LECTURE 22, May 30, 2023

MEAN-FIELD GAMES.

PROBLEM: Hard to study NASH EQUILIBRIA with MANY players.

If we look for equil. in feedback form via PDEs: state $\longrightarrow$ d state dim. for each agent, N agents. Full state: dim Nd

system of N PDES, each in $\mathbb{R}^{Nd}$

Examples
- financial markets $N \sim 10^6$
- ENERGY distribution. $N \sim 10^5$
- CROWD MOTION $N \sim 10^2$

Solving nonlinear PDES in Nd dim is NOT FEASIBLE if $Nd > 10$.

Idea. 2006 J.M. Lasry, P.L. Lions

2005-6 P. Caines, M. Huang, R. Malhamé.

Supp.: "agents are similar", and they interact only via the costs via the EMPIRICAL MEASURE of the other players $(\neq j)$: $\frac{1}{N-1} \sum_{i \neq j} \delta_{x_i}$

Similarity with MEAN-FIELD THEORIES in PHYSICS:

look for simpler MACROSCOPIC DESCRIPTION of complex phenomena instead MICROSCOPIC description.

<u>Q.</u>: How describe population of particles or rational agents?

<u>Answ.</u>: by its distribution in the state space:

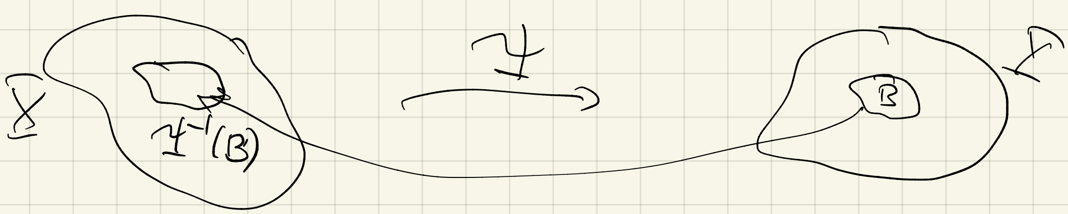
$\{$amount of agents in $B \subseteq \mathbb{R}^d$ at time $t\} = \mu_t(B) =$

$$= \int_B d\mu_t(x) \underset{\Uparrow}{=} \int_B m(x,t)\,dx$$

if μ_t has a density $m(\cdot,t)$

<u>Q</u>: How does μ_t evolve in time?

<u>Def</u>: $\Psi : X \to Y$, X, Y metric, $\mu \in \mathbb{P}(X)$



the PUSH-FORWARD measure of μ via Ψ is: $\forall B \subseteq Y$ Borel

$$(\#) \qquad \Psi\#\mu(B) := \mu(\Psi^{-1}(B))$$

$$\chi_Z(y) := \begin{cases} 1 & y \in Z \\ 0 & y \notin Z \end{cases}$$

<u>Rmk</u> $(\#) \iff \int_Y \chi_B(y)\,d(\Psi\#\mu)(y) := \int_X \chi_{\Psi^{-1}(B)}(x)\,d\mu(x)$

$$= \int_X \chi_B(\Psi(x))\,d\mu(x)$$

Any $g : Y \to \mathbb{R}$ measurable can be approximated by "simple functions" $\Rightarrow$

$$\int_Y g(y)\,d(\Psi\#\mu)(y) = \int_X g(\Psi(x))\,d\mu(x). \qquad (*)$$

<u>Q</u>: How does μ_t evolve if each agent follows
$\dot{y}(s) = f(y(s), s)$ (DS) ?

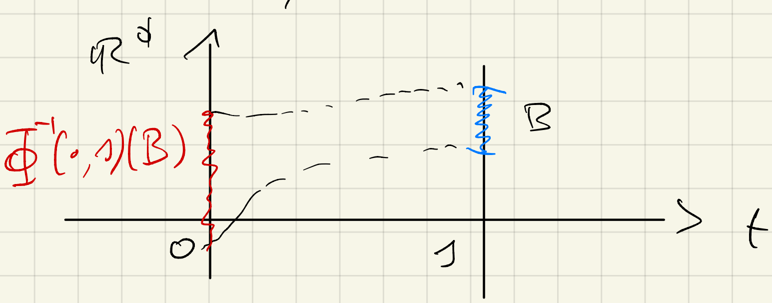
<u>Def</u>. Φ = flow ass. to (DS) i.e.
$\Phi(x,s)$ = sol. at time s of $\begin{cases} \dot{y} = f(y,s) \\ y(0) = x \end{cases}$

$\Rightarrow \dfrac{\partial \Phi}{\partial s} = f(\Phi(x,s), s)$, $\Phi(x,0) = x \quad \forall x$

<u>N.B.</u> f Lip in y, meas. in s $\overset{\text{Thm. } \exists!}{\Rightarrow}$ $x \mapsto \Phi(x,s)$ is BIJECTIVE

its inverse is $\Phi^{-1}(\cdot, s)$

<u>Def</u>: Given $\mu_0 \in \mathbb{P}(\mathbb{R}^d)$, $\mu_s := \Phi(\cdot, s)\#\mu_0$ is push forward of μ_0 by Φ (or by f ....)



Want to derive an equation for μ_s.

Take $\varphi: \mathbb{R}^d \times [0,T] \to \mathbb{R}$ meas.le (test) fn. <u>Ass.</u> $\varphi \in C^1$

$$(*) \Rightarrow \underbrace{\int_{\mathbb{R}^d} \psi(y,s)\, d\mu_s(y)}_{\Phi \# \mu_0} \overset{(+)}{=} \int_{\mathbb{R}^d} \psi(\Phi(x,s),s)\, d\mu_0(x)$$

$$\mu_s = \Phi(\cdot, s) \# \mu_0$$

$$\boxed{\frac{d}{ds} \int_{\mathbb{R}^d} \psi(y,s)\, d\mu_s(y)} = \int_{\mathbb{R}^d} \frac{d}{ds} \psi(\Phi(x,s),s)\, d\mu_0(x) =$$

$$= \int_{\mathbb{R}^d} \Big(\frac{\partial \psi}{\partial s} + \nabla_x \psi \cdot \underbrace{\Phi_s}_{f}\Big)(\Phi(x,s),s)\, d\mu_0(x) \overset{(+)}{=}$$

$$\boxed{= \int_{\mathbb{R}^d} \big(\psi_s + D_x \psi \cdot f\big)(y,s)\, d\mu_s(y)}$$

Now assume supp ψ compact in $\mathbb{R}^d \times [0,T)$

$$\int_0^T \boxed{} \Rightarrow$$

$$\int_{\mathbb{R}^d} \psi(y,s)\, d\mu_s(y) \Big|_{s=0}^{s=T} = -\int_{\mathbb{R}^d} \psi(y,s)\, d\mu_0(y) =$$

$$= \int_0^T \int_{\mathbb{R}^d} (\psi_s + D_x \psi \cdot f)(y,s)\, d\mu_s(y) \qquad \forall \psi \in C^1_c(\mathbb{R}^d \times [0,T))$$

(WCE)

Def. (WCE) is weak form (distributional) of the CONTINUITY EQUATION

$$(CE) \qquad \frac{\partial m}{\partial s} + \operatorname{div}_x (m f) = 0$$

with initial condition μ_0.

We proved:

<u>Lemma</u>: Push-forward $\mu_s = \Phi(\cdot, s) \# \mu_0$ of $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ via f satisfies (WCE).

Motivation of the name "continuity eq.":

Supp. μ_s sol. of (WCE) has density:

$$d\mu_0(x) = m_0(x)\,dx \quad , \quad d\mu_s(x) = m(x,s)\,dx \qquad \&$$

$$m \in C^1(\mathbb{R}^d \times (0,T)) \cap C(\mathbb{R}^d \times [0,T]).$$

<u>CLAIM</u>: m solves (CE). (WCE) is

$$0 = \int_{\mathbb{R}^d} \varphi(y,0) m_0(y)\,dy + \underbrace{\int_0^T \int_{\mathbb{R}^d} (\varphi_s + D_x\varphi \cdot f) m(y,s)\,dy\,ds}_{=(I)}$$

$$\int_0^T (\varphi_s m)(y,s)\,ds = \varphi m \Big|_0^T - \int_0^T \varphi m_s\,ds =$$

Integrate by parts $\rightarrow$ $= -\varphi(y,0)m_0(y) - \int_0^T (\varphi m_s)(y,s)\,ds$

$$\operatorname{div}_x(m\varphi f) = \operatorname{div}_x(mf)\varphi + D_x\varphi \cdot (mf)$$

$$(I) = \int_0^T \int_{\mathbb{R}^d} \Big[\big(-m_s - \operatorname{div}_x(mf)\big)\varphi + \operatorname{div}_x(m\varphi f)\Big]\,dy\,ds$$

$$\int_{\mathbb{R}^d} \operatorname{div}_x(m\varphi f)\,dy \overset{\text{Gauss Th.}}{=} \int_{\partial \operatorname{supp}\varphi} m\varphi f \cdot \nu\,d\sigma = 0$$

ν ext. normal.

$$\Rightarrow \quad 0 = \int_0^T (m_s + \operatorname{div}_x(mf))\varphi\,dy\,ds \qquad \forall \varphi$$

By the arbitrariness of φ $\Rightarrow$ $m_s + \mathrm{div}_x (m f) = 0$ in $\mathbb{R}^d$.

Viceversa (HW) If m solves (CE) & attains initial data, then $d\mu_s = m(\cdot, s)\, dx$ satisfies (WCE)

A HEURISTIC DERIVATION of the MFG (mean field game) System of PDEs.

Ref. • Cardaliaguet LN 2013 (also with Porretta...)
• P.L. Lions lectures at College de France.

Take a population of agents with dynamics

$$\dot{y}(s) = -a(s), \quad a(s) \in \mathbb{R}^d, \quad y(t) = x$$

Cost functional of the generic agent :

$$J(t, x, a(\cdot)) := \int_0^T \Big(L(a(s)) + F(y(s), s) \Big) ds + g(y(T))$$

Ass. $\lim_{|a| \to +\infty} \dfrac{L(a)}{|a|} = +\infty$, L convex, $H = L^*$ convex conj.

HJB associated is

$$\text{(HJB)} \quad \begin{cases} -u_t + H(D_x u) = F(x, t) & \text{in } \mathbb{R}^d \times (0, T) \\ u(x, T) = g(x) \end{cases}$$

Lemma (Verif. thm. $\approx$ Cor. in Lect. 5)

If $u \in C^1$ solves (HJB), $L \in C^1$, DL invertible, $(DL)^{-1} \in C^1$

$\Rightarrow$ $DH(D_x u(x,t))$ is an optimal feedback, i.e.,

$$\begin{cases} \dot{y}(s) = \underbrace{-DH(D_x u(y(s),s))}_{=: \alpha^*(s)} & s > t \\ y(t) = x \end{cases}$$

has a sol. & $\alpha^*(\cdot)$ minimizes $J(t, x, \cdot)$.

Consequence of (CE) & (HJB): If all agents have the same cost $L + F$, are all "rational", & $\exists u \in C^1$ sol of (HJB), then the distribution of the population is a weak solut. of

$$\begin{cases} m_t - \mathrm{div}_x(m\, DH(D_x u)) = 0 \\ m(x,0) = m_0(x). \end{cases}$$

Now supp. for N players, the cost of the N-th depends also on the empirical measure

$$\mu^N(s) = \frac{1}{N-1} \sum_{i=1}^{N-1} \delta_{y^i(s)}$$

δ = Dirac.
$y^i(s)$ = position of i-th player,

$F : \mathbb{R}^d \times \mathbb{P}(\mathbb{R}^d) \to \mathbb{R}$
$(y^N(s), \mu^N(s))$

. The value fun v^N of the N-th player solves

$$\begin{cases} -\frac{\partial v^N}{\partial t} + H(D_x v^N) = F(x, \mu^N(t)) \\ v^N(x,T) = g(x). \end{cases}$$

Supp. all players behave OPTIMALLY, all equal to the Nth. Then $\delta_{y^i(s)}$ is a weak sol. of

$$\begin{cases} \mu_t - \operatorname{div}(\mu DH(Dv^N)) = 0 \\ \mu_0 = \delta_{x^i} \end{cases}$$

, then

$$\begin{cases} \dfrac{\partial \mu^N}{\partial t} - \operatorname{div}\left(\mu^N DH(Dv^N)\right) = 0 \\ \mu^N(x,0) = \mu_0(x) \end{cases}$$

Assume as $N \to \infty$ $\mu^N_t \to \mu_t$ $\forall t$, μ_t has a density $m(\cdot,t)$, $v^N \to u$ with all derivatives, F cont.

I "expect":

$$(MFG) \quad \begin{cases} -u_t + H(D_x u) = F(x, m) & \text{in } \mathbb{R}^d \times (0,T) \\ m_t - \operatorname{div}(m\, DH(D_x u)) = 0 & \text{''} \\ m(x,0) = m_0(x), \quad u(x,T) = g(x). \end{cases}$$

N.B. system of
- backward HJB eq.
- forward cont. eq.

A rigorous proof of the limit $N \to \infty$ is not known yet.

Can give a GAME-THEORETIC justification of (MFG) as describing a Nash-type equilibrium.

Def. A pair (μ, u), $\mu : [0,T] \to \mathbb{P}(\mathbb{R}^d)$, $u : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ is a MFG equil. if:

- u is the value fn. of the opt. contr. probl. with cost $L(\alpha(s)) + F(y(s), \mu_s)$
- μ is the distrib. of a population of players all following the feedback $DH(D_x u)$, optimal for previous control. pb.

" For an agent it is NOT CONVENIENT to deviate from $DH(D_x u)$ if the rest of the population does not deviate ".