Knowledge Representation and Learning

10. First Order Logic - Herbrand Theorem

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Definition (quantifier-free formula)

A formula ϕ is quantifier-free if ϕ has no occurrence of either of the quantifiers $\forall$ or $\exists$.

Notice that a quantifier-free formula is the combination of a set of First Order Atoms using the propositional connectives.

Definition (Universal-sentence)

... A universal sentence is a sentence (closed formula of the form

$$\forall x_1 \forall x_2 \dots \forall x_n . \phi(x_1, \dots, x_n)$$

where $\phi(x_1, \dots, x_n)$ is a quantifier-free formula.

Grounding

Definition (Ground instance)

A ground instance of an universal sentence $\forall x_1 \dots \forall x_n.\phi(x_1, \dots, x_n)$ is a sentence $\phi(t_1, \dots, t_n)$ obtained by replacing each occurrence of x_i with a term t_i that does not contain variables.

Notice that a ground instance of a universal sentence is a logical consequence the universal sentence itself. i.e.,

$$\forall x_1, \dots, x_n.\phi(x_1, \dots, x_n) \models \phi(t_1, \dots, t_n)$$

Therefore if $\phi(t_1, \dots, t_n)$ is not valid, then also $\forall x_1, \dots, x_n.\phi(x_1, \dots, x_n)$

Checking validity of universal sentence

- To verify if $\forall x_1, \ldots, x_n \phi(x_1, \ldots, x_n)$ is valid, you can search for an interpretation $\mathcal{I}$ and an n-tuple of terms $t_1, \ldots, t_n$ such that $\mathcal{I} \not\models \phi(t_1, \ldots, t_n)$. If you find it, then the universal formula is not valid
- but how can we prove that a formula $\forall x_1, \ldots, x_n \phi(x_1, \ldots, x_n)$ is valid?
- we have to check that for all possible interpretations and all possible assignments to the variables $x_1, \ldots, x_n$ to the elements of the interpretation domain.

Herbrand Universe

- Jacques Herbrand (1908-1931), proposes the main idea to interpret terms in themselves.
- Herbrand poposed to consider $\Delta^{\mathcal{I}}$ as the set of all ground terms that can be built from the signature Σ.
- Since $\Delta^{\mathcal{I}}$ must contain at least one elment, Herbrand required that Σ contains at least one constant symbol.

Definition (Herbrand Universe)

The *Herbrand's universe* of a signature Σ that contains at least one constant symbol, is the set, denoted by $\Delta^{\mathcal{H}}$ of ground terms of Σ.

The Herbrand Semantics of Terms

In a Herbrand model, every constant stands for itself.
Every function symbol stands for a term-forming operation: f denotes the function that puts 'f(' ... ')' around n elements of $\mathcal{H}$.

Herbrand Interpretation

Definition

An herbrand interpretation of a signature Σ is composed by the pair $(\Delta^{\mathcal{H}}_{\Sigma}, \mathcal{H})$, where

1. $\Delta^{\mathcal{H}}_{\Sigma}$ is the Herbrand's universe of Σ;
2. $\mathcal{H}(c) = c$ for every constant symbol $c \in \Sigma$;
3. $\mathcal{H}(f) : t_1, \ldots, t_n \mapsto f(t_1, \ldots, t_n)$ is the function that maps an n-tuple of terms of $\Delta^{\mathcal{H}}_{\Sigma}$ in a term of Δ^{H}_{Σ}, for every n-ary function symbol f;
4. $\mathcal{H}(P) \subseteq (\Delta^{H}_{\Sigma})^n$ is a set of n-tuples of terms in $\Delta^{\mathcal{H}}_{\Sigma}$, for evert n-ary predicate symbol $P \in \Sigma$.

Herbrand interpretationss

Definition (Herbrand base)

The Herbrand base for a signature Σ is the set of ground atomic formulas (i.e., the set of atomic formulas that do not contain individual variables)

$$\mathcal{HB}_\Sigma \stackrel{def}{=} \{P(t_1, \ldots, t_n) | t_1, \ldots, t_n \in \Delta^{\mathcal{H}}_\Sigma\}$$

- The Herbrand base can be seen as a (possibily infinite) set of propositional variables,
- an Herbrand interpretation is a truth assignment to them

$$\mathcal{H} : \mathcal{HB}_\Sigma \rightarrow \{0, 1\}$$

- we are back to propositional logic

Example of an Herbrand Model

$$S = \left\{ \begin{array}{c} friend(x,y) \rightarrow friend(x,y) \\ friend(x,y) \rightarrow knows(x, mother(y)) \\ friend(Mary, John) \end{array} \right\}$$

$$\Sigma = \{Mary, John, mother, friend, knows\}$$

$$\Delta^{\mathcal{H}}_{\Sigma} = \left\{ \begin{array}{l} Mary,\ John,\ mother(Mary),\ mother(John), \\ mother(mother(Mary)),\ mother(mother(John)) \\ mother(\ldots mother(Mary) \ldots), \\ mother(\ldots mother(John) \ldots), \\ \ldots \end{array} \right\}$$

Example of an Herbrand Model (cont'd

$$\mathcal{HB}_\Sigma = \left\{ \begin{array}{l} friend(John, Mary),\ \ friend(Mary, John), \\ friend(John, John),\ \ friend(Mary, Mary), \\ knows(John, Mary),\ \ knows(Mary, John), \\ knows(John, John),\ \ friend(Mary, Mary), \\ friend(mother(John), Mary),\ \ friend(Mary, mother(John)), \\ friend(mother(John), mother(John)), \\ knows(mother(John), Mary),\ \ knows(Mary, mother(John)), \\ knows(mother(John), mother(John)), \\ \dots \end{array} \right\}$$

Herbrand's Theorem

Theorem (Herbrand's Theorem)

A universal formula $\forall x_1, \ldots, x_n \, \phi(x_1, \ldots, x_n)$ *is satisfiable if it is satisfied by an Herbrand interpretation on the signature* Σ *that appear in* ϕ*. If* ϕ *does not contain constant symbol we extend* Σ *with a constant symbol* a*.*

Using Herbrand's Theorem for Sat

- to check if $\Phi = \forall x_1, \ldots, x_n \, \phi(x_1, \ldots, x_n)$ is unsatisfiable we can check if it is false in all the herbrand interpretations.
- Ψ is true in an Herband interpretation $\mathcal{H}$ iff $\mathcal{H} \models \text{Ground}(\Phi)$

$$\text{Ground}(\Phi) = \{\phi(t_1, \ldots, t_n) \mid t_i \in \Delta_\Sigma^{\mathcal{H}}\}$$

- Φ is unsat iff Ground(Φ) is unsat
- By compactness theorem Ground(Φ) is unsat if a finite subset $G \subset \text{Ground}(\Phi)$ is unsat.
- we can enumerate all the finite subsets, $G_0, G_1, G_2, \ldots$ of Ground(Φ) and check for propositional satisfiability
- If Φ is unsat then we eventually discover it
- otherwise we can go on infinitely.

Skolemization

Suppose that in a formula the most internal existential quantifire falls in the scope of k universal quantifiers.

$$\forall x_1 \dots \forall x_2 \dots \forall x_k \dots \exists y \phi(y)$$

Choose a fresh k-place function symbol, say f, and replace y by $f(x_1, x_2, \dots, x_k)$.
We get

$$\forall x_1 \dots \forall x_2 \dots \forall x_k \dots \exists y \phi(f(x_1, \dots, s_n))$$

Repeat this replacement for all existential quantifiers

Skolemn's Theorem

Example (Skolemization)

Suppose that we want to check the satisfiability of the Σ-formula $\exists x F(x)$[a]

- We have to find an interpretation (Σ-structure) $\mathcal{I}$, such that

$$\mathcal{I} \models \exists x.F(x)$$

- i.e., we have to exhibit an element $d \in \Delta_{\mathcal{I}}$ such that

$$\mathcal{I} \models F(x)[a_{x \mapsto d}].$$

- This is equivalent to find an interpretation $\mathcal{I}'$ of the signature Σ' obtained by extending Σ with a new constant c, i.e. a constant that does not appear in Σ such that

$$\mathcal{I}' \models F(c)$$

- $\mathcal{I}'$ is the same as $\mathcal{I}$ with the additional interpretation $\mathcal{I}'(c) = d$;
- c is called Skolem constant.
- the transformation of $\exists x.F(x)$ into $F(c)$ is called Scolemization.

[a] A Σ-formula is a formula in the signature Σ

Skolemn's Theorem

Example (Skolemization)

Suppose that we want to check the satisfiability of the Σ-formula $\forall x, \exists x F(x, y)$

- We have to find an interpretation $\mathcal{I}$, such that $\mathcal{I} \models \forall x \exists y.F(x, y)$;
- which implies that for all $d \in \Delta_{\mathcal{I}}$

$$\mathcal{I} \models \exists y.F(x, y)[a_{x \mapsto d}]; \tag{1}$$

- to satisfy (1), for every $d \in \Delta_{\mathcal{I}}$ we have to exhibit a $d' \in \Delta_{\mathcal{I}}$ such that

$$\mathcal{I} \models F(x, y)[a_{\substack{x \mapsto d \\ y \mapsto d'}}]$$

- This is equivalent to find an interpretation $\mathcal{I}'$ of the signature Σ' obtained by extending Σ with a new unary functional symbol f_{sk}, such that

$$\mathcal{I}' \models \forall x.F(x, f_{sk}(x))$$

- $\mathcal{I}'$ is the same as $\mathcal{I}$ with the additional interpretation $\mathcal{I}'(f_{sk})$ equal to the function that maps every d into the d' that satisfies condition (1);
- f_{sk} is called Skolem function;
- the transformation of $\forall x \exists y.F(x, y)$ into $\forall x.F(x, f_{sk}(x))$ is called Scolemization.

Clause Form

Clause: a disjunction of literals

$$\neg K_1 \vee \cdots \vee \neg K_m \vee L_1 \vee \cdots \vee L_n$$

Set notation: $\{\neg K_1, \ldots, \neg K_m, L_1, \ldots, L_n\}$
Kowalski notation: $K_1, \ldots, K_m \rightarrow L_1, \ldots, L_n$
$L_1, \ldots, L_n \leftarrow K_1, \ldots, K_m$'
$\Box$ is the Empty clause:
Empty clause is equivalent to false , meaning Contradiction

Quantifier Equivalences

- If x is not free in B.

$$(\exists x A) \land B \leftrightarrow \exists x(A \land B)$$
$$(\exists x A) \lor B \leftrightarrow \exists x(A \lor B)$$

Outline of Clause Form Methods

To prove A, obtain a contradiction from $\neg A$

1. Translate $\neg A$ into CNF as $A_1 \wedge \cdots \wedge A_m$
2. This is the set of clauses $A_1 \ldots, A_m$
3. Transform the clause set, preserving consistency

Deducing the empty clause ($\Box$) refutes $\neg A$. This is like in propositional resolution

Prenex Normal Form

Rename quantified variable, so that each quantifier $\forall x$ and $\exists x$ is defined on a separated variable

$$\forall x P(x) \wedge \exists x P(x) \quad \Longrightarrow \quad \forall x_1 P(x_1) \wedge \exists x_2 P(x_2)$$

Convert to Negation Normal Form using the propositional rewriting rules plus the additional rules

$$\begin{aligned} \neg(\forall x A) & \Longrightarrow \exists x \neg A \\ \neg(\exists x A) & \Longrightarrow \forall x \neg A \end{aligned}$$

Move quantifiers to the front using (provided x is not free in B)

$$\begin{aligned} (\forall x A) \wedge B & \equiv \forall x(A \wedge B) \\ (\forall x A) \vee B & \equiv \forall x(A \vee B) \end{aligned}$$

and the similar rules for $\exists$

Example of Conversion to Clauses

For proving

$$\exists x(P(x) \rightarrow \forall x P(x))$$

$\exists x(P(x) \rightarrow \forall y P(y))$	rename variables
$\neg[\exists x[P(x) \rightarrow \forall y P(y)]]$	negated goal
$\forall x[P(x) \wedge \exists y \neg P(y)]$	conversion to NNF
$\forall x \exists y[P(x) \wedge \neg P(y)]$	pulling $\exists$ out
$\forall x[P(x) \wedge \neg P(f(x))]$	Skolem term $f(x)$
$\{P(x)\}, \{\neg P(f(x))\}$	Final clauses

Correctness of Skolemization

The formula $\forall x \exists y A$ is consistent
$\Longrightarrow$ it holds in some interpretation (Δ, I)
$\Longrightarrow$ for all $x \in \Delta$ there is some $y \in \Delta$ such that A holds
$\Longrightarrow$ some function $F : D \to D$ yields suitable values of y given x
$\Longrightarrow$ $A[f(x)/y]$ holds in some (Δ, I') extending (Δ, I) so that $I'(f) = F$.
$\Longrightarrow$ the formula $\forall x A[f(x)/y]$ is consistent.

Simplifying the Search for Models

S is satisfiable if even one model makes all of its clauses true.
Differently from propositional logic, There are infinitely many models to consider!
Also many duplicates : "states of the USA" and "the integers 1 to 50"
Fortunately, nice models exist.

- They have a uniform structure based on the language's syntax.
- They satisfy the clauses if any model does.

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