LECTURE 19, May 16, 2023

Confirm: lecture extra schedule: May 31, 12.30
room 2BC60.

LINEAR-QUADRATIC DIFFERENTIAL GAMES

$$\dot{y} = Ay + B_1 a + B_2 b, \quad t \le s \le T, \quad a = a_1 \in \mathbb{R}^{m_1}, \quad b = a_2 \in \mathbb{R}^{m_2}$$

$i = 1, 2$

$$J^i(x, t, a_1(\cdot), a_2(\cdot)) := -\int_t^T \left[y(s)^T \frac{M_i}{2} y(s) + \frac{|a_i(s)|^2}{2} \right] ds + y(T)^T \frac{Q_i}{2} y(T)$$

Pre-Hamiltonians:

$$\Phi^A(a, b, x, p_1) = p_1^T (Ax + B_1 a + B_2 b) - \frac{x^T M_1 x}{2} - \frac{|a|^2}{2} \longrightarrow -\infty \quad \text{as } |a| \to +\infty$$

$\Rightarrow \exists$ UNIQUE MAX in a

$$\Phi^B(a, b, x, p_2) = p_2^T (Ax + B_1 a + B_2 b) - \frac{x^T M_2 x}{2} - \frac{|b|^2}{2} =$$

$$= p_2^T B_2 b - \frac{|b|^2}{2} + \text{terms indep. of } b \longrightarrow -\infty$$

STR. concave in b

$\Rightarrow \exists$ UNIQUE MAX in b

$\Rightarrow \left(\operatorname{argmax}_a \Phi^A, \operatorname{argmax}_b \Phi^B \right)$ is the UNIQUE NASH equil. of (Φ^A, Φ^B, A, B).

$$D_a\left(p_1^T B_1 a - \frac{|a|^2}{2} \right) = B_1^T p_1 - a = 0 \iff a = B_1^T p_1$$

$u_1^\#(x, p_1) =$ linear in p_1, continuous.

$$D_b\left(p_2^T B_2 b - \frac{|b|^2}{2}\right) = B_2^T p_2 - b = 0 \iff b = B_2^T p_2 = $$

$$= u_2^{\#}(x, p_2) \quad \text{linear in } p_2 \Rightarrow \text{cont.}$$

$\Rightarrow$ Hyp # is verified !

Write the system of PDEs

$$\begin{cases} \dfrac{\partial w_1}{\partial t} + D_x w_1 \cdot (Ax + \overbrace{B_1 B_1^T D_x w_1}^{u_1^{\#}(p_1)} + \overbrace{B_2 B_2^T D_x w_2}^{u_2^{\#}(p_2)}) = \dfrac{x^T M_1 x}{2} + \dfrac{|\overbrace{B_1^T D_x w_1}^{u_1^{\#}(p_1)}|^2}{2} \\ \dfrac{\partial w_2}{\partial t} + D_x w_2 \cdot (\quad) = \dfrac{x^T M_2 x}{2} + \dfrac{|B_2^T D_x w_2|^2}{2} \leftarrow u_2^{\#}(p_2) \\ w_1(x, T) = \dfrac{x^T Q_1 x}{2}, \quad w_2(x, T) = \dfrac{x^T Q_2 x}{2} \end{cases}$$

ANSATZ : $w_i(x,t) = \dfrac{x^T K_i(t)\, x}{2}$ $\quad i = 1, 2$

$K_i(t) \in \mathrm{Sym}(n)$, $K_i \in C^1$ in time.

$D_x w_i = K_i(t)\, x$, $\dfrac{\partial w_i}{\partial t} = \dfrac{x^T \dot{K}_i(t)\, x}{2}$ $\qquad j = 3 - i = \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \end{cases}$

Plug into the system

$$x^T \frac{\dot{K}_i}{2} x + K_i x \cdot (Ax + \overbrace{B_i B_i^T}^{S_i =: S_i} K_i x + \overbrace{B_j B_j^T}^{=: S_j} K_j x) = \frac{x^T M_i x}{2} + \frac{|B_i^T K_i x|^2}{2}$$

$$= \frac{1}{2} x^T K_i \underbrace{B_i B_i^T}_{= S_i} K_i x$$

$$x^T\left(\frac{\dot{K}_i}{2} + \frac{K_i A + A^T K_i}{2} + \underbrace{K_i S_j K_j}_{\frac{K_i S_j K_j + K_j S_j K_i}{2}}\right) x = x^T\left(\frac{M_i}{2} - \frac{K_i S_i K_i}{2}\right) x \qquad \forall x$$

$$\iff \dot{K}_i + K_i (A + S_j K_j) + (A^T + K_j S_j) K_i = M_i - K_i S_i K_i \qquad i = 1, 2$$

$$(RDS)\ \begin{cases} \dot K_1 = M_1 - K_1 S_1 K_1 - K_1(A + S_2 K_2) - (A^T + K_2 S_2) K_1 \\ \dot K_2 = M_2 - K_2 S_2 K_2 - K_2(A + S_1 K_1) - (A^T + K_1 S_1) K_2 \\ K_1(T) = Q_1 \quad , \quad K_2(T) = Q_2 \end{cases}$$

coupling terms

Backward Cauchy problem for 2 MATRIX RICCATI ODES.

Thm. If $\exists$ sol. $K_i \in C^1((t_0,T), Sym(n))$, cont. at $t = T$, $i = 1, 2$ of (RDS) in (t_0, T). Then $w_i(x,t) = \frac{x^T K_i(t) x}{2}$ solve the system of HJB equations in the Verif. Thm. & $u_i^*(x,t) = B_i^T K_i(t) x$ $i = 1,2$ form a Nash equil. for the LQ diff. game.

Moreover such sol. K_i $\exists$ for some $t_0 < T$.

Proof We saw $u_i^\#(p_i) = B_i^T p_i$, $D_x w_i = K_i(t) x$ $\Rightarrow$ candidate Nash feedback equil. are $u_i^*(x,t) = B_i^T K_i(t) x$.

Is it an admissible pair of feedbacks? The system becomes

$$\begin{cases} \dot y(s) = A y(s) + B_1 B_1^T K_1(t) y(s) + B_2 B_2^T K_2(t) y(s) \\ y(t) = x \end{cases}$$

linear (non-autonomous) homogen. system of ODEs.

$\Rightarrow$ $\forall x \in \mathbb{R}^n$, $t \in (t_0, T)$ it has a unique sol. in $(t_0, T]$.

$\Rightarrow$ (u_1^*, u_2^*) is admissible $\overset{\text{Verif. Thm.}}{\Longrightarrow}$ it is a NASH EQ.

Last thing. (RDS) has a sol.: use local $\exists$ thm for ODES, must check $K_i(t) \in Sym(n)$ $\forall t$.

HW: check K_1^T, K_2^T solves (RDS) $\Rightarrow K_i^T = K_i$ $i=1,2$.

0-SUM DIFF. GAMES

$$\begin{cases} \dot{y} = f(y, a, b) \\ y(t) = x \end{cases} \qquad J = \int_t^T \ell(y(s), a(s), b(s))\,ds + g(y(T))$$

A maximises J, B min J. Special case of the general one: $\ell_A = \ell$, $\ell_B = -\ell$, $g_A = g$, $g_B = -g$

Simplified Pre-Hamiltonian:

$$\Phi(a, b; x, p) = p \cdot f(x, a, b) + \ell(x, a, b)$$

Hypothesis #bis: $\exists\,(u_1^\#, u_2^\#): \mathbb{R}^n \times \mathbb{R}^n \to A \times B$ CONTINUOUS
$(x, p) \mapsto (u_1^\#, u_2^\#)(x, p)$

a SADDLE point of the 0-Sum game (A, B, Φ).

(x, p parameters). i.e.

$$u_1^\#(x,p) \in \arg\max_a \Phi(a, u_2^\#; x, p)$$

$$u_2^\#(x,p) \in \arg\boxed{\min}_b \Phi(u_1^\#, b; x, p)$$

Def. (u_1^*, u_2^*) admissible feedbacks is a SADDLE POINT for the diff. game starting at (x,t) if.

$$J(x,t; u_A, u_2^*) \le J(x,t; u_1^*, u_2^*) \le J(x,t; u_1^*, u_B)$$

$\forall\,(u_A, u_2^*)$ admiss. $\qquad \forall\,(u_1^*, u_B)$ admiss.

Cor. (Verification thm. for 0-sum d.g.) Supp. $\exists\, w \in C^1([t_0, T))$ cont. at $t=T$ solution of

(IE)
Isaacs eq.

$$\begin{cases} \begin{cases} \dfrac{\partial w}{\partial t} + D_x w \cdot f(x, u_1^\#, u_2^\#) + \ell(x, u_1^\#, u_2^\#) = 0 \\ u_i^\# = u_i^\#(x, D_x w) \end{cases} \\ w(x,T) = g(x) \end{cases}$$

and

$(u_1^*, u_2^*)(x,t) = \left(u_1^\#(x, D_x w(x,t)),\ u_2^\#(x, D_x w(x,t))\right)$ is a pair of admissible feedbacks $\Rightarrow (u_1^*, u_2^*)$ is a SADDLE POINT of D.G. among admiss. feedbacks.

Pf. Use Verif. thm. for Nash equil. $w_1 = w$, $w_2 = -w$

CLAIM: (w_1, w_2) solves the HJ system of the general case:

Eq. 1

$$\frac{\partial}{\partial t} w_1 + \max_{a_1} \{ D_x w \cdot f(x, a_1, u_2^\#) + \ell(x, a_1, u_2^\#) \} = 0$$

is satisfied because $u_1^\# \in \operatorname{argmax}_a \mathcal{L}(a_1, u_2^\#; x, D_x w)$ by # bis.

Eq. 2 Take $-$ (IE)

$$\underbrace{-\frac{\partial w}{\partial t}}_{\frac{\partial w_2}{\partial t}} \underbrace{- D_x w \cdot f}_{D_x w_2} \underbrace{- \ell(\ )}_{\ell_B} = 0$$

$$\Rightarrow \quad \frac{\partial w_2}{\partial t} + D_x w_2 \cdot f(x, u_1^\#, u_2^\#) + \ell_B(x, u_1^\#, u_2^\#) = 0$$

& by Ass. #bis :

$$u_2^\# \in \underset{b}{\arg\min}\Big\{ \underbrace{D_x w}_{-D_x w_2} \cdot f(x, u_1^\#, b) + \underbrace{\ell}_{-\ell_B}(x, u_1^\#, b) \Big\}$$

$$= \underset{b}{\arg\max}\{ D_x w_2 \cdot f(x, u_1^\#, b) + \ell_B(x, u_1^\#, b) \}.$$

$\Rightarrow$ w_2 sats. 2^{nd} eq. of the system for Nash equil. $\xrightarrow{\text{Gen. Verif. thm.}}$ (u_1^*, u_2^*) adm. is also a Nash EQUIL.

$\Rightarrow$ (u_1^*, u_2^*) is a saddle of the 0-sum diff. game. ∎

COMMENTS on ISAACS' EQUATION.

$$\Phi = p \cdot f(x,a,b) + \ell(x,a,b)$$

Hyp#bis $\Rightarrow$ $\Phi(u_1^\#(x,p), u_2^\#(x,p); x, p) =$

$$= \max_a \min_b \Phi(a,b; x,p) \underset{(I)}{=} \min_b \max_a \Phi(a,b; x\, p).$$

$\longrightarrow$ Isaacs' condition.

(IE) can be re-written as

$$\frac{\partial w}{\partial t} + \max_a \min_b \{ D_x w \cdot f(x,a,b) + \ell(x,a,b) \} = 0 \qquad (I1)$$

and as

$$\frac{\partial w}{\partial t} + \min_b \max_a \{ \quad - - - - \quad \} = 0 \qquad (I2)$$

Rmks
- #bis $\Rightarrow$ (I) $(\not\Leftarrow)$
- (I1) and (I2) are upper & lower Isaacs' eqs.

they coincide if (I).

Application 1: L–Q, 0-sum diff games.

$$\dot{y} = Ay + B_1 a + B_2 b \qquad B_1 \in M_{n\times m_1}, \quad B_2 \in M_{n\times m_2}$$

$$a(\cdot) \in L^2([0,T], \mathbb{R}^{m_1}), \quad b(\cdot) \in L^2([0,T], \mathbb{R}^{m_2})$$

$$J = -\int_t^T \left(y(s)^T \frac{M}{2} y(s) + \frac{|a(s)|^2}{2} - \frac{|b(s)|^2}{2} \right) ds + y(T)^T \frac{Q}{2} y(T)$$

$M, Q \in Sym(n)$. 1st player MAXes, 2nd pl. MINies.

Ex HW check • Hyp #bis holds, compute $u_1^\#, u_2^\#$

• $w(x,t) = x^T K(t) x$ solves (IE) $\iff$ K sats.

$$(RDE\text{-}0\Sigma) \quad \begin{cases} \dot{K} = M - KA - A^T K + K\underbrace{(B_2 B_2^T - B_1 B_1^T)}_{?BB^T?} K \\ K(T) = 0 \end{cases}$$

Q: $\exists$ unique sol of in $[0,T]$?

Cor If $Q \geq 0$, $M \leq 0$, $B_2 B_2^T - B_1 B_1^T \geq 0 \Rightarrow$ (RDE-0Σ) has a sol $w \in C^1((-\infty, T), Sym(n)) \Rightarrow$ diff. game has a saddle among adm.le feedbacks $\forall$ initial $t \leq T$.

Pf Use the thm. for opt. control: it's enough to check $\exists B$: $\underbrace{B_2 B_2^T - B_1 B_1^T}_{N} = BB^T$ "B square root of N"

this is true because $N \geq 0$. ▨

Application 2   A model of ADVERTISING in a DUOPOLY

Ref [Jorgensen-Zaccour]. $y_1, y_2 \in [0,1]$ are the % of market of 2 firms in competition.

Ass. : DUOPOLY $y_1 + y_2 = 1$ α_i = advertising effort of firm i

Dynamics (Lanchester dyn.)

$$\dot{y}_i = \alpha_i (1-y_i) - \alpha_j y_i \qquad i = 1, 2$$

Payoff = Income $$J^i(x,t;\alpha_1,\alpha_2) = \int_t^T \left(y_i r_i - \frac{\alpha_i^2}{2} \right) ds + R_i y_i(T)$$

r_i = unitary price > 0

$R_i \geq 0$ $\alpha_i \geq 0$

HW • Reformulate it as a 0-sum game $y = y_1$, $y_2 = 1 - y$ Find: $J = \ldots\ldots$

• Write pre-Ham. Φ, check Hyp. # bis & compute $u_i^{\#}$ (Hint: treat separately $p \geq 0$ & $p \leq 0$)

• Look for solution of (IE): it is reasonable to restrict to $w_x \geq 0$, so (IE) simplifies

  ▸ Ansatz: $w(x,t) = k(t)x + c(t)$, find a system of ODEs for k, c that can be solved explicitely.