Best regards

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A BOUNDARY VALUE PROBLEM FOR THE MINIMUM-TIME FUNCTION*

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Abstract. A natural boundary value problem for the dynamic programming partial differential equation associated with the minimum time problem is proposed. The minimum time function is shown to be the unique viscosity solution of this boundary value problem.

Key words. nonlinear systems, time-optimal control, dynamic programming, Hamilton–Jacobi equations, viscosity solutions

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1. Introduction. Given the control system

$$\dot{y}(s)+b(y(s), z(s))=0, \qquad s\geqq 0, \quad y\in\mathbb{R}^N, \quad z\in Z\subseteq\mathbb{R}^M, \tag{1.1}$$

the minimum time function $T(x)$ associates to a point $x\in\mathbb{R}^N$ the infimum of the times that the trajectories of (1.1) satisfying $y(0)=x$ take to reach the origin. The problem of determining T and the optimal controls realizing the minimum is one of the most extensively studied in the control-theoretic literature, especially in the linear case [1], [5], [6], [8], [15], [16], [20], [22]–[26].

It is well known that Bellman's Dynamic Programming Principle implies that at points of differentiability T satisfies the following first-order fully nonlinear partial differential equation (PDE) of Hamilton–Jacobi type:

$$\sup_{z\in Z} b(x, z)\cdot DT(x)=1. \tag{1.2}$$

It is also known that in general T is not differentiable everywhere, but it satisfies (1.2) in some generalized sense [8], [24]. In the last five years the new concept of viscosity solution for Hamilton–Jacobi equations has been introduced by Crandall and Lions [11] and the theory of such solutions has developed quickly (see, e.g., [9]–[12], [17], [21] and the references therein) and has been applied to many problems in control theory and differential games (see, e.g., [3], [4], [7], [13], [21], the survey paper of Fleming [14], and its long list of references). Following Lions [21] it is not hard to show that T satisfies (1.2) in the viscosity sense as soon as it is continuous. The goal of this paper is to complement (1.2) with a natural boundary condition and prove a uniqueness result for viscosity solutions of such a boundary value problem (BVP). Ishii [17] has proved the uniqueness of viscosity solutions of the Dirichlet problem in a bounded open set for a class of equations that includes (1.2). However the Dirichlet problem does not seem to be the most natural one for the minimum time function because in general we do not know a priori the value of T on the boundary of some given bounded set. Instead we consider (1.2) in the set $\mathcal{R}\backslash\{0\}$ where $\mathcal{R}$ is the set of points x such that there is a trajectory of (1.1) starting at x and reaching the origin in finite time, i.e., the largest set where T is defined (and finite); we propose the following boundary condition:

$$T(0)=0 \quad \text{and} \quad T(x)\to+\infty \quad \text{uniformly as } x\to\partial\mathcal{R}. \tag{1.3}$$

In § 2 we will prove that under quite general assumptions T satisfies (1.2) in $\mathcal{R}\backslash\{0\}$ in the viscosity sense and (1.3). In § 3 we will show that for any open set $\mathcal{R}$ having

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zero in its interior there is at most one viscosity solution of (1.2) in $\mathcal{R}\backslash\{0\}$ satisfying (1.3) and bounded below.

The main result is the uniqueness theorem in § 3. It presents three difficulties with respect to standard uniqueness results in the Hamilton-Jacobi theory: (i) the Hamiltonian depends on the gradient of the unknown function but does not depend explicitly on the unknown function itself, while it is usually required that it be strictly monotone in such a variable; (ii) the infinite boundary condition (1.3) if $\mathcal{R} \neq \mathbb{R}^N$; and (iii) the lack of regularity of the solutions to be compared and of the Hamiltonian if b is not globally bounded and $\mathcal{R}$ is unbounded.

To overcome the first difficulty we introduce a change of the unknown variable first used for this goal by Kruzkov [18]. It turns out that this transforms the infinite boundary condition into a finite one, therefore automatically taking care of the second difficulty.

The third difficulty can be solved using the new approach to uniqueness presented in the paper of Crandall, Ishii, and Lions [10]. Indeed our uniqueness theorem has to be considered as a corollary of the methods developed in [10]. As one of the referees pointed out to us, this difficulty had already been overcome for different problems in an earlier paper by Ishii [28], whose methods could be applied effectively to our problem as well.

We remark that the proof of the uniqueness theorem does not make use of the convexity of the Hamiltonian. Therefore the methods of this paper can be employed to study the minimum time problem in games of pursuit and evasion (see Bardi and Soravia [27]).

After the completion of this work we have learned that Kruzkov's change of variables has been used recently by Lasry and Lions [19] to study the minimum time function of a differential game with state constraints in a bounded domain, and by Barles [2] for unbounded control problems. Moreover, one of the referees pointed out that a uniqueness theorem for discontinuous solutions of (1.2), (1.3) can be proved by combining the Kruzkov transform and the results by Barles and Perthame [3], provided the target to be reached is a smooth set instead of a single point.

In the last decade the use of the theory of subanalytic sets has led to very strong results on the regularity of the minimum time function and of feedback controls (see Brunovsky [6] and Sussmann [26] and the references therein). However it is also known that certain quite smooth systems exhibit very irregular behaviors that fail to fall within the theory of subanalyticity (see Lojasiewicz and Sussmann [22] or the classical Fuller's example in [23]). It is our hope that the PDE setting of the minimum time problem proposed in this paper could be of some help in the study of these problems.

2. The BVP of the minimum time function. We begin listing the assumptions to be used in the following.

(H1) $b:\mathbb{R}^N \times Z \to \mathbb{R}^N$, where $Z \subseteq \mathbb{R}^M$, is continuous and there exist $L, K \in \mathbb{R}$ such that $|b(x,z)-b(y,z)| \leqq L|x-y|$ and $|b(y,z)| \leqq K(1+|y|)$, for all $x, y \in \mathbb{R}^N$, and for all $z \in Z$.

Let $\mathcal{M}$ be the set of measurable functions $z:[0,\infty) \to Z$, and let $y(s) = y(s; x, z)$ be the solution of

$$y(s) = x - \int_0^s b(y(t), z(t))\, dt \tag{2.0}$$

for $s \geqq 0$, $x \in \mathbb{R}^N$, and $z \in \mathcal{M}$. Let $\mathcal{T} \subseteq \mathbb{R}^N$ be a given closed set, the *terminal set* (e.g.,

$\mathcal{T}=\{0\}$), and define

$$\mathcal{R}:=\{x\in\mathbb{R}^N: \exists z\in\mathcal{M},\ t\geqq 0 \text{ such that } y(t;x,z)\in\mathcal{T}\}.$$

Clearly, $\mathcal{R}\supseteq\mathcal{T}$. Now define the minimum time function

$$T:\mathcal{R}\to[0,\infty), \qquad T(x):=\inf\{t: y(t;x,z)\in\mathcal{T} \text{ for some } z\in\mathcal{M}\}.$$

We will assume the following.

(H2) $\mathcal{R}$ is open.

(H3) T is continuous in $\mathcal{R}$.

(H4) For every $x_0\in\partial\mathcal{R}$, $\lim_{x\to x_0} T(x)=+\infty$.

Conditions under which (H2)–(H4) are satisfied are well known in the literature, especially in the linear case. See, for instance, [1], [15], [20] for (H2), [25], [15], [1], [5], [8] for (H3), [8], [15] for (H4), and the references therein. Essentially (H2)–(H4) follow from some controllability around $\mathcal{T}$.

Next we define a suitable boundary condition for functions $u\in C(\mathcal{R}\backslash\mathcal{T})$:

(BC) u converges uniformly to 0 as $x\to\partial\mathcal{T}$ and to $+\infty$ as $x\to\partial\mathcal{R}$, i.e., for every ε, $M>0$ there exists $\delta>0$ such that $\operatorname{dist}(x,\partial\mathcal{T})<\delta$ implies $|u(x)|<\varepsilon$ and $\operatorname{dist}(x,\partial\mathcal{R})<\delta$ implies $u(x)>M$.

Before proving that T satisfies (BC) we need a technical lemma.

LEMMA 1. *Assume* (H1) *is true. If* $y(t;x,z)=x_1$, *then*

$$t\geqq\frac{1}{K}\log\left(1+\frac{|x-x_1|}{1+\min(|x|,|x_1|)}\right).$$

Proof. Hypothesis (H1) implies

$$|y(s)-x|\leqq Ks(1+|x|)+\int_0^s K|y(\tau)-x|\,d\tau,$$

and then, using Gronwall's inequality, we get

$$|x_1-x|\leqq(1+|x|)(e^{Kt}-1),$$

which gives

$$t\geqq\frac{1}{K}\log\left(1+\frac{|x_1-x|}{1+|x|}\right).$$

The same calculation for $\tilde{y}(s):=y(t-s)$ leads to

$$t\geqq\frac{1}{K}\log\left(1+\frac{|x_1-x|}{1+|x_1|}\right).$$ □

Remark 1. It follows easily from Lemma 1, choosing $x_1\in\mathcal{T}$, that $T(x)>0$ for all $x\in\mathcal{R}\backslash\mathcal{T}$ (since $\mathcal{T}$ is closed) and that $\mathcal{T}$ bounded implies $\lim_{|x|\to\infty} T(x)=+\infty$.

LEMMA 2. *Assume* (H1)–(H4) *and* $\mathcal{T}$ *bounded. Then* T *satisfies* (BC).

Proof. Since T is continuous, null on $\partial\mathcal{T}$, and $\mathcal{T}$ is bounded, the first part of (BC) is clearly satisfied.

To prove the second part we fix $M>0$. From Lemma 1 and Remark 1 the existence of $R>0$ such that $T(x)>M$ for $|x|>R$ follows. Now we use (H4) to get a covering of the compact set $\partial\mathcal{R}\cap\{x: |x|\leqq R\}$ made of a finite number of open balls B_i centered on $\partial\mathcal{R}$ and having small radius so that $T(x)>M$ for $x\in\mathcal{R}\cap B_i$. We conclude observing that there exists $\delta>0$ such that $|x|\leqq R$ and $\operatorname{dist}(x,\partial\mathcal{R})<\delta$ imply $x\in B_i$ for some i. □

We recall that a continuous function u defined in an open set $\mathcal{O} \subseteq \mathbb{R}^N$ is defined to be a *viscosity solution* of

$$H(x, u, Du) = 0 \quad \text{in } \mathcal{O},$$

if for every $\phi \in C^1(\mathcal{O})$ and x_0 local maximum point of $u - \phi$ we have

$$H(x_0, u(x_0), D\phi(x_0)) \leqq 0,$$

while for x_0, local minimum point of $u - \phi$ we have

$$H(x_0, u(x_0), D\phi(x_0)) \geqq 0.$$

We will now prove that T is a viscosity solution of

$$\text{(HJ)} \qquad \sup_{z \in Z} b(x, z) \cdot Du - 1 = 0 \quad \text{in } \mathcal{R} \backslash \mathcal{T}.$$

This fact is certainly known to experts, since it follows from arguments of Lions [21, Chap. 1]. We include a full proof for the sake of completeness.

Define for $z \in \mathcal{M}$, $x \in \mathbb{R}^N$, $t_x(z) := \inf \{t\colon y(t; x, z) \in \mathcal{T}\}$ and denote by $\chi_{\{t < t_x(z)\}}$ the function defined on $[0, \infty)$ which is one if $t < t_x(z)$ and zero if the opposite inequality holds.

LEMMA 3 (Dynamic Programming Principle). *Assume* (H1). *Then for all* $x \in \mathcal{R}$ *and* $t \geqq 0$

$$T(x) = \inf_{z \in \mathcal{M}} \{\min(t, t_x(z)) + \chi_{\{t < t_x(z)\}} T(y(t; x, z))\}.$$

Proof. Fix x and t and let A be the right-hand side of the above equality. To prove $T(x) \geqq A$ we fix an arbitrary $\varepsilon > 0$ and show that

$$T(x) \geqq A - \varepsilon. \tag{2.1}$$

Let $z_1 \in \mathcal{M}$ be such that

$$T(x) \geqq t_x(z_1) - \varepsilon. \tag{2.2}$$

If $t \geqq t_x(z_1)$, then (2.1) holds. Now suppose $t < t_x(z_1)$ and let $z_2(s) = z_1(t + s)$. Then

$$t_x(z_1) = t + t_{y(t; x, z_1)}(z_2) \geqq t + T(y(t; x, z_1)) \geqq A,$$

and so by (2.2) we have (2.1).

Now we want to prove

$$T(x) \leqq A + \varepsilon, \tag{2.3}$$

and for this we choose z_1 such that

$$A + \frac{\varepsilon}{2} \geqq \min(t, t_x(z_1)) + \chi_{\{t < t_x(z_1)\}} T(y(t; x, z_1)). \tag{2.4}$$

If $t \geqq t_x(z_1)$ then (2.3) holds. If $t < t_x(z_1)$ let $z_2 \in \mathcal{M}$ be such that

$$T(y(t; x, z_1)) \geqq t_{y(t; x, z_1)}(z_2) - \frac{\varepsilon}{2}. \tag{2.5}$$

Now define the control

$$z_3(s) := \begin{cases} z_1(s) & \text{if } s < t, \\ z_2(s - t) & \text{if } s \geqq t. \end{cases}$$

Clearly

$$t_x(z_3) = t + t_{y(t;x,z_1)}(z_2),$$

and therefore we get from (2.4) and (2.5)

$$A + \frac{\varepsilon}{2} \geqq t + T(y(t; x_1, z_1)) \geqq t_x(z_3) - \frac{\varepsilon}{2} \geqq T(x) - \frac{\varepsilon}{2},$$

which proves (2.3). By the arbitrariness of ε the proof is complete. □

THEOREM 1. *Assume* (H1)–(H3). *Then the minimum time function T is a positive viscosity solution of* (HJ). *If moreover* (H4) *holds and the terminal set $\mathcal{T}$ is bounded, then T satisfies the boundary condition* (BC) *as well.*

Proof. The second statement is just Lemma 2. To prove the first statement we begin by considering $x_0 \in \mathcal{R} \setminus \mathcal{T}$ and $\phi \in C^1(\mathcal{R} \setminus \mathcal{T})$ such that for all x sufficiently close to x_0,

$$T(x_0) - \phi(x_0) \geqq T(x) - \phi(x). \tag{2.6}$$

Let z_1 be any constant control, $z_1(s) \equiv \bar{z} \in Z$. By Lemma 3 we have for all $0 < s < T(x_0)$

$$T(x_0) - T(y(s; x_0, z_1)) \leqq s,$$

and so by (2.6) we have for sufficiently small positive s,

$$\frac{\phi(x_0) - \phi(y(s; x_0, z_1))}{s} \leqq \frac{T(x_0) - T(y(s; x_0, z_1))}{s} \leqq 1.$$

Now letting $s \searrow 0$ and using (2.0), we get

$$D\phi(x_0) \cdot b(x_0, \bar{z}) \leqq 1,$$

and by the arbitrariness of $\bar{z}$,

$$\sup_{z \in Z} b(x_0, z) \cdot D\phi(x_0) - 1 \leqq 0.$$

Let us now consider new x_0 and ϕ as above but such that

$$T(x_0) - \phi(x_0) \leqq T(x) - \phi(x)$$

for all x in a given neighborhood of x_0. By Lemma 1 there exists s_1 such that

$$\phi(x_0) - \phi(y(s; x_0, z_1)) \geqq T(x_0) - T(y(s; x_0, z)) \quad \forall s \leqq s_1, \quad \forall z \in \mathcal{M}.$$

Fix $\varepsilon > 0$. By Lemma 3 for every $s \leqq T(x_0)$ there is $z^* \in \mathcal{M}$ such that

$$T(x_0) \geqq s + T(y(s; x_0, z^*)) - \varepsilon s,$$

and thus for $0 < s \leqq s_2$

$$\frac{\phi(x_0) - \phi(y(s; x_0, z^*))}{s} \geqq 1 - \varepsilon. \tag{2.7}$$

Using (2.0) and the expansion

$$\phi(x) = \phi(x_0) + D\phi(x_0) \cdot (x - x_0) + m(x)|x - x_0| \quad \text{with } \lim_{x \to x_0} m(x) = 0,$$

we can write the left-hand side of (2.7) as follows:

$$\frac{1}{s} \int_0^s D\phi(x_0) \cdot b(y(t; x_0, z^*), z^*(t))\, dt - m(y(s; x_0, z^*)) \frac{1}{s} \left| \int_0^s b(y(t; x_0, z^*), z^*(t))\, dt \right|.$$

The second term in this expression can be made smaller than ε for small s by (H1) and Lemma 1; by using (H1) again, the first term can be written as $1/s \int_0^s D\phi(x_0) \cdot b(x_0, z^*(t))\, dt$ plus a correction term smaller than ε for small s. Then by (2.7),

$$\sup_{z\in Z} D\phi(x_0) \cdot b(x_0, z) \geqq \frac{1}{s} \int_0^s D\phi(x_0) \cdot b(x_0, z^*(t))\, dt \geqq 1-3\varepsilon,$$

which provides the desired inequality by the arbitrariness of ε. □

3. Uniqueness. In this section we will prove the following theorem.

THEOREM 2. *Assume* (H1), *let* $\mathcal{R}$ *be an open subset of* $\mathbb{R}^N$, *and let* $\mathcal{T} \subseteq \mathcal{R}$ *be a closed set. If* $u_1, u_2 \in C(\mathcal{R}\backslash\mathcal{T})$ *are viscosity solutions of* (HJ) *satisfying* (BC) *and bounded from below, then* $u_1 = u_2$.

The main tool of the proof is a lemma of Crandall, Ishii, and Lions [10, Lemma 1], which we report below in a version simplified for the present purpose. We say that $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies condition (H) if the following holds:

(H) H is continuous; there exist a Lipschitz continuous, everywhere differentiable function $\mu : \mathbb{R}^n \to [0, \infty)$ and a continuous, nondecreasing function $\sigma : [0, \infty) \to [0, \infty)$ satisfying $\sigma(0) = 0$, such that $H(x, p) - H(x, p + \lambda D\mu(x)) \leqq \sigma(\lambda)$, for all $x, p \in \mathbb{R}^n$, $\lambda \in [0, 1]$ and $\lim_{|x|\to\infty} \mu(x) = +\infty$.

LEMMA 4. *Let* H *satisfy condition* (H) *and let* Ω *be an open subset of* $\mathbb{R}^n$. *Let* $z \in C(\bar{\Omega})$ *be a viscosity solution of* $z + H(x, Dz) = 0$ *in* Ω, *and let* $w \in C^1(\bar{\Omega})$ *satisfy*

$$w(x) + H(x, Dw(x)) \geqq 0 \quad \textit{and} \quad |Dw(x)| \leqq C \quad \forall x \in \Omega.$$

Assume that

$$\sup_{\partial\Omega} (z - w) < \sup_{\Omega} (z - w) < \infty.$$

Then $z \leqq w$ *in* Ω.

Proof. See [10] for the proof of this lemma. □

The other key tool in the proof of Theorem 2 is a change of the unknown variable in (HJ). For this we need a slight extension of Corollary I.8 in [11].

LEMMA 5. *Let* u *be a viscosity solution of* $H(x, u, Du) = 0$ *in* $\mathcal{O}$, *open subset of* $\mathbb{R}^n$; *let* $\Phi \in C^1(\mathbb{R})$, $\Phi'(r) > 0$ *for all* r; *and let* $\Psi : \Phi(\mathbb{R}) \to \mathbb{R}$ *be the inverse function of* Φ. *Then* $v = \Phi \circ u$ *is a viscosity solution of*

$$H(x, \Psi(v), \Psi'(v)Dv) = 0 \quad \textit{in } \mathcal{O}.$$

Proof. Let $x_0 \in \mathcal{O}$ and $\zeta \in C^1(\mathcal{O})$ be such that $v - \zeta$ has a local maximum in x_0. Define $\xi(x) = \zeta(x) - \zeta(x_0) + v(x_0)$. We have

$$D\xi = D\zeta, \qquad \xi(x_0) = v(x_0) = \Phi(u(x_0)),$$

$$v(x) \leqq \xi(x) \quad \text{in a neighborhood of } x_0.$$

Since $\Phi(\mathbb{R})$ is open, $\Psi \circ \xi$ is defined in a neighborhood of x_0 and we extend it to $\eta \in C^1(\mathcal{O})$. By the monotonicity of Ψ we have

$$u(x_0) = \eta(x_0), \quad u(x) \leqq \eta(x) \quad \text{in a neighborhood of } x_0.$$

Then

$$H(x_0, u(x_0), D\eta(x_0)) \leqq 0,$$

and so

$$H(x_0, \Psi(v(x_0)), \Psi'(v(x_0))D\zeta(x_0)) \leqq 0.$$

If x_0 is a minimum point we get the desired inequality in the same way. □

Proof of Theorem 2. Define $\Phi(t) := 1 - e^{-t}$, $v_1 := \Phi \circ u_1$, $v_2 := \Phi \circ u_2$, $\mathcal{O} := \mathcal{R} \setminus \mathcal{T}$. By Lemma 5, v_1 and v_2 are viscosity solutions of

$$\sup_{z \in Z} \left\{ b(x, z) \cdot \left(\frac{1}{1-v} Dv \right) \right\} - 1 = 0 \quad \text{in } \mathcal{O},$$

and since $v_1, v_2 < 1$, they are also viscosity solutions of

$$v + \sup_{z \in Z} \{ b(x, z) \cdot Dv \} - 1 = 0 \quad \text{in } \mathcal{O}, \tag{3.0}$$

as is easy to verify from the definition. They can be extended in a unique way to v_1, $v_2 \in C(\bar{\mathcal{O}})$ by setting

$$v_i = 0 \quad \text{on } \partial\mathcal{T}, \quad v_i = 1 \quad \text{on } \partial\mathcal{R}, \quad i = 1, 2.$$

Now define

$$H(x, p) := \sup_{z \in Z} \{ b(x, z) \cdot p - 1 \}.$$

We claim that H satisfies condition (H). Fix $\varepsilon > 0$ and choose $z_1 \in Z$ such that $H(x, p) \leqq b(x, z_1) \cdot p - 1 + \varepsilon$. Then by (H1)

$$\begin{aligned} H(x, p) - H(y, q) &\leqq b(x, z_1) \cdot p - b(y, z_1) \cdot q + \varepsilon \\ &\leqq L|x-y||p| + K(1+|y|)|p-q| + \varepsilon, \end{aligned}$$

and thus

$$|H(x, p) - H(y, q)| \leqq L|x-y||p| + K(1+|y|)|p-q|, \tag{3.1}$$

which implies the continuity of H. Now let $h \in C^1(\mathbb{R})$ be such that $h(0) = h'(0) = 0$, $h(e) = 1$, $h'(e) = 1/e$, and define

$$\mu(x) := \begin{cases} h(|x|) & \text{if } |x| < e, \\ \log(|x|) & \text{if } |x| \geqq e. \end{cases}$$

Clearly, $\mu \in C^1(\mathbb{R}^N)$ and it is Lipschitz continuous since $D\mu(x) = (1/|x|^2)x$ for $|x| \geqq e$. Moreover, by (3.1),

$$H(x, p) - H(x, p + \lambda D\mu(x)) \leqq K(1+|x|)\lambda |D\mu(x)| \leqq \lambda C =: \sigma(\lambda)$$

for $\lambda > 0$, where $C := \max(2K, K(1+e) \sup |h'|)$, which proves the claim.

Next we define, following Crandall, Ishii, and Lions [10],

$$\hat{H}(x, y, p, q) := H(x, p) - H(y, -q),$$

$$z(x, y) := v_1(x) - v_2(y),$$

and note that z is a viscosity solution of

$$z + \hat{H}(x, y, D_x z, D_y z) = 0 \quad \text{in } \mathcal{O} \times \mathcal{O}.$$

Our goal is to prove that $z(x, x) \leqq 0$, because by interchanging the roles of v_1 and v_2 we get $v_1 = v_2$ and then $u_1 = u_2$. To reach our goal we are going to apply Lemma 4 to z defined above, $\Omega := \Delta \cap (\mathcal{O} \times \mathcal{O})$ where

$$\Delta := \{(x, y) \in \mathbb{R}^{2N} : |x - y| < 1\},$$

and $w = w_\varepsilon$ for suitable ε where

$$w_\varepsilon(x, y) := (\varepsilon^{4L} + |x-y|^2)^{1/2L}/\varepsilon.$$

We have to show that there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqq \varepsilon_0$, w_ε satisfies the hypotheses of Lemma 4. Once this is done we have $z(x, x) \leqq w_\varepsilon(x, x) = \varepsilon$, and letting $\varepsilon \searrow 0$ we conclude.

Since

$$D_x w_\varepsilon(x, y) = \frac{1}{\varepsilon L}(\varepsilon^{4L} + |x-y|^2)^{(1/2L)-1}(x-y) = -D_y w_\varepsilon(x, y),$$

w_ε is Lipschitz continuous in $\bar{\Delta}$ and, moreover, by (3.1),

$$\begin{aligned} w_\varepsilon(x, y) + \hat{H}(x, y, D_x w_\varepsilon, D_y w_\varepsilon) &\geqq w_\varepsilon - L|x-y|^2(\varepsilon^{4L} + |x-y|^2)^{(1/2L)-1}/\varepsilon L \\ &\quad - K(1+|y|)|D_x w_\varepsilon + D_y w_\varepsilon| \\ &\geqq w_\varepsilon - w_\varepsilon = 0. \end{aligned}$$

Since u_1 and u_2 are bounded from below, v_1 and v_2 are bounded, and

$$\alpha_\varepsilon := \sup_\Omega (z - w_\varepsilon) \leqq A < \infty \quad \text{for all } \varepsilon > 0. \tag{3.2}$$

If

$$\liminf_{\varepsilon \searrow 0} \alpha_\varepsilon \leqq 0,$$

we immediately obtain $z(x, x) \leqq 0$. Thus it remains to prove that $\alpha_\varepsilon \geqq \alpha > 0$ and $0 < \varepsilon \leqq \varepsilon_0$ imply

$$\sup_{\partial\Omega} (z - w_\varepsilon) < \alpha_\varepsilon = \sup_\Omega (z - w_\varepsilon).$$

Suppose that $(\bar{x}, \bar{y}) \in \partial\Omega$ is such that

$$z(\bar{x}, \bar{y}) - w_\varepsilon(\bar{x}, \bar{y}) \geqq \frac{\alpha}{2} > 0. \tag{3.3}$$

Fix $0 < \delta < 1$ such that $x \in \partial\mathcal{O}$ and $|x-y| < \delta$ implies $|v_i(x) - v_i(y)| < \alpha/2$, $i = 1, 2$. This can be done because v_1 and v_2 take up their boundary values uniformly as a consequence of (BC). Suppose first that $|\bar{x} - \bar{y}| < \delta$ so that either $\bar{x}$ or $\bar{y}$, say $\bar{x}$, belongs to $\partial\mathcal{O}$. Then $z(\bar{x}, \bar{x}) = 0$ and $w_\varepsilon \geqq 0$ imply

$$z(\bar{x}, \bar{y}) - w_\varepsilon(\bar{x}, \bar{y}) \leqq v_1(\bar{x}) - v_2(\bar{y}) - v_1(\bar{x}) + v_2(\bar{x}) < \frac{\alpha}{2},$$

a contradiction to (3.3). On the other hand, if $|\bar{x} - \bar{y}| \geqq \delta$, (3.2) and (3.3) imply

$$z(\bar{x}, \bar{y}) - w_\varepsilon(\bar{x}, \bar{y}) \geqq \sup_\Omega (z - w_\varepsilon) - \alpha_\varepsilon \geqq z(\bar{x}, \bar{x}) - w_\varepsilon(\bar{x}, \bar{x}) - A,$$

from which we obtain

$$v_2(\bar{x}) - v_2(\bar{y}) \geqq w_\varepsilon(\bar{x}, \bar{y}) - \varepsilon - A.$$

The right-hand side of the last inequality can be made arbitrarily large choosing ε small because

$$\liminf_{\varepsilon \searrow 0} \{w_\varepsilon(x, y) : |x-y| \geqq \delta\} = +\infty,$$

and we get a contradiction because the left-hand side is bounded. □

Remark 2. Under the stronger assumption that $|b(y, z)| \leqq K$ for all $y \in \mathcal{R} \setminus \mathcal{T}$, $z \in Z$ (which excludes the linear case if $\mathcal{R}$ is unbounded), and strengthening the boundary conditions, we can give a weaker uniqueness theorem with a much shorter proof based on the earliest uniqueness result for viscosity solutions, that is, Theorem III.1 of Crandall and Lions [11]. In fact, under such an assumption on b, (3.1) implies that the Hamiltonian $H(x, p)$ is uniformly continuous in $\mathbb{R}^N \times \{p: |p| \leqq R\}$ for all $R > 0$. The additional boundary condition is

$$\text{(ABC)} \qquad \lim_{|x| \to \infty} u(x) = +\infty,$$

which is satisfied by the minimum time function T if $\mathcal{T}$ is bounded (see Remark 1). The proof of the weaker theorem begins with the same change of variables as the proof of Theorem 2. Next, we note that if u_1, u_2 satisfy (BC) and (ABC) then $v_1, v_2 \in BUC(\bar{\mathcal{O}})$ and that (3.0) satisfies the hypotheses of Theorem III.1(ii) in [11], which implies $v_1 = v_2$.

Remark 3. If $\mathcal{R}$ is bounded, u_1 and u_2 satisfy (ABC), they are zero on $\partial\mathcal{T}$, and they tend to $+\infty$ as $x \to x_0 \in \partial\mathcal{R}$, then the hypotheses of Theorem 2 are satisfied. In fact, u_1 and u_2 are clearly bounded from below, and they satisfy (BC) by the proof of Lemma 2.

Remark 4. If we drop the assumption that u_1 and u_2 are bounded below, then the conclusion of the theorem is false. In fact, if we take $b(x, z) = z$, Z the unit ball in $\mathbb{R}^N$, $\mathcal{T} = \{0\}$, then we get the BVP

$$|Du| - 1 = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\},$$
$$u(0) = 0,$$

which has the classical solutions $u_1(x) = |x|$ and $u_2(x) = -|x|$.

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