

On a Discrete Approximation of the Hamilton-Jacobi Equation of Dynamic Programming*

I. Capuzzo Dolcetta

Istituto Matematico, Università di Roma, Rome, Italy, and Department of Mathematics, University of Maryland, College Park, MD 20742

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Abstract. An approximation of the Hamilton-Jacobi-Bellman equation connected with the infinite horizon optimal control problem with discount is proposed. The approximate solutions are shown to converge uniformly to the viscosity solution, in the sense of Crandall-Lions, of the original problem. Moreover, the approximate solutions are interpreted as value functions of some discrete time control problem. This allows to construct by dynamic programming a minimizing sequence of piecewise constant controls.

1. Introduction

The Hamilton-Jacobi equation

$$\max_{1 \leq d \leq m} \left\{ \lambda u - \sum_{i=1}^n g_i^d \frac{\partial u}{\partial x_i} - f^d \right\} = 0 \quad \text{in } \mathbb{R}^n, \tag{HJ}$$

arises (see, for example, Fleming-Rishel [8], Lions [12]) as the optimality condition for the *infinite horizon discounted optimal control problem*

$$\text{Inf} \left\{ \int_0^\infty f(y_x(s), \alpha(s)) e^{-\lambda s} ds \mid \alpha : [0, +\infty[\rightarrow \{1, \dots, m\}, \alpha \text{ measurable} \right\}, \tag{CP}$$

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where $y_x(s)$ is defined by the controlled differential equation

$$\begin{aligned} \frac{dy}{ds} &= g(y(s), \alpha(s)) \\ y(0) &= x. \end{aligned} \tag{ODE}$$

It is well known (see [8]) that the dynamic programming techniques cannot be made fully rigorous for the problem at hand, due to the fact that (HJ) does not have, in general, a C^1 solution.

A new notion of generalized solution for Hamilton-Jacobi equations has been recently introduced by Crandall-Lions [6] (see also Crandall-Evans-Lions [5]).

Namely, a bounded uniformly continuous function u is called a *viscosity solution* of (HJ) provided for each $\phi \in C^1(\mathbb{R}^n)$ the following hold:

(i) if $u - \phi$ attains a local maximum at x_0 , then

$$\text{Max}_{1 \leq d \leq m} \left\{ \lambda u - \sum_{i=1}^n g_i^d \frac{\partial \phi}{\partial x_i} - f^d \right\} \leq 0 \quad \text{at } x_0$$

(ii) if $u - \phi$ attains a local minimum at x_1 , then

$$\text{Max}_{1 \leq d \leq m} \left\{ \lambda u - \sum_{i=1}^n g_i^d \frac{\partial \phi}{\partial x_i} - f^d \right\} \geq 0 \quad \text{at } x_1.$$

The above definition, while not requiring u to be differentiable at any point, is strong enough, however, to ensure uniqueness of the solution (see [5, 6]).

On the other hand, a viscosity solution of (HJ) can be constructed as the uniform limit as $\epsilon \searrow 0$ of the solutions of the second order elliptic problems

$$-\epsilon \Delta u^\epsilon + \text{Max}_{1 \leq d \leq m} \left\{ \lambda u^\epsilon - \sum_{i=1}^n g_i^d \frac{\partial u^\epsilon}{\partial x_i} - f^d \right\} = 0,$$

for which a priori estimates are obtained via the maximum principle, (see [5, 6, 12]).

The purpose of this paper is to show how the viscosity solution of (HJ) can be constructed by means of a completely different approximation procedure, which makes no use of PDE methods.

Namely, we consider the approximate equations

$$\begin{aligned} \text{Max}_{1 \leq d \leq m} \{ u^h(x) - (1 - \lambda h) u^h(x + g^d(x)h) - hf^d(x) \} &= 0, \\ x \in \mathbb{R}^n, \quad h > 0 &\tag{HJ}^h \end{aligned}$$

and show by the contraction mapping theorem that $(HJ)^h$ has, for sufficiently small h , a unique bounded, Hölder continuous solution u^h . Uniform a priori estimates on u^h and their Hölder seminorms allow us then to show that u^h

converges locally uniformly, as $h \searrow 0$, to a function u , which solves (HJ) in the viscosity sense (see Sect. 2).

A final section is devoted to the application of the above result to the optimal control problem underlying (HJ). More precisely, the functions u^h are interpreted as *values* of some discrete time control problems and, by means of rigorous dynamic programming techniques (see Bertsekas-Shreve [1]), optimal discrete feedback controls are designed. These approximated controls are shown to form a minimizing sequence for the original problem. A key role in this is played by the remark (see Lions [12]) that the viscosity solution of (HJ) is in fact the value function for problem (CP).

The approximation considered in the present paper has been used in the particular case of the optimal stopping time problem by Capuzzo Dolcetta-Matzeu [3] (see also Capuzzo Dolcetta-Matzeu-Menaldi [4], Gawronski [9], Goletti [10]). The methods of this paper are, however, different and the convergence proof simplified, due to the use of the viscosity solution notion.

2. The Approximate Equations

We shall assume in the following that

$$\begin{aligned}
 &|g_i^d(x) - g_i^d(x')| \leq L|x - x'|, \quad |g_i^d(x)| \leq L \\
 &\text{for some constant } L > 0, \quad \text{for all } x, \quad x' \in \mathbb{R}^n, \\
 &d \in \{1, \dots, m\}, \quad i = 1, \dots, n.
 \end{aligned}
 \tag{2.1}$$

$$\begin{aligned}
 &|f^d(x) - f^d(x')| \leq D|x - x'|^\gamma, \quad |f^d(x)| \leq D \\
 &\text{for some constants } D > 0, \quad \gamma \in [0, 1], \quad x' \in \mathbb{R}^n, \quad d \in \{1, \dots, m\}
 \end{aligned}
 \tag{2.2}$$

for all x

$$\lambda > 0.
 \tag{2.3}$$

Let $h > 0$ be parameter and consider the following approximation of (HJ):

$$\text{Max}_{1 \leq d \leq m} \{u^h(x) - (1 - \lambda h)u^h(x + g^d(x)h) - hf^d(x)\} = 0, \quad x \in \mathbb{R}^n.
 \tag{HJ}^h$$

A more natural approximation would be

$$\text{Max}_{1 \leq d \leq m} \{\lambda hu^h(x) - (u^h(x) - u^h(x + g^d(x)h)) - hf^d(x)\} = 0;$$

the choice of (HJ)^h is, however, more convenient since it suggests immediately the way of proving the existence of solutions (see Theorem 2.1 below).

We shall denote by X the space of bounded, Hölder continuous functions on \mathbb{R}^n , normed by

$$\|v\|_X = \sup_x |v(x)| + \sup_{x_1 \neq x_2} \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|^\gamma}, \quad \gamma \in [0, 1]. \quad (2.4)$$

We shall also employ the notation

$$|v|_{0,\gamma} = \sup_{x_1 \neq x_2} \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|^\gamma}.$$

Theorem 2.1. *If $\lambda > \gamma L$, then (HJ)^h has a unique solution $u^h \in X$ for any $h \in [0, \frac{1}{\lambda}]$. Moreover, the following estimates hold:*

$$\sup_x |u^h(x)| \leq \frac{D}{\lambda}, \quad (2.5)$$

$$\sup_{x_1 \neq x_2} \frac{|u^h(x_1) - u^h(x_2)|}{|x_1 - x_2|^\gamma} \leq \frac{D}{\lambda - \gamma L}. \quad (2.6)$$

Proof. Let us observe that (HJ)^h is equivalent to the fixed point equation

$$u^h(x) = Tu^h(x), \quad x \in \mathbb{R}^n, \quad (2.7)$$

where T is defined by

$$Tv(x) = \text{Min}_{1 \leq d \leq m} \{(1 - \lambda h)v(x + g^d(x)h) + hf^d(x)\}, \quad x \in \mathbb{R}^n. \quad (2.8)$$

Let $v, \hat{v} \in X$ and choose d, \hat{d} (depending on x) such that the minimum in (2.8) is attained. Then

$$\begin{aligned} Tv(x) - T\hat{v}(x) &= (1 - \lambda h) \left[v(x + g^d(x)h) - \hat{v}(x + g^{\hat{d}}(x)h) \right] \\ &\quad + h \left[f^d(x) - f^{\hat{d}}(x) \right] \\ &\leq (1 - \lambda h) \left[v(x + g^{\hat{d}}(x)h) - \hat{v}(x + g^{\hat{d}}(x)h) \right] \\ &\leq (1 - \lambda h) \sup_x |(v - \hat{v})(x)|, \end{aligned}$$

and, symmetrically,

$$T\hat{v}(x) - Tv(x) \leq (1 - \lambda h) \sup_x |(v - \hat{v})(x)|.$$

Therefore,

$$\sup_x |Tv(x) - T\hat{v}(x)| \leq (1 - \lambda h) \sup_x |(v - \hat{v})(x)|. \tag{2.9}$$

Hence, by the contraction mapping principle there exists a unique bounded function u^h satisfying (HJ)^h. Let us show now that u^h belongs in fact to X . Let us observe at this purpose that for every $v \in X$ the following inequality holds:

$$\begin{aligned} Tv(x_1) - Tv(x_2) &= (1 - \lambda h) [v(x_1 + g^{d_1}(x_1)h) - v(x_2 + g^{d_2}(x_2)h)] \\ &\quad + h [f^{d_1}(x_1) - f^{d_2}(x_2)] \\ &\leq (1 - \lambda h) [v(x_1 + g^{d_2}(x_1)h) - v(x_2 + g^{d_2}(x_2)h)] \\ &\quad + h [f^{d_2}(x_1) - f^{d_2}(x_2)] \\ &\leq (1 - \lambda h) |v|_{0,\gamma} (1 + Lh)^\gamma |x_1 - x_2|^\gamma + hD |x_1 - x_2|^\gamma, \end{aligned}$$

where d_1, d_2 are such that the minimum in (2.8) is attained.

Hence, by symmetry,

$$|Tv|_{0,\gamma} \leq (1 - \lambda h)(1 + Lh)^\gamma |v|_{0,\gamma} + hD. \tag{2.10}$$

Since $\lambda > \gamma L$, the number $C_h = \frac{hD}{1 - (1 - \lambda h)(1 + Lh)^\gamma}$ is strictly positive and it is easy to check that

$$|Tv|_{0,\gamma} \leq C_h, \tag{2.11}$$

for every $v \in X$ with $|v|_{0,\gamma} \leq C_h$.

Therefore, the iterates $T^n u_0$ starting from any u_0 with $|u_0|_{0,\gamma} \leq C_h$ converge to the unique solution $u^h \in X$ of (HJ)^h.

The right-hand member of (2.11) is a decreasing function of $h > 0$. Hence,

$$|u^h|_{0,\gamma} \leq \lim_{h \rightarrow 0^+} \frac{hD}{1 - (1 - \lambda h)(1 + Lh)^\gamma} = \frac{D}{\lambda - \gamma L}$$

and (2.6) is proved.

From the inequality

$$|Tu^h(x)| \leq (1 - \lambda h) \sup_x |u^h(x)| + hD,$$

it follows at once that

$$\lambda h \sup_x |u^h(x)| \leq hD,$$

and this proves (2.5). □

The next result provides the identification of the limit as $h \searrow 0$ of u^h as the viscosity solution of (HJ).

Theorem 2.2. *As $h \searrow 0$, $u^h \rightarrow u$ locally uniformly in \mathbb{R}^n , where u is the viscosity solution of (HJ).*

Proof. In view of Thm. 2.1 and the Ascoli-Arzelà compactness criterion, there exists a subsequence $h_p \searrow 0$ as $p \rightarrow +\infty$ and a function $u \in X$ such that

$$u^{h_p} \rightarrow u \text{ as } p \rightarrow +\infty, \text{ locally uniformly in } \mathbb{R}^n. \tag{2.12}$$

We shall prove now that u solves (HJ) in the viscosity sense according to the definition of Sect. 1. To see this, let us take $\phi \in C^1(\mathbb{R}^n)$ and assume x_0 is a local maximum for $u - \phi$.

We may assume (see [2]) that there exists a closed ball B centered at x_0 such that

$$(u - \phi)(x_0) > (u - \phi)(x), \quad \text{for all } x \in B. \tag{2.13}$$

Let now $x_0^{h_p}$ be a maximum point for $u^{h_p} - \phi$ over B ; from (2.12) and (2.13) it follows that

$$x_0^{h_p} \rightarrow x_0, \quad \text{as } p \rightarrow +\infty. \tag{2.14}$$

Then, by (2.1) and (2.14), for any $d \in \{1, \dots, m\}$ the point $x_0^{h_p} + g^d(x_0^{h_p})h_p$ is in B , provided p is large enough, and therefore

$$u^{h_p}(x_0^{h_p}) - \phi(x_0^{h_p}) \geq u^{h_p}(x_0^{h_p} + g^d(x_0^{h_p})h_p) - \phi(x_0^{h_p} + g^d(x_0^{h_p})h_p), \tag{2.15}$$

$$d \in \{1, \dots, m\}.$$

Now (2.15) and (HJ) ^{h_p} yield

$$0 = \text{Max}_{1 \leq d \leq m} \left\{ u^{h_p}(x_0^{h_p}) - (1 - \lambda h_p) u^{h_p}(x_0^{h_p} + g^d(x_0^{h_p})h_p) - h_p f^d(x_0^{h_p}) \right\}$$

$$\geq \text{Max}_{1 \leq d \leq m} \left\{ \phi(x_0^{h_p}) - \phi(x_0^{h_p} + g^d(x_0^{h_p})h_p) \right.$$

$$\left. + \lambda h_p u^{h_p}(x_0^{h_p} + g^d(x_0^{h_p})h_p) - h_p f^d(x_0^{h_p}) \right\}. \tag{2.16}$$

Since $\phi \in C^1(\mathbb{R}^n)$, from (2.16) it follows that

$$0 \geq \text{Max}_{1 \leq d \leq m} \left\{ -h_p^{-1} \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(x_0^{h_p} + \beta_p^d g^d(x_0^{h_p})h_p) g_i^d(x_0^{h_p})h_p \right.$$

$$\left. + \lambda u^{h_p}(x_0^{h_p} + g^d(x_0^{h_p})h_p) - f^d(x_0^{h_p}) \right\}, \tag{2.17}$$

for some $\beta_p^d \in [0, 1]$. Using (2.12) and (2.14), we can pass to the limit in (2.17) as $p \rightarrow +\infty$. This shows that part (i) of the definition of viscosity solution is satisfied by u .

The same argument applies with obvious modification to case (ii).

Since the viscosity solution of (HJ) is unique (see [5, 6]) we have also that $u_h \rightarrow u$ as $h \searrow 0$, and the proof is complete. \square

Remark 2.1. If (2.2) holds with $\gamma = 1$ and $\lambda > L$, then $u = \lim_{h \rightarrow 0} u^h$ is Lipschitz continuous. Hence, by Rademacher's Theorem, u solves (HJ) almost everywhere (see [6]).

Remark 2.2. The above results apply to the optimal stopping time problem. To see this it is enough to augment (HJ) by setting

$$\begin{aligned} g^{m+1}(x) &= 0 & x \in \mathbb{R}^n \\ f^{m+1}(x) &= \frac{\psi(x)}{\lambda} & x \in \mathbb{R}^n, \end{aligned}$$

where the stopping cost ψ is a given function satisfying (2.2). This problem has been studied by Menaldi [13] (see also [3]) in the case where the control does not affect the dynamics (i.e., $g^d(x) = g(x)$, $d = 1, \dots, m$).

3. Interpretation of the Approximate Solutions

In this section we interpret the previous results in terms of the optimal control problem (CP).

Let us associate to every control $\alpha(\cdot)$ a vector $y_k = y_k^h(kh)$ by the recursive formula

$$\begin{aligned} y_{k+1} &= y_k + g^{d_k}(y_k)h, & k = 0, 1, 2, \dots \\ y_0 &= x, \end{aligned} \tag{ODE}^h$$

where

$$d_k = \alpha(kh), \quad k = 0, 1, 2, \dots \tag{3.1}$$

and a cost

$$J_x^h(\alpha(\cdot)) = h \sum_{k=0}^{\infty} f^{d_k}(y_k)(1 - \lambda h)^k \tag{3.2}$$

Define also a function $d_h^* : \mathbb{R}^n \rightarrow \{1, \dots, m\}$ by setting

$$d_h^*(x) = \text{Min}\{d \in \{1, \dots, m\} | u^h(x) = (1 - \lambda h)u^h(x + g^d(x)h) + hf^d(x)\} \tag{3.3}$$

and denote by $\alpha_h^*(\cdot)$ the control

$$\alpha_h^*(s) = d_h^*(y_k), \quad s \in [kh, (k+1)h), \quad k = 0, 1, 2, \dots \tag{3.4}$$

Proposition 3.1. *The solution u^h of (HJ)^h satisfies*

- (i) $u^h(x) \leq J_x^h(\alpha(\cdot)), \quad \forall x \in \mathbb{R}^n, \quad \forall \alpha(\cdot)$
- (ii) $u^h(x) = J_x^h(\alpha_h^*(\cdot)), \quad \forall x \in \mathbb{R}^n.$

Proof. It is convenient to use the equivalent formulation (2.8) of (HJ)^h. Choose any $\alpha(\cdot)$ and set $d_k = \alpha(kh)$. From (2.8) it easily follows that

$$u^h(x) \leq (1 - \lambda h)^p u^h(y_p) + h \sum_{k=0}^{p-1} f^{d_k}(y_k) (1 - \lambda h)^k, \quad p = 1, 2, \dots, \tag{3.5}$$

where y_p is given by (ODE)^h.

Since u^h is bounded and $0 < 1 - \lambda h < 1$, (i) follows from (3.5) by letting $p \rightarrow +\infty$.

To prove (ii) it is enough to observe that the particular choice (3.3), (3.4) yields equality in (3.5) for every p . □

Remark 3.1. If $f^d \geq 0, (d = 1, \dots, m)$, the iteration

$$u_1^h(x) \equiv 0$$

$$u_{p+1}^h(x) = \text{Min}_{1 \leq d \leq m} \{ (1 - \lambda h) u_p^h(x + g^d(x)h) + h f^d(x) \}$$

converge monotonically increasing as $p \rightarrow +\infty$ to the solution u^h of (HJ)^h.

The same argument of the proof of Prop. 3.1 shows that the iterates u_p^h are the value functions for the truncated costs

$$J_x^{h,p}(\alpha(\cdot)) = h \sum_{k=0}^{p-1} f^{d_k}(y_k) (1 - \lambda h)^k, \quad p = 1, 2, \dots$$

As a consequence of the above proposition and Thm. 2.2, we have that $J_x^h(\alpha_h^*(\cdot))$ converges locally uniformly, as $h \searrow 0$, to the viscosity solution u of (HJ) which, in the present setting, is the value function of problem (CP), (see [2, 12]). We have therefore proved the following

Proposition 3.2. *Let $\alpha_h^*(\cdot)$ be the control defined by (3.3), (3.4). Then, as $h \searrow 0$,*

$$J_x^h(\alpha_h^*(\cdot)) \rightarrow \text{Inf}_{\alpha(\cdot)} J_x(\alpha(\cdot)) \quad \text{locally uniformly in } \mathbb{R}^n, \tag{3.6}$$

where

$$J_x(\alpha(\cdot)) = \int_0^\infty f(y_x(s), \alpha(s)) e^{-\lambda s} ds. \quad (3.7)$$

It turns out that $\alpha_h^*(\cdot)$ is in fact a minimizing sequence for the cost functional (3.7).

Theorem 3.1. *Let $\alpha_h^*(\cdot)$ be the control defined by (3.3), (3.4). Then,*

$$J_x(\alpha_h^*(\cdot)) \rightarrow \text{Inf}_{\alpha(\cdot)} J_x(\alpha(\cdot)), \quad \text{as } h \searrow 0. \quad (3.8)$$

Proof. For every $T > 0$ and $h > 0$ we have

$$\begin{aligned} & |J_x^h(\alpha_h^*(\cdot)) - J_x(\alpha_h^*(\cdot))| \\ & \leq \left| h \sum_{k=0}^{[T/h]-1} f^{d_k^*}(y_k)(1-\lambda h)^k - \int_0^T f(y_x(s), \alpha_h^*(s)) e^{-\lambda s} ds \right| \\ & \quad + \left| h \sum_{k=[T/h]}^{+\infty} f^{d_k^*}(y_k)(1-\lambda h)^k \right| + \left| \int_T^\infty f(y_x(s), \alpha_h^*(s)) e^{-\lambda s} ds \right|, \end{aligned} \quad (3.9)$$

where $[T/h]$ denotes the largest integer less or equal than T/h .

Hence, for every $\varepsilon > 0$

$$|J_x^h(\alpha_h^*(\cdot)) - J_x(\alpha_h^*(\cdot))| \leq E(h, T) + 2\varepsilon, \quad (3.10)$$

for T large enough, where

$$E(h, T) = \left| h \sum_{k=0}^{[T/h]-1} f^{d_k^*}(y_k)(1-\lambda h)^k - \int_0^T f(y_x(s), \alpha_h^*(s)) e^{-\lambda s} ds \right|. \quad (3.11)$$

From (3.4) it follows that

$$\begin{aligned} E(h, T) = & \left| \sum_{k=0}^{[T/h]-1} \int_{kh}^{(k+1)h} [f^{d_k^*}(y_k)(1-\lambda h)^k - f^{d_k^*}(y_x(s)) e^{-\lambda s}] ds \right. \\ & \left. - \int_{[T/h]h}^T f(y_x(s), \alpha_h^*(s)) e^{-\lambda s} ds \right|. \end{aligned} \quad (3.12)$$

Hence, taking (2.1) and (2.2) into account,

$$E(h, T) \leq D \sum_{k=0}^{[T/h]-1} \int_{kh}^{(k+1)h} [|y_k - y_x(s)|^\gamma (1 - \lambda h)^k + |(1 - \lambda h)^k - e^{-\lambda s}|] ds + D(T - [T/h]h). \tag{3.13}$$

It is well-known (see Henrici [11]), that

$$|y_k - y_x(s)| \leq C_T h$$

$$|(1 - \lambda h)^k - e^{-\lambda s}| \leq C_T h, \quad s \in [kh, (k+1)h], k = 0, 1, \dots, [T/h] - 1 \tag{3.14}$$

for some positive constant C_T .

Therefore, (3.13) and (3.14) yield

$$E(h, T) \leq D\tilde{C}_T [T/h]h(h^\gamma + h) + D(T - [T/h]h)$$

which implies, since $[T/h]h \rightarrow T$ as $h \searrow 0$, that

$$E(h, T) \rightarrow 0 \quad \text{as } h \searrow 0.$$

Hence, by (3.10),

$$J_x^h(\alpha_h^*(\cdot)) - J_x(\alpha_h^*(\cdot)) \rightarrow 0 \quad \text{as } h \searrow 0$$

and the statement is proved, taking Proposition 3.2 into account. □

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