LECTURE 20, May 18, 2023

GENERAL THEORY of PDEs for DIFFERENTIAL GAMES?

- NOT for NON-ZERO SUM GAMES! because SYSTEMS of 1st. order nonlinear are NASTY.
- YES for 0-sum games & the Isaacs equat

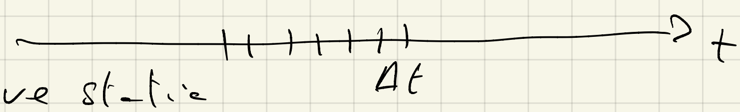
PLAN
- def. a value function.
- DYNAMIC PROGRAM. PRINCIPLE
- value is a visc. sol. of Isaacs eq. ....
  .... the UNIQUE ONE.

Q How to define the value function?

N.B. open loop vs. open loop is not good!

Hystory: • R. Isaacs 50-60 : method of characteristics
... piecewise C^1 solutions. $n \leq 2$

- W. Fleming (1961)



Euler scheme, solve static games at discrete times, let $\Delta t \to 0$.

- A. Friedman, time discretization $\Delta t \to 0$ ....

- N. Krassovski - A. Subbotin 1969 - 1977

"Positional differential games."

also involves a limit procedure.

- NON-ANTICIPATING (CAUSAL or MARKOVIAN) STRATEGIES

Varaiya, Roxin, Elliott - Kalton (1967 - 72)

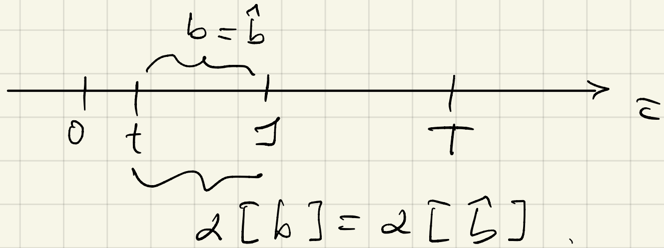
Ref: L.C. Evans - Souganidis 1984 $\Leftrightarrow$ visco Sols.

Notations : $\mathcal{A}_t := \{ a : [t,T] \to A \text{ meas.le} \}$.

$\mathcal{B}_t := \{ b : [t,T] \to B \text{ meas.le} \}$.

Def. A strategy of 1^{st} player is $\alpha : \mathcal{B}_t \to \mathcal{A}_t$, is NONANTICIPATING if. $\forall\, t \le s \le T$, $\forall\, b, \hat{b} \in \mathcal{B}_t$,

$$b(\tau) = \tilde{b}(\tau) \quad \forall\, t \le \tau \le s \implies \alpha[b](\tau) = \alpha[\hat{b}](\tau) \quad \forall\, t \le \tau \le s$$



$\Gamma_t = \{ \text{nonantic. strat. } 1^{st} \text{ player} \}$.

$\Delta_t = \{ \quad " \quad\quad " \quad 2^{nd} \quad " \quad \} \ni \beta$

$\beta : \mathcal{A}_t \to \mathcal{B}_t$, $\forall\, s \in [t,T]$, $\forall\, a, \hat{a} \in \mathcal{A}_t$:

$$a(\tau) = \hat{a}(\tau) \quad \forall\, \tau \in [t,s] \implies \beta[a](\tau) = \beta[\hat{a}](\tau) \quad \forall\, \tau \in [t,s].$$

N.B. $\forall\, b \in \mathcal{B}_t$ $\forall\, \alpha \in \Gamma_t$ $\exists$ unique trajectory of

$$\begin{cases} \dot{y}(s) = f(y(s), \alpha[b](s), b(s)) & s \ge t \\ y(t) = x \end{cases}$$

denote $y(s) = y_x(s; t, \alpha[b], b)$.

Similarly $a \in \Theta_t$, $\beta \in \Delta_t$ $\qquad \dot{y} = f(y, a, \beta[a])$.

Payoff (for 1st) – cost (for 2nd) functional

$$J(x,t;a(\cdot),b(\cdot)) = \int_t^T \ell(y_x(s), a(s), b(s))\,ds + g(y_x(T))$$

<u>DEF.</u> The LOWER VALUE of the D.G. is

$$V(t,x) := \inf_{\beta\in\Delta_t}\sup_{a\in\Theta_t} J(x,t,a,\beta[a]),$$

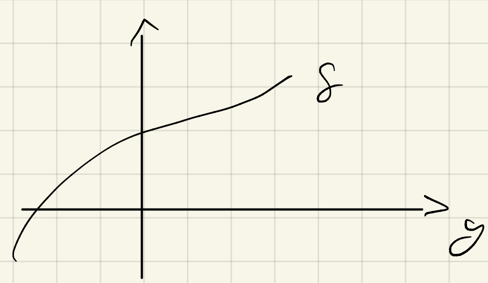
$$U(t,x) := \sup_{\alpha\in\Gamma_t}\inf_{b\in\mathcal{B}_t} J(x,t,\alpha[b],b).$$

If $V(t,x) = U(t,x)$ the D.G. has a value.

<u>Q</u>: $V \leq U$?

<u>Example</u> (Berkovitz) $\begin{cases} \dot{y} = (a-b)^2 \geq 0 \quad \text{in } \mathbb{R} \\ y(t) = x \end{cases}$

$A = B = \{0,1\}$, $\ell \equiv 0$, $g' > 0$



b MINIMIZES: she wants $\dot{y} = 0$

$\beta^*[a](s) = a(s)$ $\forall s$ $\quad \beta^* \in \Delta_t$, & $\dot{y}^* = 0$ $\forall s$

$\forall a \in \Theta_t$ $\qquad y^*(s) = x$ $\forall s$ $\forall a \in \Theta_t$

$$\Rightarrow V(t,x) \leq \sup_{a\in\Theta_t} g(\underset{x}{\underset{\shortparallel}{y^*(T)}}) = g(x)$$

a MAXIMISES, she wants $\dot{y} = 1$

$$\alpha^*[b](s) := \begin{cases} 1 & \text{if } b(s)=0 \\ 0 & \text{if } b(s)=1 \end{cases} \qquad \alpha^* \in \Gamma_t$$

$$\dot{y}^*(s) = (\alpha^*[b](s) - b(s))^2 = 1 \quad \forall s \quad \forall b \in \mathcal{B}_t$$

$$\Rightarrow \quad y^*(s) = x + s - t$$

$$U(t,x) \geq \inf_{b \in \mathcal{B}_t} g(y^*(T)) = g(x + \underbrace{T-t}_{>0}) > g(x)$$

$$\Rightarrow \quad V(t,x) < U(t,x) \quad \forall x \quad \forall t < T. \quad \blacksquare$$

Example of STRATEGIES:

Ex. 1. CONSTANT STRAT. Fix $\bar{b} \in \mathcal{B}_t$

$\beta[a] = \bar{b} \quad \forall a \in \mathcal{A}_t \Rightarrow \beta \in \Delta_t \Rightarrow \Delta_t \supseteq$ a copy of $\mathcal{B}_t$

Ex. 2 $\Psi : A \to B$, ASS: $s \mapsto \Psi \circ a(s)$ meas.le. $\forall a \in \mathcal{A}_t$

$\Rightarrow \beta[a](s) = \Psi(a(s))$ is nonanticip.

Ex. 3 Feedback: $\Phi : \mathbb{R}^n \times [0,T] \to B$ s.t.

$\forall a \in \mathcal{A}_t \ \exists$ UNIQUE sol. of $\begin{cases} \dot{q}(s) = f(q(s), a(s), \Phi(q(s), s)) & s > t. \\ q(t) = x \end{cases}$

& spse. $s \mapsto \Phi(q(s), s)$ meas.le. Then

$\beta[a](s) := \Phi(q(s), s)$ is nonanticip.? SI, $\beta \in \Delta_t$

& $y_x(s; t, a, \beta[a]) = q(s)$. $\blacksquare$

Rmk. Information pattern is not very realistic, but all more realistic values $\in [V, \mathcal{V}]$, so if $V = \mathcal{V}$ they all coincide.

Es. If $V = \mathcal{V}$ & $\exists$ saddle $\bar{V}$ among admissible feedbacks $\underset{HW}{\Rightarrow}$ $\bar{V} = V = \mathcal{V}$. ▮

Assumptions. $T > 0$, A, B metric compact.

- $f : \mathbb{R}^n \times A \times B \to \mathbb{R}^n$ cont., $|f| \le d_1$

  $|f(x, a, b) - f(\hat{x}, a, b)| \le C_1 |x - \hat{x}|$ $\quad \forall x, \hat{x} \in \mathbb{R}^n$ $a \in A, b \in B$

- $\ell : \mathbb{R}^n \times A \times B \to \mathbb{R}$ cont., $|\ell| \le d_2$

  $|\ell(x, a, b) - \ell(\hat{x}, a, b)| \le C_2 |x - \hat{x}|$ $\quad \forall \ldots$

- $g : \mathbb{R}^n \to \mathbb{R}$, $|g| \le d_3$, $|g(x) - g(\hat{x})| \le C_3 |x - \hat{x}|$.

Rmk. I can add $\dot{y}_{n+1} = \ell(y, a, b)$

$\tilde{g}(y, y_{n+1}) = g(y) + y_{n+1}$ & get an equivalent game with $\tilde{\ell} \equiv 0$ & $\tilde{g}$. ▮

DYNAMIC PROGRAMMING PRINCIPLE (TENET OF TRANSITION).

Thm. $0 \le t < t + \sigma \le T$ $\quad \forall x \in \mathbb{R}^n$

$$V(t, x) = \inf_{\beta \in \Delta_t} \sup_{a \in \mathcal{a}_t} \left\{ \int_t^{t+\sigma} \ell(y(s), a(s), \beta[a](s))\, ds + V(t+\sigma, y(t+\sigma)) \right\}$$

where $g(s) = y_x(s; t, a, \beta[a])$.

$$V(t,x) = \sup_{\alpha\in\Gamma_t}\ \inf_{b\in\mathcal{B}_t} \left\{ \int_t^{t+\sigma} \ell(y(s), \alpha[b](s), b(s)) + V(t+\sigma, y(t+\sigma)) \right\}$$

$y(s) = y_x(s; t, \alpha[b], a)$.

Pf. only for $\ell \equiv 0$, & V, only "$\le$" ("$\ge$" in the notes optional)

$\frac{1}{2}$ thesis: $V(t,x) \le W(t,x) := \inf_{\beta}\ \sup_{a} V(t+\sigma, y(t+\sigma))$

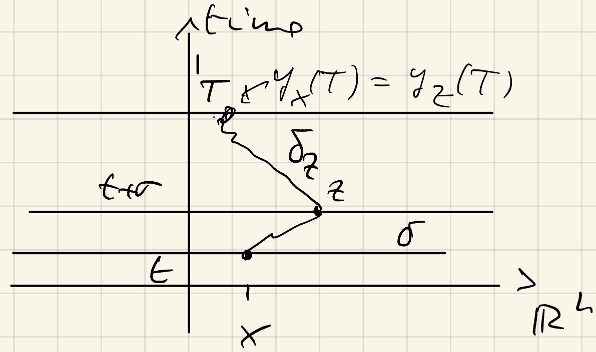
$\forall \varepsilon > 0 \quad \exists\, \delta \in \Delta_t$: (def. of inf).

$$W(t,x) \overset{(1)}{\ge} \sup_{a\in\mathcal{A}_t} V(t+\sigma, \underbrace{y_x(t+\sigma; t, a, \delta[a])}_{z}) - \varepsilon$$

$\forall z \in \mathbb{R}^n$ def of inf. $\Rightarrow \exists\, \delta_z \in \Delta_{t+\sigma}$

$$V(t+\sigma, z) \overset{(2)}{\ge} \sup_{a\in\mathcal{A}_{t+\sigma}} g(y_z(T; t+\sigma, a, \delta_z[a])) - \varepsilon$$

Def. $$\bar\beta[a](s) = \begin{cases} \delta[a](s) & t \le s \le t+\sigma \\ \delta_z[a](s) & t+\sigma \le s \le T \end{cases}$$



with $z = y_x(t+\sigma; t, a, \delta[a])$

$\bar\beta \in \Delta_t$? YES, (1) & (2) $\Rightarrow$

$$W(t,x) \ge \underbrace{\sup_{a\in\mathcal{A}_t}\ \sup_{a\in\mathcal{A}_{t+\sigma}}}_{\sup_{a\in\mathcal{A}_t}} g(\underbrace{y_z(T; t+\sigma, a, \delta_z[a])}_{y_x(T;\, t, a, \bar\beta[a])}) - 2\varepsilon$$

$$\ge \sup_{a\in\mathcal{A}_t} g(\cancel{z}\quad) - 2\varepsilon \qquad \Rightarrow$$

$$W(t,x) \geq \underbrace{\inf_{\beta \in \mathcal{A}_t} \sup_{a \in \mathcal{a}_t} g(y_x(T; t, a, \beta[a]))}_{V(t,x)} - 2\varepsilon$$

Let $\varepsilon \to 0$ & get $W \geq V$. $\forall t, x$ $\boxtimes$

Estimates on V & $\mathcal{V}$

Thm. $\exists\, C_4$ dep. on T, c_1, c_2, c_3 s.t.

1. $|V(t,x)|, |\mathcal{V}(t,x)| \leq C_4$ $\quad \forall (t,x) \in [0,T] \times \mathbb{R}^n$

2. $|V(t,x) - V(\hat{t},\hat{x})| \leq C_4 (|t-\hat{t}| + |x-\hat{x}|)$

$|\mathcal{V}(t,x) - \mathcal{V}(\hat{t},\hat{x})| \leq$ same.

Part of proof: 1. $\left| \int_t^T \ell(\;)\,ds + g(\;) \right| \leq (T-t)C_2 + C_3 \leq$

$\leq TC_2 + C_3 =: C_4$ $\boxtimes$

2. optional, see Notes. $\boxtimes$

Isaacs' Hamiltonians

$$H^+(x,p) := \min_{b \in B} \max_{a \in A} \{ p \cdot f(x,a,b) + \ell(x,a,b) \}$$

$$H^-(x,p) := \max_{a \in A} \min_{b \in B} \{ \text{ same } \} \leq H^+(x,p).$$

Prop. $H^+, H^- : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are continuous and for $F = H^+$ or H^-

$\exists K > 0$:

$$(L_x) \quad |F(x,p) - F(\hat{x},p)| \le K|x - \hat{x}|(1+|p|)$$

$$(L_p) \quad |F(x,p) - F(x,\hat{p})| \le c_1 |p - \hat{p}|$$

Pf. 1. (L_x) for $F = H^+$ (H^- HW).

$H^+(\hat{x},p) - H^+(x,p) \le \ldots$ choose $b' \in B$:

$$H^+(x,p) = \max_a \{p \cdot f(x,a,b') + \ell(x,a,b')\}$$

$$\ge p \cdot f(x,a,b') + \ell(x,a,b') \quad \forall a \in A.$$

$$H^+(\hat{x},p) \le \max_a \{p \cdot f(\hat{x},a,b') + \ell(\hat{x},a,b')\} \overset{\exists a'}{=} p \cdot f(\hat{x},a',b') + \ell(\hat{x},a',b')$$

$$H^+(\hat{x},p) - H^+(x,p) \le p \cdot f(\hat{x},a',b') + \ell(\hat{x},a',b') + p \cdot f(x,a',b') + \ell(x,a',b')$$

$$\le |p| c_1 |x - \hat{x}| + c_2 |x - \hat{x}| \le K|x - \hat{x}|(1+|p|)$$

if $K := c_1 \vee c_2$. Exchange x & $\hat{x}$ $\Rightarrow$

(L_x) for H^+. ∎

2. (L_p) HW

3. $(L_x) + (L_p) \Rightarrow$ cont. ty of H^+ & H^-. ∎