Knowledge Representation and Learning

9. Model counting in First Order Logic

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Model Counting in First Order Logic

- Is it possible to count the model of a first order formula?
- Without further specifications the answer is: a formula has 0 models, if it is not satisfiable, and it has an infinite number of models, it is satisfiable;
- indeed if a formula has one model with domain $\Delta_{\mathcal{I}}$ then it has an infinite set of models with domains isomorphic to $\Delta_{\mathcal{I}}$.
- What if we change our question in:
- is it possible to count the models of a first order formula, on a domain of a given finite, or countable cardinality?
- This question makes more sense and can be answered.
- We consider the special case in which the cardinality of the domain is finite, i.e., a natural number n.

First order model counting

Definition (First order model counting)

The problem of first order model counting is the problem of computing the number of Σ-interpretations that satisfies a first order sentence ϕ on a given finite domain of $n > 0$ elements. The problem is denoted as

$$\text{FOMC}(\phi, n)$$

Counting problems in FOL

- the number of undirected graphs with n nodes $\textsc{fomc}(UG, n)$

$$UG \triangleq \forall x \forall y (\neg R(x,x) \wedge (R(x,y) \leftrightarrow R(y,x)))$$

- the number of 3-colored undirected graphs with n nodes $\textsc{fomc}(UG \wedge 3C, n)$

$$3C \triangleq \forall xy((C_1(c) \veebar C_2(x) \veebar C_3(x)) \wedge R(x,y) \rightarrow \bigwedge_{i=1}^{3} (\neg C_i(x) \wedge C_i(y)))$$

- the number of graphs with n vertexes every pair of nodes are connected with a path with length $\leq k$. $\textsc{fomc}(UG \wedge R_{\leq k}, n)$

$$R_{\leq k} \triangleq \forall x \forall y ((\bigvee_{i=1}^{k} R_k(x,y)) \wedge (R_1(x,y) \leftrightarrow R(x,y)) \wedge \bigwedge_{i=1}^{k-1} (R_{i+1}(x,y) \leftrightarrow \exists z (R_i(x,z) \wedge R(z,y))))$$

$R_i(x,y)$, x is connected with y with a path of lenght i

Formulating counting problems in FOL

Example

Count the possible configuraiton of a group of n people composed of Ph.D students and professor knowling that every student has a supervisor, every professor supervises at leas one student.

$$\textsc{fomc}(SP, n)$$

$$SP = \left\{ \begin{array}{l} Prof(x) \veebar Stud(x) \\ super(x,y) \rightarrow Stud(x) \wedge Prof(y) \\ Stud(x) \rightarrow \exists y (Prof(y) \wedge super(x,y)) \end{array} \right\}$$

First order model counting

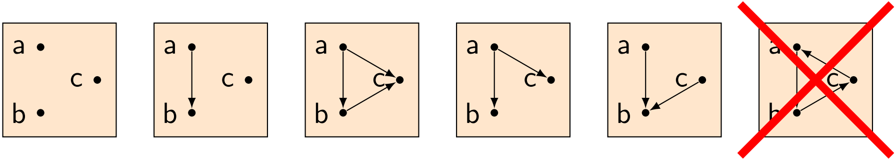
Example

Consider the definition of partial order. A binary relation, R, on a set A is a strict partial order if

- $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$
- if $(a, b) \in R$ then $(b, a) \notin R$

To determine how many partial orders can be defined on a set of n elements we can solve the problem:

$$\text{FOMC}\left(\begin{array}{c} \forall x, y, z.(R(x, y) \wedge R(y, z) \rightarrow R(x, z)) \\ \wedge \forall x, y.(R(x, y) \rightarrow \neg R(y, x)) \end{array}, n\right)$$



Number of partial orders on a set of n elements

n	Number of partial orders on a set of n elements
0	1
1	1
2	4
3	29
4	355
5	6942
6	209527
7	9535241
8	642779354
9	63260289423
10	8977053873043
11	1816846038736192
12	519355571065774021
13	207881393656668953041
14	115617051977054267807460
15	88736269118586244492485121
16	93411113411710039565210494095
17	134137950093337880672321868725846
18	261492535743634374805066126901117203

Taken from The On-Line Encyclopedia of Integer Sequences (OEIS)

Example

The number of total relations on a set of n elements can be computed by a formula:

$$\textsc{fomc}(\forall x \exists y.R(x, y), n) = (2^n - 1)^n$$

Proof.

- For every node x we can select a non empty subset S of $\{1, \ldots, n\}$ such that $R(x, y)$ is true if $y \in S$.
- the number of non empty subsets of n elements are $2^n - 1$.
- since there are n nodes. we have $(2^n - 1)^n$ possibilities.

□

First order model counting - systematic solution

- in all the previous example we use different methodologies to count the models;
- the strategies that we as humans use cannot be easily implemented in a algorithms;
- the question is: is there a systematic way to compute $\textsc{fomc}(\phi, n)$

First order model counting via grounding

Since we are dealing which finite domains we can reduce first order formulas to equivalent propositional formulas by grounding quantifiers:

Grounding

$Ground(\phi, C)$ for every formula ϕ and set of constants C is defined as follows:

- $Ground(\phi, C) = \phi$ if ϕ does not contain quantifiers;
- $Ground(\forall x.\phi(x), C) = \bigwedge_{c \in C} Ground(\phi(c), C)$
- $Ground(\exists x.\phi(x), C) = \bigvee_{c \in C} Ground(\phi(c), C)$
- $Ground(\phi \circ \psi, C) = Ground(\phi, C) \circ Ground(\phi, C)$ for every connective $\circ$
- $Ground((\neg\phi, C) = \neg Ground(\phi, C)$

First order model counting via grounding

Example

$Ground(\forall x(A(x) \rightarrow \exists y(R(x,y) \wedge B(y))), \{a, b\})$

$$
\begin{array}{l}
A(a) \rightarrow (R(a,a) \wedge B(a)) \vee (R(a,b) \wedge B(b)) \ \wedge \\
A(b) \rightarrow (R(b,a) \wedge B(a)) \vee (R(b,b) \wedge B(b))
\end{array}
$$

Example

$Ground(\forall x, y.(R(x,y) \rightarrow R(y,x)), C) =$

$$
\bigwedge_{c \in C} \bigwedge_{c' \in C} R(c, c') \rightarrow R(c', c)
$$

First order model counting via grounding

Proposition

If ϕ is a first order sentence with no constant and function symbols

$$\text{FOMC}(\phi, n) = \#\text{SAT}(ground(\phi, \{c_1, \dots, c_n\}))$$

Proof Outline.

For every model $\mathcal{I}$ of ϕ on the domian of $\{1, \dots, n\}$ We define the following bijection:

$$\mathcal{I}_{FOL} \models p(x_1, dots, x_n)[a_{x_1 \leftarrow d_1, \dots, x_n \leftarrow d_n}] \text{ iff } \mathcal{I}_{PROP}(p(c_{d_1}, \dots, c_{d_n})) = 1$$

One can easily show that this mapping is an isomorphism between the set fo FOL interrpetatins on $\{1, \dots, n\}$ and the propositional assignment $\mathcal{I}_{PROP}$ and thast $\mathcal{I}_{FOL} \models \phi$ if and only if $\mathcal{I}_{PROP} \models ground(\phi, \{c_1 \dots, c_n\})$ □

Size of $Ground(\phi, C)$

- If ϕ contains an n ary predicate $R(x_1, \dots, x_k)$ then $\mathrm{Ground}(\phi, C)$ contains $|C|^k$ propositional variables.
- For instance $Ground(\forall x(A(x) \rightarrow \exists y.(R(x,y) \wedge B(y))), C)$ contains $2|C| + |C|^2$ propositional variables.
  - $|C|$ variables of the form $A(c)$;
  - $|C|$ variables of the form $B(c)$;
  - $|C|^2$ variables of the form $R(c, c')$.
- with 10 constant whe have 120 variables, i.e., 2^{120} models.

fomc via grounding

$$\textsc{fomc}(\phi, n) = \#\textsc{sat}(Ground(\phi, \{1, \dots, n\}))$$

The complexity of $fomc(\phi, n)$ grows exponential w.r.t, the size of the domain n. In practice it is hard to go beyond 10 objects even with simple formulas.

Lifrability

- is there a method to compute $\text{FOMC}(\phi, n)$ such that the complexity is not explonential in n?
- if such a method exists we will say that the problem of $\text{FOMC}(\phi, n)$ for ϕ is liftable.
- A method to achieve liftability is to exploit symmetries of First Order Formula.

The language $\mathcal{L}^2$

Definition

For every $k \geq 1$ the language $\mathcal{L}^k$ contains all the first order formulas that can be build using only k individual variables.

Example

The following are formulas of $\mathcal{L}^2$;

- $\forall x \exists y (R(x, y) \wedge A(x) \wedge B(y) \wedge \neg x = y)$
- $\exists x (A(x) \wedge \forall y (R(x, y) \rightarrow \exists x R(y, x) \wedge B(x))$

Example

$\forall x, y, z. R(x, y) \wedge R(y, z) \rightarrow R(x, z)$ is a formula in $\mathcal{L}^3$, that formalizes the fact that R is a transitive relation. Such a condition cannot be expressed in $\mathcal{L}^2$.

The above example shows that $\mathcal{L}^2$ is less expressive thatn $\mathcal{L}^3$.

Definition (1-type)

Given a FOL signature Σ a 1-type is a conjunction of maximally consistent set of literals containing exactly one variable and no constants.

Example (1-type)

Let $\Sigma = \{A/1, R/2, S/3\}$ (the notation X/n means that X is a predicate with arity equal to n)

$$
\begin{array}{ll}
A(x) \wedge R(x,x) \wedge S(x,x,x) & A(x) \wedge R(x,x) \wedge \neg S(x,x,x) \\
A(x) \wedge \neg R(x,x) \wedge S(x,x,x) & A(x) \wedge \neg R(x,x) \wedge \neg S(x,x,x) \\
\neg A(x) \wedge R(x,x) \wedge S(x,x,x) & \neg A(x) \wedge R(x,x) \wedge \neg S(x,x,x) \\
\neg A(x) \wedge \neg R(x,x) \wedge S(x,x,x) & \neg A(x) \wedge \neg R(x,x) \wedge \neg S(x,x,x)
\end{array}
$$

Proposition

If Σ contains n predicates there are 2^n 1-types.

We use natural number $1(x)$, $2(x)$, ..., $2^n(x)$ to denote the 1-types.

Definition (2-table)

Given a FOL signature Σ a 2-*table* is the conjunction of a maximally consistent set of literals containing exactly two distinct variables x, y and no constants and the literal $x \neq y$.

Example (2-table)

Let $\Sigma = \{A/1, R/2\}$

$$R(x,y) \wedge R(y,x) \wedge x \neq y \qquad R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$
$$\neg R(x,y) \wedge R(y,x) \wedge x \neq y \qquad \neg R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$

Proposition

if Σ contains n_i predicates with arity equal to i, then there are $2^{\sum_i n_i(2^i-2)}$

We use $1(x,y), 2(x,y), \ldots$ to denote 2-table.

Definition (2-type)

Given a FOL signature Σ a 2-*type* is the conjunction of a maximally consistent set of literals containing at most two distinct variables x, y and no constants and the literal $x \neq y$.

Proposition

A 2-type is the conjuction of two 1-types $i(x)$ and $j(y)$ and a 2-table $l(x, y)$.

Example

$\Sigma = \{R/2\}$

$$R(x,,x) \wedge R(y,y) \wedge R(x,y) \wedge R(y,x) \wedge x \neq y$$
$$\neg R(x,,x) \wedge R(y,y) \wedge R(x,y) \wedge R(y,x) \wedge x \neq y$$
$$R(x,,x) \wedge R(y,y) \wedge \neg R(x,y) \wedge R(y,x) \wedge x \neq y$$
$$\dots$$

The two type $i(x) \wedge j(y) \wedge l(x, y)$ is denoted by $ijl(x, y)$.

Realization of types and tables

Given a Σ structure, we say that

- a constant c *realizes* a 1-type i if $\mathcal{I} \models i(c)$
- a pair of constants (c, d) *realizes* a 2-table $l(x, y)$ if $\mathcal{I} \models l(a, b)$
- a pair of constants (c, d) *realizes* a 2-type $ijl(x, y)$ if $\mathcal{I} \models ijl(x, y)$.

Proposition

1. *Every constant realizes a single 1-type;*
2. *Every pair of constants realizes a single 2-table;*
3. *Every pair of constants realizes a single 2-type*

Example

1-types

$1(x) \triangleq R(x,x),$

$2(x) \triangleq \neg R(x,x),$

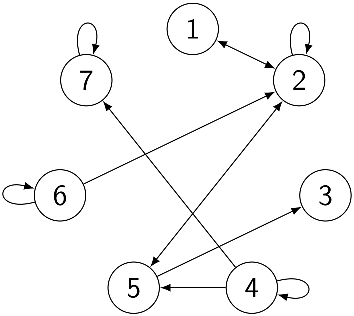
2-tables

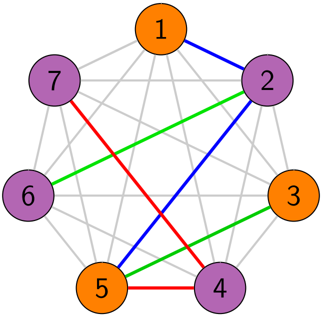
$1(x,y) \triangleq R(x,y) \wedge R(y,x) \wedge x \neq y$

$2(x,y) \triangleq R(x,y) \wedge \neg R(y,x) \wedge x \neq y$

$3(x,y) \triangleq \neg R(x,y) \wedge R(y,x) \wedge x \neq y$

$4(x,y) \triangleq \neg R(x,y) \wedge \neg R(y,x) \wedge x \neq y$





Proposition

Let $\Sigma = \{R_1, \dots, R_k)$ be a FOL signature containing only relational symbols. Every Σ-interpretation on the domain $[n]$ is uniquely described by

1. *partition $[n]$ in u-sets $N_1, \dots, N_u$ one for each one type;*
2. *assign a two table to every ordered pair $c < d \in N_i$;*
3. *assign a two table to every ordered pair $c \in N_i$ and $d \in N_j$ with $i < j$.*

Cardinality vectors $(\boldsymbol{k}, \boldsymbol{h})$

- for every interpretation $\mathcal{I}$ we define the vector $\boldsymbol{k} = (k_1, \dots, k_u)$ such that k_i is the number of elements of $\mathcal{I}$ that realizes the i-th 1-type.
- for every $i \leq j$ we define the vector of integers $\boldsymbol{h}^{ij} = (h^{ij}_1, \dots, h^{ij}_b)$, such that h^{ij}_l contains a pair that realizes the 2-type ijl.

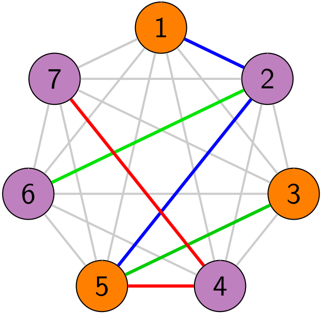
Example

1-types

$$1(x) \triangleq R(x,x),$$
$$2(x) \triangleq \neg R(x,x),$$

2-tables

$$1(x,y) \triangleq R(x,y) \wedge R(y,x) \wedge x \neq y$$
$$2(x,y) \triangleq R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$
$$3(x,y) \triangleq \neg R(x,y) \wedge R(y,x) \wedge x \neq y$$
$$4(x,y) \triangleq \neg R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$



$$\boldsymbol{k} = (4, 3)$$

$$\boldsymbol{h}^{11} = (0, 1, 1, 4)$$
$$\boldsymbol{h}^{12} = (2, 1, 0, 9)$$
$$\boldsymbol{h}^{11} = (0, 0, 1, 2)$$

Facts about cardinality vectores

- $\sum \boldsymbol{k} = \sum_{i=1}^{u} k_i = n$
- $\sum \boldsymbol{h} = \sum_{i \leq j} \sum_{l=1}^{b} k_l^{ij} = \frac{n(n-1)}{2}$
- $\sum \boldsymbol{h}^{ii} = \sum_l h_l^{ii} = \frac{k_i(k_i-1)}{2}$
- $\sum \boldsymbol{h}^{ij} = \sum_l h_l^{ij} = k_i \cdot k_j$ (if $i \neq j$)

for every cardinality vector $(\boldsymbol{k}, \boldsymbol{h})$ there are

$$\binom{n}{\boldsymbol{k}} \prod_i \binom{\frac{k_i(k_i-1}{2}}{\boldsymbol{h}^{ii}} \prod_{i<j} \binom{k_i k_j}{\boldsymbol{h}^{ij}}$$

distinct interpretations that have the cardinality vector $(\boldsymbol{k}, \boldsymbol{h})$ where for every positive integers $a, b_1, \ldots, b_m$ with $\sum_i b_i = a$

$$\binom{a}{b_1, \ldots, b_m} = \frac{a!}{b_1! \cdot b_2! \cdots b_n!}$$

FOMC for pure universal formula in $\mathcal{L}^2$

Definition

a 2-type $ijl(x, y)$ is consistent w.r.t. a universal formula ϕ if

$$\phi(x, x) \wedge \phi(x, y) \wedge \phi(y, x) \wedge \phi(y, y) \wedge ijl(x, y)$$

is satisfiable. $2t(\phi)$ denotes The set of 2-types consistent with $\forall xy\phi(x, y)$.

Proposition

A pure universal formula $\forall x \forall y\, \phi(x, y)$ in the language FO^2 can be rewritten in the following form:

$$\forall x \forall y \left(x \neq y \rightarrow \bigvee_{i \leq j} \bigvee_{iji \in 2t(\phi)} ijl(x, y) \right) \tag{1}$$

The formula (1) is equivalent to the original one on models that contains 2 or more lements.

A simle method for finding $2t(\phi)$ is to find the truth assignments that satisfies $\phi(x,x) \wedge \phi(x,y) \wedge \phi(y,x) \wedge \phi(y,y)$

Example

$\forall x \forall y (R(x,x) \wedge R(x,y) \rightarrow \neg R(y,x)))$ is equivalent to

$$
\begin{aligned}
\forall x \forall y (x \neq y \rightarrow\ & \neg R(x,x) \wedge \neg R(y,y) \wedge R(x,y) \wedge R(y,x) \wedge x \neq y\ \wedge \\
& \neg R(x,x) \wedge \neg R(y,y) \wedge \neg R(x,y) \wedge R(y,x) \wedge x \neq y\ \wedge \\
& \neg R(x,x) \wedge \neg R(y,y) \wedge R(x,y) \wedge \neg R(y,x) \wedge x \neq y\ \wedge \\
& \neg R(x,x) \wedge \neg R(y,y) \wedge \neg R(x,y) \wedge \neg R(y,x) \wedge x \neq y
\end{aligned}
$$

Using the notation for 1- and 2-types we have that the initial formula is equivalent to

$$\forall x \forall y (x \neq y \rightarrow 221(x,y) \vee 222(x,y) \vee 223(x,y) \vee 224(x,y))$$

Example

$\forall x \forall y (R(x,x) \wedge R(x,y) \rightarrow R(y,y))$ is equivalent to:

$$\begin{aligned}\forall x \forall y \, (x \neq y \rightarrow\ & 111(x,y) \vee 112(x,y) \vee 113(x,y) \vee 114(x,y) \vee \\ & 123(x,y) \vee 114(x,y) \\ & 221(x,y) \vee 222(x,y) \vee 223(x,y) \vee 224(x,y) \vee\end{aligned}$$

We have that the original formula $\forall x \forall y \phi(x, y)$ is equivalent to

$$\Phi = \forall x \forall y \left(x \neq y \rightarrow \bigvee_{i \leq j} \bigvee_{ijl \in 2t(\phi)} ijl(x, y) \right)$$

implies that if $n \geq 2$

$$\mathrm{Ground}(\Phi, [n]) \leftrightarrow \bigwedge_{c \neq d = 1}^{n} \left(\bigvee_{i \leq j} \bigvee_{ijl \in 2t(\phi)} ijl(c, d) \right)$$

This implies that

$$\mathcal{I} \models \forall xy \phi(x, y) \text{ iff } h_l^{ij} \neq 0 \Rightarrow ijl \in 2t(\phi)$$

Furthermore the condition "$h_l^{ij} \neq 0 \Rightarrow ijl \in 2t(\phi)$" can be represented with

$$\mathbb{1}_{ijl \in 2t(\phi)}^{h_l^{ij}} = \begin{cases} 1 & \text{if } h_l^{ij} = 0 \text{ or } n_{ij} \neq 0 \\ 0 & \text{Otherwise} \end{cases}$$

where $\mathbb{1}_{ijl \in 2t(\phi)}$ is the indicator function for the set $2t(\phi)$.

Putting everything together

$$
\begin{aligned}
\text{FOMC}(\forall x, y.\phi(x,y), n) &= \sum_{\boldsymbol{k},\boldsymbol{h}} \binom{n}{\boldsymbol{k}} \prod_{i \le j = 1}^{u} \binom{\boldsymbol{k}(i,j))}{\boldsymbol{h}^{ij}} \prod_{l} \mathbb{1}_{ijl \in 2t(\phi)}^{h_l^{ij}} \\
&= \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{i \le j = 1}^{u} \sum_{\boldsymbol{k}} \binom{\boldsymbol{k}(i,j)}{\boldsymbol{h}^{ij}} \prod_{l} \mathbb{1}_{ijl \in 2t(\phi)}^{h_l^{ij}} \\
&= \sum_{\boldsymbol{k},\boldsymbol{h}} \binom{n}{\boldsymbol{k}} \prod_{i \le j = 1}^{u} \left(\sum_{l=1}^{b} \mathbb{1}_{ijl \in 2t(\phi)} \right)^{\boldsymbol{k}(i,j)} \\
&= \sum_{\boldsymbol{k},\boldsymbol{h}} \binom{n}{\boldsymbol{k}} \prod_{i \le j = 1}^{u} n_{ij}^{\boldsymbol{k}(i,j)}
\end{aligned}
$$

with

$$
n_{ij} = \sum_{l=1}^{b} \mathbb{1}_{ijl \in 2t(\phi)}
$$

First Order Model Counting in universal fragment of $\mathcal{L}^2$

Theorem

Let $\phi(x, y)$ a quantifier free formula that contains p predicate symbols and the two free variables x and y and no constant and functional symbolss;

$$\mathrm{FOMC}(\forall x, y.\phi(x, y), n) = \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{1 \leq i \leq u} n_{ij}^{\boldsymbol{k}(i,j)}$$

- $\boldsymbol{k} = (k_1, k_1, \ldots, k_u)$, *s.t.*, $\sum_{i=1}^{u} k_i = n$;
- $n_{ij} = \#\mathrm{SAT}(Ground(\phi_0(x, y) \wedge i(x) \wedge j(y), [2])$
- $\boldsymbol{k}(i,j) = \begin{cases} \frac{k_i \cdot (k_j - 1)}{2} & \text{if } i = j \\ k_i \cdot k_j & \text{Otherwise} \end{cases}$

Complexity

- Notice that to compute $\sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{0 \leq i \leq 2^p - 1} n_{ij}^{k(i,j)}$ you have to do a number of operations which are exponential in p but polinomial in n, since the only impact of n is in the computation of the binomial coefficient and the summation.
- it is known that computing the binomial coefficient $\binom{n}{k_1, \ldots, k_h}$ is polinomial in n.

Exercizes

Using the previous formula compute $\textsc{fomc}(\Phi, n)$ where Φ is the following fomrula:

1. $\forall x \forall y (R(x, y) \rightarrow R(y, x))$;
2. $\forall x \forall y (R(x, y) \rightarrow \neg R(y, x))$;
3. $\forall x \forall y (R(x, y) \rightarrow \neg R(x, x))$;
4. $\forall x \forall y (R(x, x) \rightarrow (R(x, y) \rightarrow R(y, z)))$;
5. $\forall x \forall y (S(x) \wedge F(x, y) \rightarrow S(y))$.

Cardinality constraint

A cardinality constraint is an expression that imposes constraints on the cardinality of the interpretation of predicates

Example (Cardinality constraint)

- $|A| = 3$ states that $\mathcal{I}(A)$ contains exaclty two elements;
- $|R| = 2$ states that $\mathcal{I}(R)$ contains more than 3 pairs of elements;
- $|A| > |R|$ states that $|\mathcal{I}(A)| > |\mathcal{I}(R)|$
- $|A| = 2 \rightarrow |B| < |C| \vee |B| < |C|$ states ....

Satisfaction of cardinality constraints

For every predicate P we can compute $|\mathcal{I}(P)|$ from the cardinality vectors $\boldsymbol{k}, \boldsymbol{h}$ of $\mathcal{I}$ as follows:

$$|\mathcal{I}(A)| = \boldsymbol{k}(A) = \sum_{i=1}^{u} \mathbb{1}_{A(x) \in i(x)} \cdot k_i \qquad A \text{ unary predicate}$$

$$\boldsymbol{k}(R) = \sum_{i=1}^{u} \mathbb{1}_{R(x,x) \in i(x)} \cdot k_i \qquad R \text{ binary predicate}$$

$$|\mathcal{I}(R)| = (\boldsymbol{k}, \boldsymbol{h})(R) = \boldsymbol{k}(R) + \sum_{i \leq j, l} \mathbb{1}_{R(x,y) \in ijl(x,y)} \cdot h_l^{ij} + \sum_{i \leq j, l} \mathbb{1}_{R(y,x) \in ijl(x,y)} \cdot h_l^{ij} \qquad R \text{ binary predicate}$$

If χ is a cardinality constraint then $(\boldsymbol{k}, \boldsymbol{h}) \models \chi$ holds if the expression obtained replacing $|A|$ with the value of $\boldsymbol{k}(A)$ and $|R|$ with the value of $(\boldsymbol{k}, \boldsymbol{h})(R)$ is true.

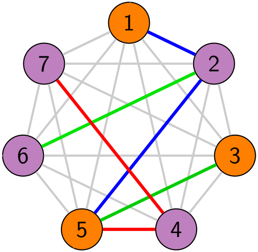
Example

1-types

$$1(x) \triangleq R(x,x),$$
$$2(x) \triangleq \neg R(x,x),$$

2-tables

$$1(x,y) \triangleq R(x,y) \wedge R(y,x) \wedge x \neq y$$
$$2(x,y) \triangleq R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$
$$3(x,y) \triangleq \neg R(x,y) \wedge R(y,x) \wedge x \neq y$$
$$4(x,y) \triangleq \neg R(x,y) \wedge \neg R(y,x) \wedge x \neq y$$



$$\boldsymbol{k} = (4, 3)$$

$$\boldsymbol{k}(R) = 4$$

$$\boldsymbol{h}^{11} = (0, 1, 1, 4)$$
$$\boldsymbol{h}^{12} = (2, 1, 0, 9)$$
$$\boldsymbol{h}^{11} = (0, 0, 1, 2)$$

$$(\boldsymbol{k}, \boldsymbol{h})(R) = 4 + 4 + 4 = 12$$

Dealing with Existential Quantifiers

- how can we count the models for formulas that contains also existential quantifiers and two variables?
- Scott's reduction: Transform it in the form:

$$\Phi = \forall x \forall y . \phi(x, y) \wedge \bigwedge_{i=1}^{m} \forall x \exists y . \psi_i(x, y)$$

Where $\phi(x, y)$ and $\phi_i(x, y)$ do not contain quantifiers.

Scott's Normal reduction

To transform a formula ϕ is Scott's normal form you have to apply the following transformations until ϕ does not contain subformulas of the form $Qy.\alpha(x,y)$, $Q \in \{\forall, \exists\}$

- Replace $Qy.\alpha(x,y)$ with a new predicate $A(x)$ and *define* $A(x)$ as $Qy.\alpha(x,y)$

$$\phi \Longrightarrow \phi[Qy.\alpha(x,y)/A(x)] \wedge \forall x.(A(x) \leftrightarrow Qy.\alpha(x,y))$$

- Transform $\forall x.(A(x) \leftrightarrow Qy.\alpha(x,y))$ in the $\forall x \forall y \dots$ or $\forall x \exists y \dots$ form

$$\begin{aligned}
\forall x(A(x) \leftrightarrow Qy.\alpha(x,y)) &\Longrightarrow \forall x(A(x) \rightarrow Qy.\alpha(x,y)) \wedge \\
&\qquad \forall x(\neg A(x) \rightarrow \bar{Q}y.\neg\alpha(x,y)) \\
\forall x.(A(x) \rightarrow \forall y.\alpha(x,y)) &\Longrightarrow \forall x,y.(A(x) \rightarrow \alpha(x,y)) \\
\forall x.(\neg A(x) \rightarrow \exists y.\neg\alpha(x,y)) &\Longrightarrow \forall x \exists y.(\neg A(x) \rightarrow \neg\alpha(x,y)) \\
\forall x.(A(x) \rightarrow \exists y.\alpha(x,y)) &\Longrightarrow \forall x \exists y(A(x) \rightarrow \alpha(x,y)) \\
\forall x.(\neg A(x) \rightarrow \forall y.\neg\alpha(x,y)) &\Longrightarrow \forall x \forall y(\neg A(x) \rightarrow \neg\alpha(x,y))
\end{aligned}$$

Fomc of formulas with Existential Quantifiers

$$\Phi = \forall x \forall y.\phi(x, y) \wedge \bigwedge_{i=1}^{m} \forall x \exists y.\psi_i(x, y)$$

Fomc of formulas with Existential Quantifiers

Let us consider the simple case with $m = 1$

$$\Phi = \forall x \forall y.\phi(x, y) \wedge \forall x \exists y.\psi(x, y)$$

- Introduce a new predicate $P(x)$ and the additional formula;

$$\phi_P = \forall x \forall y. (P(x) \rightarrow \neg\psi(x, y))$$

- ϕ_P implies that for each element a if $P(a)$ is true, then $\forall y.\neg\psi(a, y)$ is also true.
- Therefore is $|P| = k$ then there are at least k elements $a_1, \ldots, a_k$ for which $\forall x.\neg\psi(a_i, x)$

- Let M_i be the set of models of $\phi \wedge \phi_P$ such that there are exactly i elements that satisfies $\forall y \neg\psi(x, y)$
- Let M_{ij} the subset of M_i such that $|P| = j$. Clearly M_{ij} if $j > i$ M_{ij} is empty;
- Therefore $M_i = \cup_{0 \leq j \leq i} M_{ij}$
- we have that:

$$
\begin{aligned}
\text{FOMC}(\Phi, n) = |M_0| &= |M_{00}| \\
&= \text{FOMC}(\forall x \forall y \phi(x, y) \wedge \phi_p(x, y)) - \left| \bigcup_{1 \leq j \leq i = 1}^{n} M_{ij} \right|
\end{aligned}
$$

- We apply the inclusion exclusion principle that states:

$$
\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{1 \leq j_1 < \cdots < j_k \leq n} (-1)^{k+1} \left| A_{j_1} \cap A_{j_2} \cdots \cap A_{j_k} \right| \qquad \text{(IEP)}
$$

- obtaining

$$
\left| \bigcup_{1 \leq j \leq i = 1}^{n} M_{ij} \right| = \left| \bigcup_{i=1}^{n} \left(\bigcup_{j=i}^{n} M_{ij} \right) \right| \stackrel{IEP}{=} \sum_{j=1}^{n} (-1)^j \left| \bigcup_{j=i}^{n} M_{ij} \right|
$$

First order model counting in $\mathcal{L}^2$

$$\text{FOMC}\left(\forall x, y.\phi(x,y) \wedge \forall x \exists y \psi(x,y), n\right) = \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} (-1)^{\boldsymbol{k}(P)} \prod_{0 \leq i \leq j \leq 2^{p+1}-1} n_{ij}^{k(i,j)}$$

where $\boldsymbol{k}(P) = \sum_{i=1, i_b=1}^{2^{p+1}-1} k_i$ where b is the index of the predicate P.
Generalizing to m existentially quantifiers, we introduce a predicate P_i for every formula $\forall x \exists y \psi_i(x,y)$ and the corresponding formula ϕ_{P_i}.

$$\text{FOMC}\left(\forall x, y.\phi(x,y) \wedge \bigwedge_{i=1}^{m} \forall x \exists y \psi_i(x,y), n\right) = \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} (-1)^{\sum_{i=1}^{m} \boldsymbol{k}(P_i)} \prod_{0 \leq i \leq j \leq 2^{p+m}-1} n_{ij}^{k(i,j)}$$

Exercizes

Using the previous formula compute $\text{FOMC}(\Phi, n)$ where Φ is the following fomrula:

- $\forall x.\exists y.R(x.y)$
- $\forall x.\exists y.(R(x, y) \vee R(y, x))$
- $\forall x, y.(R(x, y) \rightarrow R(y, x)) \wedge \forall x \neg R(x, x) \wedge \forall x.\exists y.R(x.y)$

Solution of the first exercize

$$\forall x, y P(x) \rightarrow \neg R(x, y)$$

$P(a)$	$R(a,a)$	$P(b)$	$R(b,b)$	n_{ij}
1	1	1	1	$n_{11,11} = n_{3,3} = 0$
1	0	1	1	$n_{10,11} = n_{2,3} = 0$
1	0	1	0	$n_{10,10} = n_{2,2} = 1$
0	1	1	1	$n_{01,11} = n_{1,3} = 0$
0	1	1	0	$n_{01,10} = n_{1,2} = 2$
0	1	0	1	$n_{01,01} = n_{1,1} = 4$
0	0	1	1	$n_{00,11} = n_{0,3} = 0$
0	0	1	0	$n_{00,10} = n_{0,2} = 2$
0	0	0	1	$n_{00,01} = n_{0,1} = 4$
0	0	0	0	$n_{00,00} = n_{0,0} = 4$

$$\sum_{k_0,k_1,k_2} \binom{n}{k_0, k_1, k_2} (-1)^{k_2} n_{00}^{\frac{k_0(k_0-1)}{2}} n_{01}^{k_0 k_1} n_{02}^{k_0 k_2} n_{11}^{\frac{k_1(k_1-1)}{2}} n_{12}^{k_1 k_2} n_{22}^{\frac{k_2(k_2-1)}{2}}$$

$$= \sum_{k_0,k_1,k_2} \binom{n}{k_0, k_1, k_2} (-1)^{k_2} 4^{\frac{k_0(k_0-1)}{2}} 4^{k_0 k_1} 2^{k_0 k_2} 4^{\frac{k_1(k_1-1)}{2}} 2^{k_1 k_2} 1^{\frac{k_2(k_2-1)}{2}}$$

$$= \sum_{k_0,k_1,k_2} \binom{n}{k_0, k_1, k_2} (-1)^{k_2} 2^{k_0(k_0-1)+2k_0k_1+k_0k_2+k_1(k_1-1)+k_1k_2}$$

$$= \sum_{k_0,k_1,k_2} \binom{n}{k_0, k_1, k_2} (-1)^{k_2} 2^{(k_0+k_1)(n-1)} \sum_{k_2} \binom{n}{k_2} (-1)^{k_2} \sum_{k_0,k_1} \binom{n-k_2}{k_0, k_1} 2^{(k_0+k_1)(n-1)}$$

$$= \sum_{k_2} \binom{n}{k_2} (-1)^{k_2} (2^n)^{n-k_2} = (2^n - 1)^n$$

First order weighted model counting

- as in propositional logic interpretations are associated to a weight;
- In propositinal logic each proposition is associated with a weight;
- In First Order Logic weights are associated to predicates;
- This type of weight function is called symmetric weight functions;
- it is not the most general, but in this class we will limit to symmetric weight.

Symmetric weight function

Definition

A symmetric weight function for a first order language with signature Σ is specified by a pair of functions w and $\bar{v}$ that associate a real number $w(P)$ and $\bar{w}(P)$ to every n-ary predicate $P \in \Sigma$.

$$w(P(a_1)) = w(P(a_2)) = \cdots = w(P(a_n)) = w(P)$$
$$w(\neg P(a_1)) = w(\neg P(a_2)) = \cdots = w(\neg P(a_n)) = \bar{w}(P)$$

For every interpretation $\mathcal{I}$ on a finite domain $\Delta_{\mathcal{I}}$

$$W(\mathcal{I}) = \prod_{P \in \Sigma} w(P)^{|P|_{\mathcal{I}}} \cdot \bar{w}(P)^{|\Delta_{\mathcal{I}}|^{arity(P)} - |P|_{\mathcal{I}}}$$

$$\text{WFOMC}(\phi, w, \bar{w}, n) = \sum_{\substack{\mathcal{I} \models \phi \\ \Delta_{\mathcal{I}} = \{1, \ldots, n\}}} W(\mathcal{I})$$

First order weighted model counting in $\mathcal{L}^2$

oblem: Is it possible to adapt the formula for $\text{FOMC}\phi, n$ for ϕ in $\mathcal{L}^2$ to compute $\text{WFOMC}(\phi, w, \bar{w}, n)$?

nswer: Yes! Replace n_{ij} with w_{ij}

$$w_{ij} = \text{WMC}(Ground(\phi_0(x, y) \wedge \alpha_i(x) \wedge \alpha_j(y), \{a, b\}))$$

$$\alpha_i(x) = \bigwedge_{\substack{b=1 \\ i_b=0}}^{p} \neg A_b(x) \wedge \bigwedge_{\substack{b=1 \\ i_b=1}}^{p} A_b(x)$$

- $$\text{WFOMC}(\forall x, y.\phi_0(x, y), w, \bar{w}, n) = \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{0 \leq i \leq 2^p - 1} w_{ij}^{k(i,j)}$$