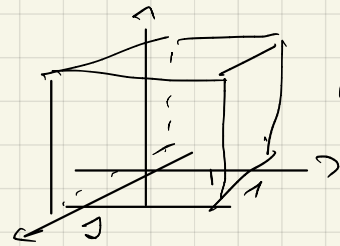
$$d_1(\mu,\nu) = \sup \left\{ \int_Q \varphi(x)\, d(\mu-\nu)(x),\ \ \varphi : Q \to \mathbb{R} \text{ Lip. with } \operatorname{Lip}\varphi = 1 \right\}$$

Pf see refs. above. $\boxtimes$

We will study (MFG) UNDER PERIODICITY ASSUMPTIONS on the data

$$F(x+k,\mu) = F(x,\mu) \quad \forall k \in \mathbb{Z}^d$$

$$g(x+k) = g(x), \quad m_0(x+k) = m_0(x)$$



$Q = \mathbb{T}^d$

FLAT TORUS $\quad \mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$

$g \in C(\mathbb{T}^d)$ means g continuous in $\mathbb{R}^d$ & $\mathbb{Z}^d$-periodic.

Basic cont.ty on F

(F) $\quad F : \mathbb{T}^d \times \mathbb{P}(\mathbb{T}^d) \to \mathbb{R}$ cont. w.r.t. $|\cdot|_2 \times d_1$

Example $\quad F(x,\mu) = \int_{\mathbb{T}^d} K(x-y)\, d\mu(y) \qquad \mu \in \mathbb{P}(\mathbb{T}^d)$

kernel $K \in \operatorname{Lip}(\mathbb{T}^d)$, $\operatorname{Lip} K = L_K$.

Claim $F : \mathbb{T}^d \times \mathbb{P}(\mathbb{T}^d) \to \mathbb{R}$ is Lip.

$$\underset{\pm F(z,\mu)}{|F(x,\mu) - F(z,\nu)|} \le \left| \int_{\mathbb{T}^d} \underbrace{\big(K(x-y) - K(z-y)\big)}_{\le L_K |x-z|} d\mu(y) \right| +$$

$$+ \left| \int_{\mathbb{T}^d} K(z-y)\, d\mu(y) - \int_{\mathbb{T}^d} K(z-y)\, d\nu(y) \right|$$

$$\le L_K |x-z| \underbrace{\int_Q d\mu(y)}_{=1} + L_K \left| \int_{\mathbb{T}^d} \underbrace{\frac{K(z-y)}{L_K}}_{\operatorname{Lip} = 1} d(\mu-\nu)(y) \right| \overset{\text{last Thm.}}{\le}$$

$\leq L_K(|x-z|+d_1(\mu,\nu))$ $\Rightarrow$ F is Lip. with cst. L_K ▢

— o —

We study

(MFG) $\begin{cases} -u_t + H(D_x u) = F(x,m) & \text{in } \mathbb{T}^d \times]0,T[\\ m_t - \operatorname{div}_x(m\, DH(D_x u)) = 0 \\ m(x,0) = m_0(x), \quad u(x,T) = g(x) \end{cases}$

<u>Def</u> A pair (u,m) is a sol. of (MFG) in $\mathbb{T}^d \times (0,T)$ if

- $u \in Lip_{loc}(\mathbb{T}^d \times [0,T])$
- $m(\cdot,t)$ is the density of $\mu_t \in \mathbb{P}(\mathbb{T}^d)$, $\mu_\cdot \in C([0,T], \mathbb{P}(\mathbb{T}^d))$
  $\|m(\cdot,t)\|_\infty \leq C \quad \forall t$
- u solves 1^{st} eq. visc. sense
- m solves 2^{nd} eq. WEAK SENSE
- $m(x,0) = m_0(x)$, $u(x,T) = g(x) \quad \forall x \in \mathbb{T}^d$.

For UNIQUENESS we make a MONOTONICITY ASS. on F:
(Lasry-Lions).

(M) $$\int_{\mathbb{T}^d} (F(x,\mu_1) - F(x,\mu_2))\, d(\mu_1 - \mu_2) > 0$$
$$\forall \mu_1, \mu_2 \in \mathbb{P}(\mathbb{T}^d) \quad \mu_1 \neq \mu_2$$

<u>Ex. 1</u> $\mu = m\,dx \qquad F(x,\mu) = f(x, m(x))$

$f: \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$. Then (M) holds if $r \mapsto f(x,r)$ is strictly increasing $\forall x$:

$$\int_{\mathbb{T}^d} \underbrace{\left(f(x,m_1(x)) - f(x,m_2(x))\right)(m_1(x) - m_2(x))}_{>0 \quad \text{if } m_1(x) \neq m_2(x)} dx > 0 \qquad \text{if for some } x \; m_1(x) \neq m_2(x)$$

N.B. $\Rightarrow$ (M) means "CROWDED PLACES are COSTLY"

Rmk The last ex does NOT satisfy (F).

Rmk Without (M) $\exists$ examples of (MFG) with multiple solutions.

Ex. 2 $\quad F(x,\mu) = f(\cdot, m * \xi(\cdot)) * \xi(x) \qquad \xi \in C^1(\mathbb{T}^d)$ & EVEN

$\mu = m\,dx \qquad f$ as in Ex. 1, $\dfrac{\partial f}{\partial m} > 0$.

HW: check (F)

$$F(x,m) = \int_{\mathbb{T}^d} f\left(y, \int_{\mathbb{T}^d} \xi(y-z)\, m(z)\, dz\right) \xi(x-y)\, dy \qquad \text{Q: (M)?}$$

$$\int_{\mathbb{T}^d} (F(x,m_1) - F(x,m_2))(m_1 - m_2)\, dx = \text{HW} \qquad \text{use Fubini \& } \xi \text{ EVEN}$$

$$= \int_{\mathbb{T}^d} \left[f(y, m_1 * \xi) - f(y, m_2 * \xi) \right] (m_1 * \xi - m_2 * \xi)\, dy > 0 \quad \text{if } m_1 \neq m_2,$$

by $\dfrac{\partial f}{\partial m} > 0$.

Theorem Ass. $H \in C^1$ & convex, F sats. (F) & (M). $(u_1, m_1), (u_2, m_2)$ are solutions of (MFG) $\Rightarrow$ $u_1 = u_2$, $m_1 = m_2$.

Pf. For simplicity only for classical solutions: this assumption can be removed by an approximation argument.

$u := u_1 - u_2$, $m := m_1 - m_2$ satisfy.

$$\begin{cases} -u_t = -H(Du_1) + H(Du_2) + F(m_1) - F(m_2) & (1) \\ m_t = \mathrm{div}_x \underbrace{\left(m_1 DH(Du_1) - m_2 DH(Du_2)\right)}_{=:G} & (2) \\ m\big|_{t=0} \equiv 0, \quad u\big|_{t=T} \equiv 0 & (BC) \end{cases}$$

Mult. (2) by u $\int_0^T \int_{\mathbb{T}^d}$:

$$(*) \underbrace{\int_0^T \int_{\mathbb{T}^d} m_t u \, dx dt}_{''} = \int_0^T \int_{\mathbb{T}^d} \underbrace{u \, \mathrm{div}\, G}_{= \mathrm{div}(uG) - G \cdot \nabla u} \, dx dt$$

$$\frac{d}{dt} \int_{\mathbb{T}^d} mu \, dx - \int_{\mathbb{T}^d} m u_t \, dx$$

$$(*) = \underbrace{-\int_{\mathbb{T}^d} mu \, dx \Big|_{t=0}^{t=T}}_{=0 \text{ by (BC)}} - \int_0^T \int_{\mathbb{T}^d} m u_t \, dx = -\int_0^T \int_{\mathbb{T}^d} G \cdot \nabla u \, dx dt + \underbrace{\int_0^T \int_{\partial \mathbb{T}^d} u G \cdot \nu \, d\sigma}_{=0 \text{ by periodicity}}$$

ext. norm.

$$\Rightarrow 0 = \int_0^T \int_{\mathbb{T}^d} (m u_t - G \cdot \nabla u) \, dx dt =$$

$$\stackrel{(1)}{=} \int_0^T \int_{\mathbb{T}^d} m \left(H(Du_1) - H(Du_2) + F(m_2) - F(m_1)\right) dx dt -$$

$$- \int_0^T \int_{\mathbb{T}^d} \nabla u \cdot \left(m_1 DH(Du_1) - m_2 DH(Du_2)\right) dx dt =$$

$$= \int_0^T \int_{\mathbb{T}^d} (m_1 - m_2)(F(m_2) - F(m_1)) \, dx dt +$$

$$+ \int_0^T \int_{\mathbb{T}^d} \left[(m_1 - m_2)(H(Du_1) - H(Du_2)) - (Du_1 - Du_2) \cdot \left(m_1 DH(Du_1) - m_2 DH(Du_2)\right)\right] dx dt$$

CLAIM : $[\dots] \le 0$. Pf. of Claim : H convex & C^1

$\Rightarrow \forall p_1, p_2 \in \mathbb{R}^d$ $\quad H(p_1) - H(p_2) \ge DH(p_2)\cdot(p_1 - p_2)$ $\quad \times\ m_2 \ge 0$

$H(p_2) \ge H(p_1) + \dots$ $\quad H(p_1) - H(p_2) \le DH(p_1)\cdot(p_1 - p_2)$ $\quad \times\ m_1 \ge 0$

Substract 1st from 2nd :

$$(m_1 - m_2)(H(p_1) - H(p_2)) \le (p_1 - p_2)(m_1 DH(p_1) - m_2 DH(p_2))$$

$p_1 = Du_1$, $p_2 = Du_2$ $\Rightarrow$ CLAIM

$$\Rightarrow \quad 0 \le \int_0^T \underbrace{\int_{\mathbb{T}^d} \left(F(x, m_2) - F(x, m_1)\right)(m_1 - m_2)\, dx}_{\le 0 \ \forall t \ \& \ < 0 \text{ at } \bar{t} : \ m_1(\cdot, \bar{t}) \ne m_2(\cdot, \bar{t})} dt$$

Integral $\int_{\mathbb{T}^d}(\ )(\ )$ is cont. in time

$\Rightarrow \quad m_1(x,t) = m_2(x,t) \quad \forall x \ \forall t$.

$\Rightarrow \quad F(x, m_1) = F(x, m_2) \quad \Rightarrow u_i$ solve the same H-J eq, by uniq. of visco. sols. I get $u_1 = u_2$. $\blacksquare$

AN EXISTENCE THEOREM for (MFG). Assume

(F2) : $\exists\, C > 0$: $\quad \forall \mu \in \mathbb{P}(\mathbb{T}^d)$

$$\sup_x \left(|F(x,\mu)| + |D_x F(x,\mu)| + |D^2_{xx} F(x,\mu)| \right) \le C$$

Ex. $F(x,\mu) = \int_{\mathbb{T}^d} K(x-y)\, d\mu(y)$ $\quad K \in C^2(\mathbb{T}^d)$

$$|F| \le \sup|K| \underbrace{\int_{\mathbb{T}^d} d\mu(y)}_{=1} = \|K\|_\infty \qquad \forall \mu$$

$$D_x F(x,\mu) = \int_{\mathbb{T}^d} DK(x-y)\, d\mu(y) \Rightarrow |D_x F(x,\mu)| \le \|DK\|_\infty$$

$$D^2_{xx} F(x,\mu) = \int_{\mathbb{T}^d} D^2 K(x-y)\, d\mu(y) \Rightarrow |D^2_{xx} F(\;)| \le \|D^2 K\|_\infty .$$

$\Rightarrow$ F sats. (F2) $\quad \boxed{\S}$

Take (MFG) with $H(p) = \frac{|p|^2}{2}$ $\Rightarrow DH(p) = p$

(MFG) becomes

$$(\text{MFG}') \begin{cases} -u_t + \frac{|Du|^2}{2} = F(x,m) \\ m_t - \operatorname{div}(m\, Du) = 0 \\ m|_{t=0} = m_0, \quad u|_{t=T} = g \end{cases}$$

Thm. Ass. (F) & (F2), $m_0 \in L^\infty(\mathbb{T}^d)$, $g \in C^2(\mathbb{T}^d)$ ($\Rightarrow Dg$ & $D^2 g$ bdd)

$\Rightarrow$ $\exists$ a solution to (MFG').

Outline of the proof:

Step 1 $\mathcal{C} := \{ m \in C([0,T], \mathcal{P}(\mathbb{T}^d)) : m(0) = m_0 \}$

it is $\subseteq$ Banach space $C([0,T], \mathcal{M}(\mathbb{T})^d)$ with sup-norm

signed measures

& it is CONVEX.

Step 2 For $m \in \mathcal{C}$ solve

(HJ) $\begin{cases} -u_t + \frac{|Du|^2}{2} = F(x, m(t)) \\ u(\cdot, T) = g \end{cases}$ has a visc. sol.n

$f(x,t)$ cont.

$\Psi_1 : m \mapsto u$

Step. 3 Given u find μ $\Psi_2 : u \mapsto \mu$

(CE) $\begin{cases} \mu_t - \operatorname{div}(\mu Du) = 0 \\ \mu(x, 0) = m_0 \end{cases}$

HARDEST STEP.

Step. 4 $\mu \in \mathcal{C} \Rightarrow \Psi := \Psi_2 \circ \Psi_1 : \mathcal{C} \to \mathcal{C}$

a fixed point of $\Psi \Rightarrow$ sol. of (MFG)

Step 5 Use Schauder to get $\exists$ of fixed point.