LECTURE 18, May 11, 2023

LAST LECTURES of the course:

May 16, 18 + ? 19 at 12.30 ? (2BC60)

23 NO 25

30 June 1st.

NON-ZERO SUM GAMES in MIXED STRATEGIES

$\mu \in \mathbb{P}(A)$, $\nu \in \mathbb{P}(B)$ prob. measures

$$\tilde{\Phi}^A(\mu, \nu) = \iint_{A\times B} \Phi^A(a,b)\, d\mu(a)\, d\nu(b)$$

$$\tilde{\Phi}^B(\mu, \nu) = \iint_{A\times B} \Phi^B(\quad) \quad " \quad " \quad .$$

Cor (Nash) All BI-MATRIX Games (i.e. A & B finite sets) have at least one Nash Equil. in MIXED STRATEGIES.

Pf $A = \{1, \dots, m\}$, $B = \{1, \dots, n\}$

$\mathbb{P}(A) \longleftrightarrow \Delta_m$ $\qquad$ $\mathbb{P}(B) \longleftrightarrow \Delta_n$

$\mu \longleftrightarrow x$ discrete density $\qquad$ $\nu \longleftrightarrow y$

$\Phi^A(i,j) = (M^A)_{ij}$ $\qquad$ $\Phi^B(i,j) = (M^B)_{ij}$

$$\tilde{\Phi}^A(x,y) = x^T M^A y, \quad \tilde{\Phi}^B(x,y) = x^T M^B y \qquad \forall x,y \in \Delta_m \times \Delta_n$$

bilinear

Ass. of Nash thm. are verified (cont. + CONCAVITY)

$\Rightarrow \exists$ Nash Eq. for $(\Delta_m, \Delta_n, \tilde{\Phi}^A, \tilde{\Phi}^B)$. $\square$

<u>Thm.</u> (Nash gener.) A, B compact, $\Phi^A, \Phi^B \in C(A \times B)$

$\Rightarrow$ the game in mixed strategies has an equil.

<u>Pf</u> as for von Neumann..., see [Bressan]. $\square$

DYNAMIC GAMES : i.e. INVOLVING TIME :

- repeated (static) games.
- discrete-time games.
- continuous-time : differential games.

2-person diff. games : Dynamical system with controls of 2 players:

$$\begin{cases} \dot{y}(s) = f(y(s), \underbrace{a(s), b(s)}_{\text{controls}}) & s > t \\ y(t) = x \end{cases}$$

$f : \mathbb{R}^n \times A \times B \to \mathbb{R}^n$ cont. & Lip in y unif in a, b

$T > 0$ fixed. $\mathcal{A} := \{ a(\cdot) : [0,T] \to A \text{ meas. le} \}$

$\mathcal{B} := \{ b(\cdot) : [0,T] \to B \text{ meas. le} \}$.

Know: $\exists$ UNIQUE trajectory $y(\cdot)$ $\forall$ fixed $a \in \mathcal{A}, b \in \mathcal{B}$

$$y(s) := y_x(s; t, a(\cdot), b(\cdot))$$

Payoffs !

$$J^A(t,x,a(\cdot),b(\cdot)) := \int_t^T \ell_A(y(s),a(s),b(s))\,ds + g_A(y(T))$$

$$J^B(\ldots) := \int_t^T \ell_B(\ldots)\,ds + g_B(y(T))$$

1^{st} player wants to MAXIMISE J^A

2^{nd} " " " " J^B

Game is 0-sum if $J^A = -J^B$, i.e. if

$$\ell_A = -\ell_B, \quad g_A = -g_B .$$

Other possible payoffs :

- INFINITE HORIZON $\quad \lambda \geq 0$

$$J^i := \int_0^{+\infty} \ell_i(y(s),a(s),b(s))\,e^{-\lambda s}\,ds \qquad i = A,B$$

- games with a TARGET : given $\mathcal{T} \subseteq \mathbb{R}^n$

$$t_x(a(\cdot),b(\cdot)) := \begin{cases} \min\{s : y_x(s;0,a,b) \in \mathcal{T}\} & \text{if } \{\ \} \neq \emptyset \\ +\infty & \text{if } \{\ \} = \emptyset \end{cases}$$

PURSUIT - EVASION $\qquad y = (y_A, y_B) \in \mathbb{R}^k \times \mathbb{R}^k$

$$\begin{cases} \dot y_A = f_A(y_A, a) \\ \dot y_B = f_B(y_B, b) \end{cases} \qquad \mathcal{T} = \{ y : y_A = y_B \}$$

$$J^B = t_x(a(\cdot), b(\cdot))$$

A = pursuer

B = evader

$$J^A = -J^B = -t_x(a,b)$$

RUFUS ISAACS ~ 1954-6

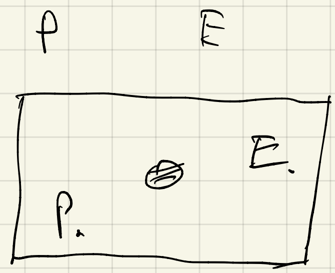
(Pontryagin URSS) Rand Corp.

FUNDAMENTAL QUESTION : where do I choose the control functions ?

1st try $a \in \mathcal{A}$, $b \in \mathcal{B}$ → "STANDARD" GAME

$(\mathcal{A}, \mathcal{B}, J^A, J^B)$. Is this an interesting model?

Example CHASING GAME (Acchiapparella)



$$\dot y_A = a \in \bar B_1(0)$$

$$\dot y_B = b \in \bar B_1(0)$$

$$J^A = -t_x \,, \quad J^B = t_x$$

$n = 4$

PURS-EVASION GAME

$$V^- = \sup_{b \in \mathcal{B}} \underbrace{\inf_{a \in \mathcal{A}} t_x(a(\cdot), b(\cdot))}_{\leq 4} \leq 4$$

$$V^+ = \inf_{a} \underbrace{\sup_{b} t_x(a(\cdot), b(\cdot))}_{+\infty \ \forall a} = +\infty$$

CONCLUSION: the information pattern is NOT realistic, nor interesting! We want a better model!

Recall Feedback controls for 1 player (MARKOVIAN STRATEGIES)

$\Phi : [t,T] \times \mathbb{R}^n \to A$ is ADMISSIBLE if $\exists$ UNIQUE solut. $\forall\, t, x$ of

$$\begin{cases} \dot y(s) = f(y(s), \Phi(s, y(s))) & s > t \\ y(t) = x \end{cases}$$

& $a_\Phi(s) = \Phi(s, y(s))$ measurable

Rmk. Q: $V(x,t) := \inf_{a \in \mathcal{A}} J(x, t, a(\cdot)) \overset{?}{=}$

$$\overset{?}{=} \inf_{\Phi \text{ adm. feedback}} J(x, t, a_\Phi)$$

Ans.: YES. Pf HW.

Notations : $u_A : [0,T]\times\mathbb{R}^n \to A$ feed. for 1^{st} pl.,

$u_B : [0,T]\times\mathbb{R}^n \to B$ feedb. for 2^{nd} pl.

Def. a pair (u_A, u_B) of MEAS.LE feedbacks is ADMISSIBLE if $\forall\, x, t$ $\exists$ UN. sol. in $[t,T]$ of
$$\begin{cases} \dot y(s) = f(y(s), u_A(s,y(s)), u_B(s, y(s))) \\ y(t) = x \end{cases}$$

Note $s \mapsto u_A(s, y(s))$ $s \mapsto u_B(s, y(s))$ meas.le.

Def. (u_A^*, u_B^*) admiss. pair is Nash equil. for x_0, t_0 among adm. feedb. pairs if

- u_A^* is opt. for 1^{st} pl. if 2^{nd} player plays u_B^* i.e., u_A^* maximizes $u_A \mapsto J^A(x_0, t_0, u_A, u_B^*)$ among all u_A: (u_A, u_B^*) is admissible.
- u_B^* is optimal for 2^{nd} pl. if 1^{st} uses u_A^*: u_B^* maximizes $u_B \mapsto J^B(x_0, t_0, u_A^*, u_B)$ among u_B: (u_A^*, u_B) is admiss.

Plan: build such Nash equil via a Verif. thm.

1 player:

Pre-Hamiltonian : $p \cdot f(x,a) + \ell(x,a)$ param. x, p

For 2 players 2 pre-Ham. with 3 parameters: x, P_1, P_2

$$\Phi^A(a,b; x, P_1) := P_1 \cdot f(x,a,b) + \ell_A(x,a,b)$$

$$\Phi^B(a,b; x, P_2) := P_2 \cdot f(x,a,b) + \ell_B(x,a,b)$$

Hypothesis # : $\exists$ continuous fn. $(u_1^\#, u_2^\#) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to A \times B$ s.t. $\forall x, p_1, p_2 \in \mathbb{R}^n$

$(u_1^\#, u_2^\#)(x, p_1, p_2)$ is a Nash eq. of the static game

(A, B, Φ^A, Φ^B), i.e.

$$u_1^\#(x, p_1, p_2) \in \operatorname{argmax}_a \Phi^A(a, u_2^\#(x, p_1, p_2), x, p_1)$$

$$u_2^\#(x, p_1, p_2) \in \operatorname{argmax}_b \Phi^B(u_1^\#(x, p_1, p_2), b, x, p_2).$$

Suff. conditions for ass. # from Nash thm.,... see Lemma in the Notes ...

<u>VERIFICATION THM.</u> Supp. $\exists$ $w_1, w_2 \in C^1((t_0, T), \mathbb{R}^n)$ cont. at $t = T$

$$\begin{cases} \dfrac{\partial w_1}{\partial t} + D_x w_1 \cdot f(x, u_1^\#, u_2^\#) + \ell_A(x, u_1^\#, u_2^\#) = 0 & \text{in } (t_0, T) \times \mathbb{R}^n \\ \dfrac{\partial w_2}{\partial t} + D_x w_2 \cdot f(x, u_1^\#, u_2^\#) + \ell_B(x, u_1^\#, u_2^\#) = 0 & \text{"} \\ u_i^\# = u_i^\#(x, D_x w_1, D_x w_2) & i = 1, 2. \\ w_1(x, T) = g_A(x), \quad w_2(x, T) = g_B(x) & \end{cases}$$

& $(t, x) \mapsto (u_1^\#(x, Dw_1(x,t), Dw_2(x,t)), u_2^\#(\text{same}))$ is a pair of admissible feedbacks. $\Longrightarrow$

such pair is Nash equil. for diff. game (among adm. feeds.)

<u>Rmk</u> 1st. eq. is

$$\frac{\partial w_1}{\partial t} + \max_{a \in A} \{ D_x w_1 \cdot f(x, a, u_2^\#) + \ell_A(x, a, u_2^\#) \} = 0$$

& 2nd eq. is

$$\frac{\partial w_2}{\partial t} + \max_{b\in B}\{D_x w_2 \cdot f(x, u_1^\#, b) + \ell_B(x, u_1^\#, b) = 0$$

2 H-J-B equations coupled via $u_2^\#$, $u_1^\#$.

<u>Proof</u> def. $\tilde{f}(x,t,a) := f(x, a, u_2^\#(x, D_x w_1(x,t), D_x w_2(x,t)))$

$\tilde{\ell}_A(x,t,a) = \ell_A(x, a, u_2^\#(\quad))$ $\Rightarrow$ 1st eq. is

$$\begin{cases} \frac{\partial w_1}{\partial t} + \max_a\{D_x w_1 \cdot \tilde{f}(x,t,a) + \tilde{\ell}_A(x,t,a)\} = 0 \\ w_1(x,T) = g_A(x) \end{cases}$$

a standard H-J-B eq. for 1st player. Can apply a Verif. thm. with dependence on time in f & ℓ

(see L.Notes, pf. ... same as without t ... HW).

$\Rightarrow$ $u_1^\#(x, D_x w_1, D_x w_2)$ is optimal feedback for a if 2nd uses $u_2^\#(x, D_x w_1, D_x w_2)$, among pairs $(u_A, u_2^\#(\quad))$ admiss. $\Rightarrow$ 1st part of being a Nash equil.

For 2nd player: use 2nd eq. & 2nd terminal condition use verif. thm. with fixed $u_1^\#(x, D_x w_1, D_x w_2)$

$$\begin{cases} \frac{\partial w_2}{\partial t} + \dots = 0 \\ w_2(x,T) = g_B(x) \end{cases}$$

$\Rightarrow$ $u_2^\#$ is opt .... $\Rightarrow$ 2nd part of Nash equil. ∎

APPLICATION : LINEAR - QUADRATIC DIFF. GAMES.

$$\begin{cases} \dot{y} = Ay + B_1 a + B_2 b \\ y(t) = x \end{cases}$$

$A \in \mathcal{M}_{n \times n}$ $\quad a \in \mathbb{R}^{m_1}$

$b \in \mathbb{R}^{m_2}$

$B_1 \in \mathcal{M}_{n \times m_1}$ $\quad B_2 \in \mathcal{M}_{n \times m_2}$

$$a(\cdot) \in L^2([0,T], \mathbb{R}^{m_1}) \ , \ b \in L^2([0,T], \mathbb{R}^{m_2})$$

$$J^A(x, t, a(\cdot), b(\cdot)) := -\int_t^T \left[y(s)^T \frac{M_1}{2} y(s) + \frac{|a(s)|^2}{2} \right] ds + y(T)^T \frac{Q_1}{2} y(T)$$

$$J^B(x, t, a(\cdot), b(\cdot)) := -\int_t^T \left[y(s)^T \frac{M_2}{2} y(s) + \frac{|b(s)|^2}{2} \right] ds + y(T)^T \frac{Q_2}{2} y(T)$$

$M_1, Q_1, M_2, Q_2 \in Sym(n)$

Q1. Hyp. # holds ? $\max_a \Phi^A$ & $\max_b \Phi^B$?

$$\Phi^A(a, b; x, P_1) = P_1^T (Ax + B_1 a + B_2 b) - x^T \frac{M_1}{2} x - \frac{|a|^2}{2} =$$

$$= \underbrace{P_1^T B_1 a - \frac{|a|^2}{2}}_{\to -\infty \text{ as } |a| \to \infty} + \text{ terms indep. of } a.$$

$\Rightarrow \exists$ UNIQUE MAX

Φ^B . . . . . .