LECTURE 21, May 23, 23

LAST LECTURES of the course:

May 30 at 10.30, 31 at 12.30, June 1 at 8.30

all in 2AB40.

0-sum differential games with dynamics

$$\begin{cases} \dot{y} = f(y, a, b) \\ y(t) = x \end{cases}, \quad f \text{ bdd., Lip. in } x \text{ unif. in } a, b$$

payoff-cost $\quad J = \int_t^T \ell(y, a, b)\, ds + g(y(T))$

ℓ, g bdd. & Lip. $\quad a \in A, \; b \in B$ compact.

lower value $\quad V(t,x) = \inf_{\beta \in \Delta_t} \sup_{a \in \mathcal{O}_t} J(t, x, a, \beta[a])$

upper " $\quad \mathcal{V}(t,x) = \sup_{\alpha \in \Gamma_t} \inf_{b \in \mathcal{B}_t} J(t, x, \alpha[b], b)$

Isaacs' Hamiltonians:

$$H^+(x,p) := \min_{b \in B} \max_{a \in A} \{ p \cdot f(x, a, b) + \ell(x, a, b) \}$$

$$H^-(x,p) := \max_a \min_b \{ \text{same} \} \qquad T > 0$$

Theorem (Evans-Souganidis '84) Under the assumptions on A, B, f, ℓ, g:

a) the upper value $\mathcal{V}$ is the UNIQUE VISCOSITY solution of

$$(I^+)\begin{cases} -\left(\dfrac{\partial v}{\partial t} + H^+(x, D_x v)\right) = 0 & \text{in }]0,T[\times\mathbb{R}^n \\ v(T,x) = g(x) \end{cases}$$

b) the lower value V is the UNIQUE v.s. sol. in $BUC(\bar\Omega)$ of

$$(I^-)\begin{cases} -\left(\dfrac{\partial V}{\partial t} + H^-(x, D_x V)\right) = 0 & \text{in }]0,T[\times\mathbb{R}^n =: \Omega \\ V(T,x) = g(x) \end{cases}$$

Proof 1. $v(T,x) = g(x) = V(T,x)$ $\forall x$ because $y_x(T) = x$

2. Comparison Principle holds for $F = H^-$ or H^+

$$(HJ) \to \begin{cases} -(W_t + F(x, D_x W)) = 0 \\ W(T,x) = g(x) \end{cases}$$

because H^- & H^+ satisfy (L_x) & (L_p). Then $u, v \in BUC(\bar\Omega)$ resp. sub. & supersol. of (HJ), $u \le v$ at $t=T$, then $u \le v$ in $\bar\Omega$.

v, V are bdd. & Lip. in $\bar\Omega$ $\Rightarrow$ each, if it solves (I^+) or (I^-), it is the unique solution. ∎

3. Are v & V sols.? Need

Technical Lemma. $\Lambda \in C(A\times B)$, $\sigma > 0$, $t_0 \in \mathbb{R}$ $\Rightarrow$

$$\underbrace{\max_{a\in A}\min_{b\in B}\Lambda(a,b)}_{=:\, \lambda} = \underbrace{\inf_{\beta\in\Delta_{t_0}}\sup_{\alpha\in\mathcal{A}_{t_0}} \frac{1}{\sigma}\int_{t_0}^{t_0+\sigma}\Lambda(a(s), \beta[a](s))\,ds}_{=:\, \theta}$$

Pf. $d \le v$ is easy & don't need it!

We need $d \ge v$. Idea $\varphi(a) = \arg\min_{b \in B} \Lambda(a,b)$, so

$$\Lambda(a, \varphi(a)) = \min_b \Lambda(a,b) \quad \forall a$$

Def β^* strat. for b: $\beta^*[a](s) := \varphi(a(s)) \quad \forall a \in \mathcal{A}_{t_0}$

If $\beta^* \in \Delta_{t_0}$:

$$v \le \sup_{a \in \mathcal{A}_{t_0}} \frac{1}{\sigma} \int_{t_0}^{t_0+\sigma} \underbrace{\Lambda(a(s), \beta^*[a](s))}_{\Lambda(a(s), \varphi(a(s)))} ds$$

$$\Lambda(a(s), \varphi(a(s))) = \min_{b \in B} \Lambda(a(s), b)$$

$$\le \max_{a \in A} \min_{b \in B} \Lambda(a,b) = d \qquad \forall a \in \mathcal{A}_{t_0}$$

$$\Rightarrow \quad v \le \frac{1}{\sigma} \int_{t_0}^{t_0+\sigma} d \, ds = d$$

Q: Does $\beta^* \in \Delta_{t_0}$? NONANTICIP. is OK

but $s \mapsto \beta^*[a](s) = \varphi(a(s))$ is MEASURABLE ?

VARIANT: CLAIM: $\forall \varepsilon > 0 \quad \exists \varphi = \varphi_\varepsilon : A \to B$

MEASURABLE:

$$\Lambda(a, \varphi_\varepsilon(a)) \le \min_b \Lambda(a,b) + \varepsilon \quad \forall a \in A.$$

then $\beta^*[a](s) := \varphi_\varepsilon(a(s)) \Rightarrow \beta^* \in \Delta_{t_0}$

Previous argument $\Rightarrow$ $v \le d + \varepsilon$. $\varepsilon \searrow 0+$ ∎

Pf of CLAIM: see Notes on [E-S]. ∎ T. Lang

Proof of thm.: For simplic. $\ell \equiv 0$, so

$$(I^-) \quad -\left(V_t + \max_{a\in A} \min_{b\in B} f(x,a,b)\cdot\nabla_x V\right) = 0 \quad \text{in } \Omega$$

$$(DPP) \quad V(t,x) = \inf_{\beta\in\Delta_t} \sup_{a\in\mathcal{A}_t} V(t+\sigma, y_x(t+\sigma; t, a, \beta[a]))$$

$$\forall\ 0\le t < t+\sigma \le T$$

I'll prove only V SUBSOLUTION.

GOAL $\forall \phi\in C^1(\Omega)$, $(t_0,x_0)\in\Omega$ max point of $V-\phi$:

$$-\left(\phi_t + \max_b \min_a f(x,a,b)\cdot D_x\phi\right)\Big|_{(t_0,x_0)} \le 0$$

By contradiction: $\exists\ \delta>0$ ... $\geq \delta$ i.e.

$$\phi_t(t_0,x_0) + \max_b \min_a \underbrace{f(x_0,a,b)\cdot D_x\phi(t_0,x_0)}_{\Lambda(a,b) := } \le -\delta$$

Use technical lemma.

$$= \phi_t(t_0,x_0) + \inf_{\beta\in\Delta_{t_0}} \sup_{a\in\mathcal{A}_{t_0}} \frac{1}{\sigma}\int_{t_0}^{t_0+\sigma} f(x_0, a(s), \beta[a](s))\cdot D_x\phi(t_0,x_0)\,ds$$

$(\le -\delta)$

Consider $y(s) := y_{x_0}(s; t_0, a, \beta[a])$. Recall

$|y(s)-x_0| \le C_1(s-t_0)$. Put $(s, y(s))$ in the place of (t_0, x_0) in $\int_{t_0}^{t_0+\sigma}$, use cont. fy of f, ϕ_t, $\nabla\phi$, for $\sigma>0$ small.

$$\inf_{\beta\in\Delta_{t_0}} \sup_{a\in\mathcal{A}_{t_0}} \frac{1}{\sigma}\int_{t_0}^{t+\sigma}\left[\phi_t(s,y(s)) + f(y(s), a(s), \beta[a](s))\cdot D_x\phi(s,y(s))\right]ds \le -\frac{\delta}{2}$$

$$[\dots] = \frac{d}{ds}\phi(s, y(s)) \quad \text{a.e. } s.$$

Use Fund. Thm of Calc. $\Rightarrow$

$$\inf_{\beta \in \Delta_{t_0}} \sup_{a \in \mathcal{A}_{t_0}} \left(\phi(t_0+\sigma, y(t_0,\sigma)) - \phi(t_0, x) \right) \leq -\frac{\delta\sigma}{2}$$

$$(V-\phi)(t_0, x_0) \geq (V-\phi)(s, y(s)) \quad \Rightarrow \qquad s = t_0+\sigma$$

$$V(t_0+\sigma, y(t_0+\sigma)) - V(t_0, x_0) \leq \phi(t_0+\sigma, y(t_0+\sigma)) - \phi(t_0, x_0)$$

$$\Rightarrow \underbrace{\inf_{\beta \in \Delta_{t_0}} \sup_{a \in \mathcal{A}_{t_0}} \left(V(t_0+\sigma, y(t_0+\sigma)) - V(t_0, x_0) \right)}_{=0 \text{ by (DPP)}} \leq -\frac{\delta\sigma}{2} < 0 \qquad \times \qquad \blacksquare$$

<u>Corollary</u> In the ass. of the Thm.:

(i) $V(t,x) \leq U(t,x) \quad \forall (t,x) \in \overline{\Omega}$

(ii) If moreover Isaacs cond. holds, i.e.

$H^+ = H^- \quad \forall x, p \quad \Rightarrow V = U$ & the game has a value. $\forall t, x$.

<u>Proof</u>: (i) $\quad H^- \leq H^+ \qquad V$ solves

$$= -V_t - H^-(x, D_x V) \geq -V_t - H^+(x, D_x V)$$

$\Rightarrow V$ is a subsol. of (I^+). Comparison Principle for (I^+) $\Rightarrow V \leq U =$ the sol. of (I^+) $\blacksquare$

(ii) $H^- = H^+ \;\forall x, p \Rightarrow (I^+) = (I^-) \Rightarrow V = U$. $\blacksquare$

<u>Rmk.</u> When does Isaacs' cond. hold? $\quad H^- \overset{?}{=} H^+$

Main example "separated data":

$$f(x,a,b) = f_1(x,a) + f_2(x,b), \quad \ell(x,a,b) = \ell_1(x,a) + \ell_2(x,b)$$

$$\Rightarrow H^- = H^+ = \max_{a\in A} \{f_1(x,a)\cdot p + \ell_1(x,a)\} + \min_{b\in B} \{f_2(x,b)\cdot p + \ell_2(x,b)\}.$$

$\blacksquare$

Last application: What to do if $H^-(x,p) < H^+(x,p)$ for some (x,p)? It can be that $V(t,x) < \bar{V}(t,x)$ some t,x.

Use MIXED STRATEGIES: Extend f, ℓ to $\mathbb{P}(A)$, $\mathbb{P}(B)$

$\mu \in \mathbb{P}(A)$, $\nu \in \mathbb{P}(B)$ $\quad \tilde{f}(x,\mu,\nu) := \iint_{A\times B} f(x,a,b)\, d\mu(a)\, d\nu(b)$,

$$\tilde{\ell}(x,\mu,\nu) := \iint_{A\times B} \ell(x,a,b)\, d\mu\, d\nu. \qquad \underline{N.B.}\ \tilde{f}(x,\delta_a,\delta_b) = f(x,a,b)$$

$\underline{\text{Lemma}}$. $\tilde{f}$ & $\tilde{\ell}$ satisfy same ass. as f, ℓ. (i.e. bdd, Lip in x unif in μ,ν, jointly cont.)

$\underline{\text{Pf}}$ HW (see later). $\blacksquare$

$\underline{\text{Recall}}$ A, B compact $\Rightarrow$ $\mathbb{P}(A)$, $\mathbb{P}(B)$ compact w.r.t. $\xrightarrow{*}$ convergence.

The diff. game with data $\mathbb{P}(A)$, $\mathbb{P}(B)$, $\tilde{f}$, $\tilde{\ell}$, g

sats the ass. of Ev.-Soug. thm., for

$$\tilde{J}(t,x,\mu(\cdot),\nu(\cdot)) := \int_t^T \tilde{\ell}(\tilde{y}(s),\mu(s),\nu(s))\, ds + g(\tilde{y}(T)).$$

$$\dot{\tilde{y}}(s) = \tilde{f}(\tilde{y},\mu,\nu)$$

$$\tilde{V}(t,x) = \inf_{\beta\in\tilde{\Delta}_t} \sup_{\alpha\in\tilde{\alpha}_t} \tilde{J}(t,x,\mu(\cdot),\beta[\mu](\cdot))$$

$\leftarrow$ RELAXED Controls

is the UNIQUE sol of $(\tilde{I}^-)$ where

$$\tilde{H}^-(x,p) = \max_{\mu \in \mathbb{P}(A)} \min_{\nu \in \mathbb{P}(B)} \{ p \cdot \tilde{f}(x,\mu,\nu) + \tilde{\ell}(x,\mu,\nu) \}.$$

& $\tilde{u} = \dots \quad \tilde{H}^+ \dots\dots \quad \underline{Q}: \tilde{V} = \tilde{U}$??

<u>Remark</u> SINGLE PLAYER. (A = singleton)

$$\tilde{V}(t,x) = \inf_{\nu(\cdot) \in \tilde{\mathcal{B}}_t} \tilde{J}(t,x;\nu(\cdot)).$$

problem with RELAXED CONTROLS or CHATTERING "

Motivation of REL. controls: can prove ∃ of OPTIMAL ONE (open loop). [Fleming–Rishel].

<u>Prop.</u> $\tilde{V}(t,x) = V(t,x) := \inf_{a(\cdot) \in \mathcal{B}_t} J(t,x;a(\cdot))$.

<u>Pf.</u> HW: $H = \tilde{H}$ = Ham. with relaxed controls. ∎

Answer to Q: DIFF. GAMES in MIXED STRATEGIES.

<u>Def</u> $\tilde{\mathcal{a}}_t, \tilde{\mathcal{B}}_t, \tilde{\Delta}_t, \tilde{\Gamma}_t$ as before with $\mathbb{P}(A), \mathbb{P}(B)$, instead of A, B. $\tilde{V} = \dots$

$$\tilde{U}(t,x) = \sup_{\alpha \in \tilde{\Gamma}_t} \inf_{\nu \in \tilde{\mathcal{B}}_t} \tilde{J}(t,x,\alpha[\nu],\nu)$$

<u>Cor.</u> In the standing ass. $\tilde{V}(t,x) = \tilde{U}(t,x) \ \forall (t,x) \in \Omega$, i.e., the diff. game has a value in MIXED STRATEGIES.

Pf. Goal: $\tilde{H}^+ = \tilde{H}^-$, then can use UNIQUENESS of sol. to $(\tilde{I}^+) = (\tilde{I}^-)$. $F(\mu,\nu) = p \cdot \tilde{f}(x,\mu,\nu) + \tilde{\ell}(x,\mu,\nu)$.

$$\tilde{H}^+(x,p) = \min_{\nu \in \mathbb{P}(B)} \max_{\mu \in \mathbb{P}(A)} F(\mu,\nu)$$

Want to use Von-Neumann thm.

$$\tilde{H}^-(x,p) = \max_{\mu} \min_{\nu} F$$

Check ass.: $\mathbb{P}(A)$, $\mathbb{P}(B)$ ok, F concave-convex?

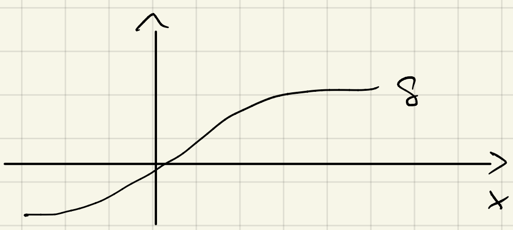
$$F(\mu,\nu) = \iint_{A \times B} [f(x,a,b) \cdot p + \ell(x,a,b)] \, d\mu(a) \, d\nu(b)$$

is BILINEAR $\Rightarrow$ CONCAVE in μ, CONVEX in ν.

V.N. Thm $\Rightarrow H^+ = H^- \Rightarrow \tilde{V} = \tilde{v}$ by E-S. thm. $\blacksquare$

Example (Exerc. HW). Game without a value:

$\dot{y} = (a-b)^2$ in $\mathbb{R}$, $\ell \equiv 0$, $A, B = \{0,1\}$, $g \in C^1$, $g' > 0$



HW: • $H^-(p)$, $H^+(p)$

• Check $V(t,x) = g(x)$, ($\leq$ known)

$v(t,x) = g(x+T-t)$ ($\geq$ known)

• $\tilde{H}(p) = ?$

• $\tilde{V}(t,x) = \tilde{v}(t,x) = g\left(x + \frac{T-t}{2}\right)$ $\blacksquare$

END of 0-SUM.