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THE NONCONVEX MULTI-DIMENSIONAL RIEMANN PROBLEM FOR HAMILTON–JACOBI EQUATIONS*

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Abstract. Simple inequalities are presented for the viscosity solution of a Hamilton-Jacobi equation in N space dimension when neither the initial data nor the Hamiltonian need be convex (or concave). The initial data are uniformly Lipschitz and can be written as the sum of a convex function in a group of variables and a concave function in the remaining variables, therefore including the nonconvex Riemann problem. The inequalities become equalities wherever a "maxmin" equals a "minmax" and thus a representation formula for this problem is then obtained, generalizing the classical Hopf's formulas.

Key words. Hamilton-Jacobi equations, viscosity solutions, Riemann problem, Godunov's scheme, Hopf's representation formulas

AMS(MOS) subject classifications. 35L99, 35L65, 65M15, 65M10

1. Introduction. We are concerned with viscosity solutions (see Crandall and Lions [3], Crandall, Evans, and Lions [2], Lions [12]) to the following partial differential equation:

$$\varphi_t + H(D_x\varphi) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \tag{H-J}$$

satisfying the initial data

$$\varphi(x, 0) = \varphi_0(x) \quad \text{in } \mathbb{R}^N, \tag{IC}$$

where $H \in C(\mathbb{R}^N)$, $D_x\varphi = (\varphi_{x_1}, \cdots, \varphi_{x_N})$ is the spatial gradient of φ, and φ_0 is at least uniformly continuous. This Cauchy problem has, for any $T > 0$, a unique viscosity solution $\varphi(x, t)$ in the space $UC_x(\mathbb{R}^N \times [0, T])$ of the continuous functions which are uniformly continuous in $x \in \mathbb{R}^N$ uniformly in $t \in [0, T]$, see Ishii [10] or Crandall and Lions [5].

We are interested in giving explicit pointwise upper and lower bounds for the solution, providing in some cases a representation formula for φ, for some special initial data but without extra assumptions on the Hamiltonian H.

Some general representation formulas for viscosity solutions of Cauchy problems for Hamilton-Jacobi equations are due to Evans [6] and Evans and Souganidis [7]. However, they either involve an infinite number of max-min operations over $\mathbb{R}^N$ [6], or a single max-min operation over infinite-dimensional sets of "controls" and "strategies" [7]. Two simpler formulas solving almost everywhere (H-J)(IC), one dual of the other, were derived by Hopf [9] for two special cases. The first one holds for convex Hamiltonians and general (Lipschitz) initial data, and it is well known in the theory of conservation laws in the case $N = 1$ (it is often called the Lax formula). It was shown to give the viscosity solution to the problem by Lions [12], Evans [6], Bardi and Evans [1], with different proofs and slightly different assumptions. The second

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Hopf's formula is valid for general Hamiltonians and convex or concave (Lipschitz) initial data φ_0, and it is

$$\varphi(x, t) = \sup_{v\in\mathbb{R}^N} \{x \cdot v - \varphi_0^*(v) - tH(v)\} \tag{1.1}$$

for φ_0 convex, and

$$\varphi(x, t) = \inf_{v\in\mathbb{R}^N} \{x \cdot v - \varphi_0^*(-v) - tH(v)\} \tag{1.2}$$

for φ_0 concave, where φ_0^* is the Legendre transform (or Fenchel conjugate) of φ_0, that is

$$\varphi_0^*(v) := \sup_{x\in\mathbb{R}^N} \{x \cdot v - \varphi_0(x)\} \leqq +\infty$$

for φ_0 convex, while for φ_0 concave it is

$$\varphi_0^*(v) := -(-\varphi_0)^*(v) = \inf_{x\in\mathbb{R}^N} \{-x \cdot v - \varphi_0(x)\} \geqq -\infty.$$

Osher [14] rederived for the viscosity solution of (H-J) the special case of formula (1.1) occurring when the initial data are of Riemann type (and convex), i.e., they are piecewise affine with one jump in the derivative across a plane. Bardi and Evans [1] showed the connection between Osher's formulas for convex Riemann data and Hopf's formulas, and proved that (1.1) and (1.2) give the viscosity solution of (H-J)(IC) in the general case. Lions and Rochet [13] gave a different proof under slightly more general assumptions.

We are now going to describe our main result. Let j be an integer, $0 \leqq j \leqq N$, and for any $v \in \mathbb{R}^N$ set

$$v = (v_A, v_B), \quad v_A := (v_1, \cdots, v_j) \in \mathbb{R}^j, \quad v_B := (v_{j+1}, \cdots, v_N) \in \mathbb{R}^{N-j}.$$

THEOREM 1. *Assume $H \in C(\mathbb{R}^N)$, $\varphi_1 : \mathbb{R}^j \to \mathbb{R}$ uniformly Lipschitz and convex, $\varphi_2 : \mathbb{R}^{N-j} \to \mathbb{R}$ uniformly Lipschitz and concave. Then the unique viscosity solution $\varphi \in UC_x(\mathbb{R}^N \times [0, T])$ of* (H-J) *taking on the initial data*

$$\varphi(x, 0) = \varphi_1(x_A) + \varphi_2(x_B)$$

satisfies for all $x \in \mathbb{R}^N$ and $t \geqq 0$

$$\sup_{v_A\in\mathbb{R}^j} \inf_{v_B\in\mathbb{R}^{N-j}} G(v, x, t) \leqq \varphi(x, t) \leqq \inf_{v_B\in\mathbb{R}^{N-j}} \sup_{v_A\in\mathbb{R}^j} G(v, x, t), \tag{1.3}$$

where

$$G(v, x, t) := x \cdot v - \varphi_1^*(v_A) - \varphi_2^*(-v_B) - tH(v).$$

Note that the pointwise estimate (1.3) gives a representation formula for the solution whenever the first and last terms are equal (as they are for $t = 0$). A trivial case where this occurs is for $j = N$ or $j = 0$, because (1.3) reduces to Hopf's formulas (1.1) or (1.2). A more interesting case occurs when the Hamiltonian separates the variables v_A and v_B, that is,

$$H(v) = H_1(v_A) + H_2(v_B).$$

In this case we get

$$\varphi(x, t) = \sup_{v_A\in\mathbb{R}^j} \{x_A \cdot v_A - \varphi_1^*(v_A) - tH_1(v_A)\} + \inf_{v_B\in\mathbb{R}^{N-j}} \{x_B \cdot v_B - \varphi_2^*(-v_B) - tH_2(v_B)\},$$

which is the superposition of the solutions to the problems

$$\varphi_t + H_i(D_x\varphi) = 0, \qquad \varphi(\cdot, 0) = \varphi_i$$

for $i = 1, 2$.

Next we specialize formula (1.3) to a particular class of (Riemann) initial data. Let A, u_i^+, u_i^- be constants and define

$$u_i(x) := \begin{cases} u_i^+ & \text{if } x_i > 0, \\ u_i^- & \text{if } x_i < 0 \end{cases}$$

for $i = 1, \cdots, N$. Then take

$$\varphi_0(x) = A + \sum_{i=1}^{N} x_i u_i(x) = A + x \cdot u(x). \tag{1.4}$$

These data correspond to a Riemann problem for the system of conservation laws satisfied (formally) by the spatial gradient of φ; see Remark 2.2. Let, for $i = 1, \cdots, N$,

$$\Omega_i := \{s \mid \min(u_i^+, u_i^-) \leqq s \leqq \max(u_i^+, u_i^-)\},$$
$$\chi_i := \operatorname{sign}(u_i^+ - u_i^-),$$

and reorder the indices, without loss of generality, so that

$$\chi_i = 1 \quad \text{for } i = 1, \cdots, j; \qquad \chi_i = -1 \quad \text{for } i = j+1, \cdots, N \tag{1.5}$$

$(0 \leqq j \leqq N)$. Finally set

$$\Omega_A := \Omega_1 \times \cdots \times \Omega_j; \quad \Omega_B := \Omega_{j+1} \times \cdots \times \Omega_N; \quad \Omega := \Omega_A \times \Omega_B.$$

COROLLARY 2. *The viscosity solution to* (H-J)(IC) *with the initial data given by* (1.4) *under the convention* (1.5) *satisfies*

$$A + \max_{v_A \in \Omega_A} \min_{v_B \in \Omega_B} \{x \cdot v - tH(v)\} \leqq \varphi(x, t) \leqq A + \min_{v_B \in \Omega_B} \max_{v_A \in \Omega_A} \{x \cdot v - tH(v)\}. \tag{1.6}$$

The rest of the paper is organized as follows. In § 2, as motivation, we show how formula (1.6) was first (formally) derived in connection with numerical approximation schemes for Hamilton–Jacobi equations and for conservation laws. In § 3 we give the proofs of Theorem 1 and Corollary 2, which are quite different from the previous derivation, and rather simple, in that they make use only of Hopf's formulas (1.1), (1.2) and a comparison argument.

2. A derivation of (1.6) by means of Godunov's Hamiltonians. The purpose of this section is to motivate Corollary 2 and to explain its connection with approximation schemes for (H-J). The rigorous proofs will be given in § 3. We assume that the solutions of (H-J) have the following properties:

(P1) The solution $\varphi(x, t)$ is a nondecreasing function of the initial data.

(P2) The partial derivatives φ_{x_i} satisfy a maximum principle at points of continuity, i.e., for $i = 1, \cdots, N$:

$$\min(u_i^-, u_i^+) \leqq \varphi_{x_i} \leqq \max(u_i^-, u_i^+).$$

(P3) The speed of propagation is finite.

(P4) If $\psi(x_2, \cdots, x_N, t)$ is a viscosity solution of

$$\psi_t + H(v_1, \psi_{x_2}, \cdots, \psi_{x_N}) = 0$$

for a constant v_1 then

$$\varphi(x, t) = v_1 x_1 + \psi(x_2, \cdots, x_N, t)$$

is a viscosity solution to (H-J).

It is easy to see formally that the solution to the Cauchy problem (H-J)(IC), with initial data given by (1.4), satisfies

$$\varphi(x,t)=tg\left(\frac{x}{t}\right)+A=tg(\zeta)+A, \tag{2.1}$$

where g satisfies:

$$g=\zeta\cdot D_\zeta g-H(D_\zeta g) \tag{2.2}$$

whenever $D_\zeta g$ is continuous.

In (H-J), we let $\tau=t$, $y_i=x_i-\zeta_i t$ for ζ fixed. (H-J) becomes

$$\text{(H-J}^1\text{)}\qquad \varphi_\tau+H(D_y\varphi)-\zeta D_y\varphi=\varphi_\tau+H^1(D_y\varphi)=0 \qquad (\text{defining } H^1(D_y\varphi))$$

with the same initial data (1.4).

Thus, by (2.2), to evaluate $g(\zeta)$ we need only evaluate $-H^1(D_y(g(y)))$ at $y=0$ for any $t>0$. From (P2) above we know that $(D_yg)_{y=0}$ lies in Ω for $t>0$. Moreover, if we integrate $(\text{H-J})^1$ from $\tau=0$ to $\tau=\Delta t$ we have

$$\begin{aligned}\varphi(0,\Delta t)&=A-\Delta tH^1((D_yg)_{y=0})\\&=\varphi_0(0)-\Delta t\tilde{H}^1(D_+^{x_1}\varphi_0(0),D_-^{x_1}\varphi_0(0);D_+^{x_2}\varphi_0(0),D_-^{x_2}\varphi_0(0);\cdots\\&\qquad\cdots;D_+^{x_N}\varphi_0(0),D_-^{x_N}\varphi_0(0)).\end{aligned} \tag{2.3}$$

Here

$$D_\pm^{x_i}\varphi_0(0)=\pm\frac{(\varphi_0(\pm he_i)-\varphi_0(0))}{h}=u_i^\pm \tag{2.4}$$

where $e_i=\{0,0,\cdots,1,0,\cdots\}$, the ith unit vector, and $\tilde{H}^1(u_1^+,u_1^-;\ u_2^+,u_2^-;\ \cdots;\ u_N^+,u_N^-)$ is determined by (2.3).

This formula can be interpreted as a numerical algorithm. Suppose we are given a grid

$$x_{j_i}^i=j_ih,\qquad i=1,\cdots,N;\qquad j_i=0,\pm1,\cdots$$

and values of a discrete function $\psi_j=\psi_{j_1j_2\cdots j_N}$. Then for each j, we construct the piecewise affine function which, in each of the 2^N orthants centered at j, interpolates ψ_j and its N nearest neighbors, $\psi_{j\pm e_i}$ for $i=1,\cdots,N$. From (P3), if

$$\text{(CFL)}\qquad \frac{\Delta t}{h}\max_{\substack{v\in\Omega^{(j)}\\ i=1,\cdots,N}}|H^1_{u_i}|\leqq\frac{1}{N^{1/2}}, \tag{2.5}$$

where $\Omega^{(j)}$ is the same as Ω with each u_i^-, u_i^+ replaced by $D_-^{x_i}\psi_j$, $D_+^{x_i}\psi_j$, then the solution to the initial value problem (H-J^1) with the above affine initial data in the diamond centered at j when evaluated at $x=x_j$ and $t=\Delta t$ is independent of the values of the initial data outside of this diamond.

Thus (2.3) (with $\varphi_0(0)$ replaced by ψ_j^n and $\varphi(0,\Delta t)$ by ψ_j^{n+1}), gives us a monotone finite difference scheme approximating (H-J^1) which is in differenced form with numerical Hamiltonian $\tilde{H}^1$. These concepts were introduced in [4]. The scheme is monotone, which means that the right side of (2.3) is an increasing function of all the $\varphi_{j\pm e_i}$, because of property (P1). The function $\tilde{H}^1$ is called Godunov's Hamiltonian by analogy with the definition of Godunov's scheme for conservation laws in one space dimension [8]. The scheme is consistent, which means

$$\tilde{H}^1(u_1,u_1;u_2,u_2;\cdots;u_N,u_N)=H^1(u_1,u_2,\cdots,u_N).$$

Monotonicity implies that

$$\tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; \cdots; u_N^+, u_N^-)$$

is a nonincreasing function of all the u_i^+ and a nondecreasing function of all the u_i. In particular, for $N=1$, this means for any $v_1 \in \Omega = \Omega_1$:

$$\begin{aligned} \operatorname{sgn}(u_1^+ - u_1^-)[\tilde{H}^1(u_1^+, u_1^-) - H^1(v_1)] &= \operatorname{sgn}(u_1^+ - v_1)[\tilde{H}^1(u_1^+, u_1^-) - \tilde{H}^1(v_1, u_1^-)] \\ &\quad + \operatorname{sgn}(v_1 - u_1^-)[\tilde{H}^1(v_1, u_1^-) - \tilde{H}^1(v_1, v_1)] \\ &\leqq 0. \end{aligned} \tag{2.6}$$

But, by (P2), $\tilde{H}^1(u_1^+, u_1^-) = H^1(\tilde{u}_1)$ for some $\tilde{u}_1$ in Ω. Thus we have

$$\tilde{H}^1(u_1^+, u_1^-) = \chi_1 \min_{v_1 \in \Omega_1} \chi_1 H^1(v_1). \tag{2.7}$$

(This formula was obtained earlier in [14].) Now we proceed inductively. Suppose, for $N \leqq M-1$, we have

$$\begin{aligned} &\max_{v_{j+1} \in \Omega_{j+1}} \cdots \max_{v_N \in \Omega_N} \min_{v_1 \in \Omega_1} \cdots \min_{v_j \in \Omega_j} H^1(v_1, v_2, \cdots, v_N) \\ &\leqq \tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; \cdots; u_N^+, u_N^-) \\ &\leqq \min_{v_1 \in \Omega_1} \cdots \min_{v_j \in \Omega_j} \max_{v_{j+1} \in \Omega_{j+1}} \cdots \max_{v_N \in \Omega_N} H^1(v_1, v_2, \cdots, v_N), \end{aligned} \tag{2.8}$$

where

$$\chi_i = 1, \qquad i = 1, \cdots, j,$$

$$\chi_i = -1, \qquad i = j+1, \cdots, N.$$

Next we have, $N = M$ and for any $v_1 \in \Omega_1$:

$$\chi_1[\tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; \cdots; u_M^+, u_M^-) - \tilde{H}^1(v_1, v_1; u_2^+, u_2^-; \cdots; u_M^+, u_M^-)] \leqq 0, \tag{2.9}$$

using the same argument as in (2.6).

Now, for any fixed v_1, $\tilde{H}^1(v_1, v_1; u_2^+ u_2^-, \cdots; u_M^+, u_M^-)$ is Godunov's Hamiltonian when the initial data for (H-J^1) has a constant x_1 derivative,

$$\frac{\partial \varphi_0}{\partial x_1}(x) \equiv v_1.$$

Then it follows from (P4) that

$$g\left(\frac{x}{t}\right) = \frac{x_1}{t} v_1 + \tilde{g}\left(\frac{x_2}{t}, \frac{x_3}{t}, \cdots, \frac{x_M}{t}\right)$$

(where $\tilde{g}$ also depends on v_1).

By the induction hypothesis, this means we have

$$\begin{aligned} &\chi_1 \tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; \cdots; u_M^+, u_M^-) \\ &\leqq \chi_1 \tilde{H}^1(v_1, v_1; u_2^+, u_2^-; \cdots; u_M^+, u_M^-) \\ &= -\chi_1 \tilde{g}(0, 0, \cdots, 0) \\ &\leqq \chi_1 \chi_2 \min_{v_2 \in \Omega_2} \chi_2 \cdots \chi_M \min_{v_M \in \Omega_M} \chi_M H^1(v_1, v_2 \cdots v_M) \\ &= \chi_1 H^1(v_1, \tilde{v}_2, \cdots, \tilde{v}_M) \end{aligned} \tag{2.10}$$

where the extrema is taken on at $\tilde{v}_2, \cdots, \tilde{v}_M$, which depends on v_1. The vector $(v_1, \tilde{v}_2, \cdots, \tilde{v}_N) \in \Omega_1$, where $v_1 \in \Omega_1$ is arbitrary. We next take $\min_{v_1}$ of the expression in (2.10). If all the $\chi_i \equiv 1$ or all the $\chi_i \equiv -1$ we have equality by (P2). Otherwise, $\chi_i \equiv 1$, $1 \leqq i \leqq j$, $\chi_i \equiv -1$, $j+1 \leqq i \leqq M$, and we have the right-hand inequality in (2.8). Next we have, for any $v_{j+1} \in \Omega_{j+1}$, following the argument above:

$$
\begin{aligned}
&\tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; \cdots; u_M^+, u_M^-) \\
&\quad \geqq \tilde{H}^1(u_1^+, u_1^-; \cdots; v_{j+1}; \cdots; u_M^+, u_M^-) \\
&\quad \geqq \chi_{j+2} \min_{v_{j+2} \in \Omega_{j+2}} \chi_{j+2} \cdots \chi_M \min_{v_M \in \Omega_M} \chi_M \chi_1 \min_{v_1 \in \Omega_1} \chi_1 \cdots \chi_j \\
&\qquad \cdot \min_{v_j \in \Omega_j} \chi_j H^1(v_1, v_2, \cdots, v_M).
\end{aligned}
\tag{2.11}
$$

We next take the $\max_{v_{j+1}}$ of the expression in (P4) which gives us the left-hand inequality in (2.8).

We have now obtained formula (2.8) for any N; using (2.1) and (2.2) gives us our intuitive derivation of (1.6).

Remark 2.1. We note that (2.8) validates the conjecture about Godunov's Hamiltonian in [15] when the inequalities in (2.10) and (2.11) become equalities. That paper also discusses the high-order accurate nonoscillatory numerical solution of (H-J) in some detail. See also [16] for a further discussion of these issues.

Remark 2.2. If we take the space gradient of (H-J) and call $u_1 = \varphi_{x_1}$, $u_2 = \varphi_{x_2}$, etc., we arrive at the system of conservation laws

$$
(u_i)_t + \frac{\partial}{\partial_{x_i}} H(u_1, \cdots, u_N) = 0, \qquad i = 1, \cdots, N \tag{2.12}
$$

with initial data:

$$
u_i(x, 0) = \begin{cases} u_i^+ & \text{if } x_i > 0 \\ u_i^- & \text{if } x_i < 0, \end{cases} \qquad i = 1, \cdots, N.
$$

Thus (1.6) gives us information about the solution to this special Riemann problem for a special system of conservation laws.

3. Proofs.

Proof of Theorem 1. Since φ_2 can be written as the Legendre transform of its Legendre transform

$$
\varphi_2(x_B) = \inf_{v_B \in \mathbb{R}^N} \{-x_B \cdot v_B - \varphi_2^*(v_B)\},
$$

we will first solve (H-J) with the initial data

$$
\psi_0(v_B, x) = \varphi_1(x_A) - x_B \cdot v_B - \varphi_2^*(v_B), \tag{3.1}
$$

and then take the infimum as v_B varies in $\mathbb{R}^{N-j}$. Since the initial data ψ_0 are convex in x for each choice of v_B, we can write Hopf's formula for the solution $\psi(v_B, x, t)$ of (H-J) plus

$$
\psi(v_B, x, 0) = \psi_0(v_B, x) \quad \text{for all } x \in \mathbb{R}^N.
$$

To do this we compute the Legendre transform with respect to x of ψ_0:

$$
\begin{aligned}
\psi_0^*(v_B, y) &= \sup_{x \in \mathbb{R}^N} \{x_A \cdot y_A + x_B \cdot (y_B + v_B) - \varphi_1(x_A) + \varphi_2^*(v_B)\} \\
&= \begin{cases} +\infty & \text{if } y_B \neq -v_B, \\ \varphi_1^*(y_A) + \varphi_2^*(v_B) & \text{if } y_B = -v_B, \end{cases}
\end{aligned}
$$

and then apply (1.1) to get

$$\psi(v_B, x, t) = \sup_{y_A \in \mathbb{R}^j} \{x_A \cdot y_A - x_B \cdot v_B - \varphi_1^*(y_A) - \varphi_2^*(v_B) - tH(y_A, -v_B)\}$$
$$= \sup_{v_A \in \mathbb{R}^j} G(-v, x, t).$$

Since $\psi(v_B, \cdot, \cdot) \in UC_x(\mathbb{R}^N \times [0, T])$ for all v_B and $\psi(v_B, x, 0) \geqq \varphi(x, 0)$, a standard comparison theorem for unbounded viscosity solutions [10], [5] gives

$$\psi(v_B, x, t) \geqq \varphi(x, t) \quad \text{for all } (x, t) \in \mathbb{R}^N \times [0, T], \quad v_B \in \mathbb{R}^{N-j}.$$

Then

$$\inf_{v_B \in \mathbb{R}^{N-j}} \sup_{v_A \in \mathbb{R}^j} G(v, x, t) = \inf_{v_B \in \mathbb{R}^{N-j}} \psi(v_B, x, t) \geqq \varphi(x, t),$$

which is the second inequality in (1.3).

The first inequality is proved in a similar way. We apply Hopf's formula (1.2) to compute the solution $\psi(v_A, x, t)$ of (H-J) with the concave initial condition

$$\psi(v_A, x, 0) = \varphi_2(x_B) + x_A \cdot v_A - \varphi_1^*(v_A) \leqq \varphi(x, 0).$$

Since

$$\psi^*(v_A, y, 0) = \begin{cases} -\infty & \text{if } y_A \neq -v_A, \\ \varphi_1^*(v_A) + \varphi_2^*(y_B) & \text{if } y_A = -v_B, \end{cases}$$

we get

$$\psi(v_A, x, t) = \inf_{v_B \in \mathbb{R}^{N-j}} G(v, x, t),$$

and, as before, we conclude by means of a comparison theorem. □

Remark 3.1. The first and the third member of (1.3) coincide with φ at $t = 0$, but in general it is not clear whether they are continuous. However, they are anyway respectively a subsolution and a supersolution of (H-J) in the generalized viscosity sense of Ishii [11]. This follows from Proposition 2.4 in [11], because they are, respectively, a supremum and an infimum of solutions of (H-J).

Proof of Corollary 2. We set

$$\varphi_1(x_A) = A + x_A \cdot u(x_A), \qquad \varphi_2(x_B) = x_B \cdot u(x_B),$$

and compute the Legendre transforms

$$\varphi_1^*(v_A) = -A + \sup_{x_A \in \mathbb{R}^j} x_A \cdot (v_A - u(x_A))$$
$$= \begin{cases} +\infty & \text{if } v_i > u_i^+ \text{ or } v_i < u_i^- & \text{for some } 1 \leqq i \leqq j, \\ -A & \text{if } u_i^- \leqq v_i \leqq u_i^+ & \text{for all } i = 1, \cdots, j \end{cases}$$
$$= \begin{cases} +\infty & \text{if } v_A \notin \Omega_A, \\ -A & \text{if } v_A \in \Omega_A; \end{cases}$$
$$\varphi_2^*(-v_B) = \begin{cases} -\infty & \text{if } v_B \notin \Omega_B, \\ 0 & \text{if } v_B \in \Omega_B; \end{cases}$$

which substituted in (1.3) give immediately (1.6). □

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