

# ON HOPF'S FORMULAS FOR SOLUTIONS OF HAMILTON–JACOBI EQUATIONS

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## 1. INTRODUCTION

GIVEN a function  $H: \mathbb{R}^n \rightarrow \mathbb{R}$ , we consider the Hamilton–Jacobi equation

$$u_t + H(Du) = 0 \tag{1.1}$$

in  $\mathbb{R}_+^{n+1} \equiv \mathbb{R}^n \times (0, \infty)$ . This PDE admits a particularly simple class of solutions, namely the linear functions

$$\alpha \cdot x - tH(\alpha) + \beta \tag{1.2}$$

for fixed  $\alpha \in \mathbb{R}^n$ ,  $\beta \in \mathbb{R}$ . Hopf in [8] addresses the possibility of constructing more general solution of (1.1) as an envelope of (1.2) as  $\alpha$  and  $\beta$  vary appropriately. He investigates the initial value problem

$$\left. \begin{aligned} u_t + H(Du) &= 0 && \text{in } \mathbb{R}_+^{n+1} \\ u(\cdot, 0) &= u_0(\cdot) && \text{on } \mathbb{R}^n, \end{aligned} \right\} \tag{1.3}$$

where  $u_0: \mathbb{R}^n \rightarrow \mathbb{R}$  is given, and proposes two formulas for the solution:

$$(i) \quad u(x, t) \equiv \inf_z \sup_y \{u_0(z) + y \cdot (x - z) - tH(y)\}$$

and

$$(ii) \quad u(x, t) \equiv \sup_y \inf_z \{u_0(z) + y \cdot (x - z) - tH(y)\}.$$

(Here and elsewhere unless otherwise noted the “inf” and “sup” are taken over  $\mathbb{R}^n$ .) Notice that (i) and (ii) differ only in the interchange of the inf and the sup, and that each expression in the braces  $\{ \}$  has the form (1.2). Consequently we expect  $u$  given by either formula to solve (1.1) a.e. and so be a candidate for a weak solution of (1.3). Indeed, under the basic

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assumptions for (i) that  $H$  be convex and for (ii) that  $u_0$  be convex. Hopf demonstrates that these formulas do in fact yield in some sense reasonable generalized solutions of (1.3)<sup>†</sup>.

What is missing, however, in his analysis is a good notion of a weak solution of (1.3): the function  $u$  given by (i) or (ii) does indeed solve the PDE a.e. and takes on the proper initial values, but there are in general many other functions doing this as well. It is thus not clear that (i) or (ii) gives the “correct” generalized solution. In this paper we resolve the difficulty by showing—under various hypotheses—that (i) or (ii) gives the unique solution of (1.3) in the “viscosity” sense, a concept introduced by Crandall and Lions [3] and reformulated by Crandall, Evans and Lions [2]. This has already been carried out for (i) by Lions [12], whose proof we reproduce in Section 2 for the reader’s convenience and motivation for our study in Section 3 of formula (ii). Roughly speaking, if we assume  $H$  is convex we can analyse (i) by control theory methods, whereas to study (ii) when  $H$  is not convex we must use game theory techniques. These last have been developed by Evans and Souganidis [6] for fairly general  $H$  and  $u_0$ , so that the point of Section 3 is the simplification resulting in the formulas of [6] when  $u_0$  is convex.

Two applications appear in Section 4. Here we show in particular that some expressions for solutions recently obtained by Osher [14] (for what amounts to the Riemann problem for (1.3)) are special cases of Hopf’s formula (ii). We demonstrate also in Section 4 that (1.3) has a classical solution existing for all  $t > 0$  under the main hypotheses that  $H$  and  $u_0$  are convex; this is the Hamilton–Jacobi analogue of the familiar assertion that scalar conservation laws in one dimension have classical solutions if the nonlinearity is convex and the initial function is nondecreasing.

*Notation.* Throughout we let  $\Phi^*$  denote the Legendre transform of  $\Phi$ ; that is,

$$\Phi^*(x) \equiv \sup_y \{x \cdot y - \Phi(y)\} \leq \infty. \quad (x \in \mathbb{R}^n).$$

We refer the reader to Crandall, Evans and Lions [2] for the definition and properties of viscosity solutions.

2. CONVEX HAMILTONIANS: FORMULA (i)

In this section we extend slightly the analysis of formula (i) in [12, p. 216–219]<sup>‡</sup>. Let us assume:

$$(H_1) \begin{cases} \text{(a) } H : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is convex;} \\ \text{(b) } u_0 : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is uniformly Lipschitz.} \end{cases}$$

THEOREM 2.1. Under these hypotheses

$$u(x, t) \equiv \inf_z \sup_y \{u_0(z) + y \cdot (x - z) - tH(y)\} \quad (t \geq 0, \quad x \in \mathbb{R}^n) \quad (2.1)$$

is the unique uniformly continuous viscosity solution of

$$\left. \begin{aligned} u_t + H(Du) &= 0 && \text{in } \mathbb{R}^{n-1} \\ u(\cdot, 0) &= u_0(\cdot) && \text{on } \mathbb{R}^n. \end{aligned} \right\} \quad (2.2)$$

<sup>†</sup> The analogue of (i) for scalar conservation laws is due to Hopf [7] (for a special case), Lax [11], and Oleinik [13]; formula (i) appears in Kružkov [10]. See also Conway and Hopf [1].

<sup>‡</sup> Equation (i) is called the “Lax formula” in [12].

The proof of theorem 2.1 follows the lemma below. For each given  $t > 0$  define

$$N \equiv L^1(0, t; \mathbb{R}^n).$$

LEMMA 2.1. Fix  $t \geq 0$ ,  $x \in \mathbb{R}^n$ . Let

$$\hat{u}(x, t) \equiv \inf_{z(\cdot) \in N} \left\{ \int_0^t H^*(z(s)) \, ds + u_0(x(t)) \right\}, \tag{2.3}$$

where for each  $z(\cdot) \in N$  we set

$$x(t) \equiv x - \int_0^t z(s) \, ds. \tag{2.4}$$

Then  $\hat{u}$  is the unique uniformly continuous viscosity solution of (2.2).

*Proof.* It is not particularly difficult to check that  $\hat{u}$  is finite and uniformly Lipschitz in  $\mathbb{R}^{n+1}$ . Furthermore, arguing as in [12], we see that  $\hat{u}$  is a viscosity solution of the dynamic programming equation

$$\hat{u}_t + \sup_z \{z \cdot D\hat{u} - H^*(z)\} = 0 \quad \text{in } \mathbb{R}^{n+1}.$$

However

$$\sup_z \{z \cdot D\hat{u} - H^*(z)\} = H(D\hat{u})$$

and thus  $\hat{u}$  is a viscosity solution of (2.2). Uniqueness is a consequence of Ishii [9, theorem 2.1]. ■

*Proof of theorem 2.1.* Select  $t > 0$ ,  $x \in \mathbb{R}^n$ .

We may rewrite (2.1) to read

$$u(x, t) = \inf_z \left\{ u_0(z) + tH^*\left(\frac{x-z}{t}\right) \right\}. \tag{2.5}$$

Now recall formula (2.3) and define

$$z \equiv x(t) = x - \int_0^t z(s) \, ds$$

for each  $z(\cdot) \in N$ . Then Jensen's inequality gives

$$\begin{aligned} \hat{u}(x, t) &\geq \inf_{z(\cdot) \in N} \left\{ tH^*\left(\frac{1}{t} \int_0^t z(s) \, ds\right) + u_0(z) \right\} \\ &= \inf_z \left\{ tH^*\left(\frac{x-z}{t}\right) + u_0(z) \right\} \\ &= u(x, t). \end{aligned}$$

On the other hand specializing to constant controls  $z(\cdot)$  of the form

$$z(s) \equiv \frac{x-z}{t} \quad (0 \leq s \leq t),$$

we obtain

$$\hat{u}(x, t) \leq \inf_z \left\{ tH^* \left( \frac{x-z}{t} \right) + u_0(z) \right\} \\ = u(x, t). \quad \blacksquare$$

*Remark.* See [5, theorem 6.2] for a direct proof of theorem 2.1 that does not use control theory.

3. CONVEX INITIAL CONDITIONS: FORMULA (ii)

For this section we assume

$$(H_2) \begin{cases} \text{(a) } H : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is continuous;} \\ \text{(b) } u_0 : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is uniformly Lipschitz and convex.} \end{cases}$$

THEOREM 3.1. Under these hypotheses

$$u(x, t) \equiv \sup_y \inf_z \{ u_0(z) + y \cdot (x - z) - tH(y) \} \quad (t > 0, \quad x \in \mathbb{R}^n) \quad (3.1)$$

is the unique uniformly continuous viscosity solution of

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^{n+1} \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \mathbb{R}^n. \end{cases} \quad (3.2)$$

The proof of theorem 3.1 follows the lemmas below.

Let us temporarily suppose in addition to  $(H_2)$  that

$$H : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is uniformly Lipschitz.} \quad (3.3)$$

We may as well assume  $L$ , the Lipschitz constant for  $H$ , exceeds  $K$ , the Lipschitz constant for  $u_0$ .

For each fixed  $t > 0$  define

$$M \equiv L^\infty(0, t; B(K)), \quad N \equiv L^\infty(0, t; B(L)),$$

and let  $\Delta$  denote the collection of all mappings

$$\beta : M \rightarrow N$$

with the property that for each  $0 \leq s \leq t$

$$\begin{cases} y(\tau) = \hat{y}(\tau) & \text{for a.e. } 0 \leq \tau \leq s \text{ implies} \\ \beta[y](\tau) = \beta[\hat{y}](\tau) & \text{for a.e. } 0 \leq \tau \leq s. \end{cases}$$

Such a  $\beta$  is called a strategy (cf. Elliot and Kalton [4]).

LEMMA 3.2. Fix  $t \geq 0, x \in \mathbb{R}^n$ . Let

$$\hat{u}(x, t) \equiv \inf_{\beta \in \Delta} \sup_{y(\cdot) \in M} \left\{ \int_0^t y(s) \cdot \beta[y](s) - H(y(s)) \, ds + u_0(x(t)) \right\}. \quad (3.4)$$

where for each  $\beta \in \Delta$ ,  $y(\cdot) \in M$  we set

$$x(t) \equiv x - \int_0^t \beta[y](s) \, ds. \tag{3.5}$$

Then  $\hat{u}$  is the unique uniformly continuous viscosity solution of (3.2).

*Proof.* It is easy to verify that  $\hat{u}$  is uniformly Lipschitz on  $\mathbb{R}^{n+1}$  with

$$\|D\hat{u}\|_{L^*(\mathbb{R}^{n+1})} = \|Du_0\|_{L^*(\mathbb{R}^n)} = K. \tag{3.6}$$

Furthermore  $\hat{u}$  is a viscosity solution of the dynamic programming PDE

$$\left. \begin{aligned} \hat{u}_t + \min_{y \in Y} \max_{z \in Z} \{z \cdot D\hat{u} + H(y) - z \cdot y\} &= 0 \quad \text{in } \mathbb{R}^{n+1} \\ \hat{u}(\cdot, 0) &= u_0(\cdot) \quad \text{in } \mathbb{R}^n. \end{aligned} \right\} \tag{3.7}$$

for

$$Y = B(K), \quad Z = B(L);$$

a proof of this assertion may be found in [6]. However if  $|p| \leq K$ ,

$$\min_{y \in Y} \max_{z \in Z} \{z \cdot p + H(y) - z \cdot y\} = H(p);$$

and thus (3.6) and (3.7) imply  $\hat{u}$  is a viscosity solution of (3.2). Uniqueness follows from Ishii [9]. ■

LEMMA 3.3. Assume  $\Lambda : Y \times Z \rightarrow \mathbb{R}$  is Lipschitz. Then for each  $t > 0$

$$\max_{y \in Y} \min_{z \in Z} \Lambda(y, z) = \inf_{\beta \in \Delta} \sup_{y(\cdot) \in M} \frac{1}{t} \int_0^t \Lambda(y(s), \beta[y](s)) \, ds.$$

*Proof.* Set

$$A \equiv \max_{y \in Y} \min_{z \in Z} \Lambda(y, z).$$

Choose  $y^* \in Y$  such that

$$A = \min_{z \in Z} \Lambda(y^*, z).$$

Then

$$\begin{aligned} &\inf_{\beta \in \Delta} \sup_{y(\cdot) \in M} \frac{1}{t} \int_0^t \Lambda(y(s), \beta[y](s)) \, ds \\ &\geq \inf_{\beta \in \Delta} \frac{1}{t} \int_0^t \Lambda(y^*, \beta[y^*](s)) \, ds \\ &\geq A. \end{aligned}$$

On the other hand as in the proof of lemma 4.3 in [6] there exists for each  $\varepsilon > 0$  a measurable, finite valued mapping  $\varphi : Y \rightarrow Z$  such that

$$\Lambda(y, \varphi(y)) \leq \min_{z \in Z} \Lambda(y, z) + \varepsilon \quad (y \in Y).$$

Define  $\beta^* \in \Delta$  by  $\beta^*[y](s) = \varphi(y(s))$ . Then

$$\begin{aligned} & \inf_{\beta \in \Delta} \sup_{y \in M} \frac{1}{t} \int_0^t \Lambda(y(s), \beta[y](s)) \, ds \\ & \leq \sup_{y \in M} \frac{1}{t} \int_0^t \Lambda(y(s), \beta^*[y](s)) \, ds \\ & \leq A + \varepsilon. \quad \blacksquare \end{aligned}$$

*Proof of theorem 3.1.* Select  $t > 0, x \in \mathbb{R}^n$ . We may rewrite (3.1) to read

$$\begin{aligned} u(x, t) &= \sup_y \{-u_0^*(y) + y \cdot x - H(y)t\}. \\ &= \max_{y \in U} \{-u_0^*(y) + y \cdot x - H(y)t\}, \end{aligned} \tag{3.8}$$

where

$$U \equiv \{y \in \mathbb{R}^n \mid u_0^*(y) < \infty\}$$

is bounded and nonempty. Since for each  $y \in U$  the mapping  $w(x, t) \equiv -u_0^*(y) + y \cdot x - H(y)t$  is a  $C^1$  solution of

$$w_t + H(Dw) = 0 \quad \text{in } \mathbb{R}_+^{n+1},$$

$u$  is therefore a viscosity subsolution (see Crandall, Evans and Lions [2, proposition 1.4]). Hence using the methods of Ishii [9] we see

$$u \leq \hat{u} \quad \text{in } \mathbb{R}_+^{n+1}. \tag{3.9}$$

On the other hand recall that  $u_0$  is convex; hence (3.4) and Jensen's inequality imply

$$\hat{u}(x, t) \leq \inf_{\beta \in \Delta} \sup_{y(\cdot) \in M} \left\{ \frac{1}{t} \int_0^t \Lambda(y(s), \beta[y](s)) \, ds \right\},$$

for

$$\Lambda(y, z) = ty \cdot z - tH(y) + u_0(x - tz).$$

Using now lemma 3.3 we obtain

$$\begin{aligned} \hat{u}(x, t) &\leq \max_{y \in Y} \min_{z \in Z} \{u_0(x - tz) + ty \cdot z - tH(y)\}. \\ &= \max_{y \in Y} \min_{w \in W} \{u_0(w) + y \cdot (x - w) - tH(y)\}, \end{aligned}$$

where

$$W \equiv x - tZ.$$

This holds with  $Z = B(L)$  for all sufficiently large  $L$  and so

$$\begin{aligned} \hat{u}(x, t) &\leq \sup_{y \in Y} \inf_w \{u_0(w) + y \cdot (x - w) - tH(y)\} \\ &\leq \sup_y \inf_w \{u_0(w) + y \cdot (x - w) - tH(y)\} \\ &= u(x, t). \end{aligned}$$

This and (3.9) complete the proof, should  $H$  satisfy (3.3). In the general case we approximate  $H$  by a sequence of smooth functions  $H^m, H^m \rightarrow H$  uniformly on compact subsets of  $\mathbb{R}^n$ . Set

$$\begin{aligned} u^m(x, t) &= \sup_y \inf_z \{u_0(z) + y \cdot (x - z) - tH^m(y)\} \\ &= \sup_{y \in U} \{-u_0^*(y) + y \cdot x - tH^m(y)\}. \end{aligned}$$

Then

$$\begin{cases} u_t^m + H^m(Du^m) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ u^m(\cdot, 0) = u_0(\cdot) & \text{on } \mathbb{R}^n \end{cases}$$

in the viscosity sense, and

$$\|Du^m\|_{L^\infty(\mathbb{R}^{n+1})} = K.$$

Thus

$$\begin{aligned} u(x, t) &\equiv \lim_{m \rightarrow \infty} u^m(x, t) \\ &= \sup_{y \in U} \{-u_0^*(y) + y \cdot x - tH(y)\} \\ &= \sup_y \inf_z \{u_0(z) + y \cdot (x - z) - tH(y)\} \end{aligned}$$

is a viscosity solution of (3.2): see [2, theorem 1.4]. Once again we refer to Ishii [9] for uniqueness. ■

*Remark.* Note that in view of (3.8)  $u$  defined by (3.1) is convex as a function of  $x$  and  $t$ .

#### 4. APPLICATIONS

(a) *Osher's formulas*

Choose  $\alpha, \beta \in \mathbb{R}^n, |\alpha| = 1$ , and then define

$$u_0(x) \equiv \begin{cases} (x \cdot \alpha)u^L + x \cdot \beta & \text{if } x \cdot \alpha \leq 0 \\ (x \cdot \alpha)u^R + x \cdot \beta & \text{if } x \cdot \alpha \geq 0, \end{cases}$$

where  $u^L, u^R \in \mathbb{R}$ . Assuming

$$u^L \leq u^R, \tag{4.1}$$

we see that  $u_0$  is convex and uniformly Lipschitz on  $\mathbb{R}^n$ . Hence for any continuous  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  the viscosity solution of

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^{n+1} \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \mathbb{R}^n \end{cases}$$

is

$$\begin{aligned} u(x, t) &= \sup_y \inf_z \{u_0(z) + y \cdot (x - z) - tH(y)\} \\ &= \sup_y \{-u_0^*(y) + y \cdot x - tH(y)\}. \end{aligned}$$

A calculation shows

$$u_0^*(y) = \begin{cases} +\infty & \text{if } y \notin U \\ 0 & \text{if } y \in U, \end{cases}$$

where

$$U \equiv \{\lambda\alpha + \beta \mid u^L \leq \lambda \leq u^R\}.$$

Thus

$$\begin{aligned} u(x, t) &= \sup_{y \in U} \{y \cdot x - tH(y)\} \\ &= \beta \cdot x + \max_{u^L \leq \lambda \leq u^R} \{\lambda\alpha \cdot x - tH(\lambda\alpha + \beta)\}. \end{aligned} \tag{4.2}$$

Similarly

$$u(x, t) = \beta \cdot x + \min_{u^R \leq \lambda \leq u^L} \{\lambda\alpha \cdot x - tH(\lambda\alpha + \beta)\} \tag{4.3}$$

if  $u^R \leq u^L$ .

Formulas (4.2) and (4.3) are essentially those derived by different methods in Osher [14].

(b) *Classical solutions of Hamilton–Jacobi equations*

Next we make an extremely strong assumption on  $H$  and  $u_0$ :

$$(H_3) \begin{cases} \text{(a) } (H_1) \text{ and } (H_2) \text{ hold} \\ \text{(b) } |D^2 u_0| \in L^\infty(\mathbb{R}^n) \end{cases}$$

PROPOSITION 4.1. Assume  $(H_3)$ . Then

$$\begin{aligned} u(x, t) &= \inf_z \sup_y \{u_0(z) + y \cdot (x - z) - tH(y)\} \\ &= \sup_y \inf_z \{u_0(z) + y \cdot (x - z) - tH(y)\} \end{aligned}$$

is a classical solution of

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \mathbb{R}^n. \end{cases}$$

In addition  $D^2u$ ,  $Du$ , and  $u_{tt}$  are bounded in  $\mathbb{R}_+^{n-1}$ .

*Proof.* Since

$$u(x, t) = \sup_y \{-u_0^*(y) + y \cdot x - H(y)t\},$$

$u$  is convex. On the other hand since

$$u(x, t) = \inf_z \left\{ u_0(z) + tH^* \left( \frac{x-z}{t} \right) \right\},$$

we have for each unit vector  $e$  and  $h > 0$

$$\begin{aligned} & u(x + he, t) - 2u(x, t) + u(x - he, t) \\ & \sup_z \{u_0(z - he) - 2u_0(z) + u_0(z - he)\}. \end{aligned}$$

Thus

$$|D^2u| \leq C \quad \text{a.e.} \quad (4.4)$$

But also

$$u_t + H(Du) = 0 \quad \text{a.e. in } \mathbb{R}_+^{n-1}. \quad (4.5)$$

Since  $H$  is locally Lipschitz and  $|Du|$  is bounded, (4.4) implies

$$|Du_t| \leq C \quad \text{a.e.}$$

But then we can use (4.5) once more to conclude

$$|u_{tt}| \leq C \quad \text{a.e.} \quad \blacksquare$$

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