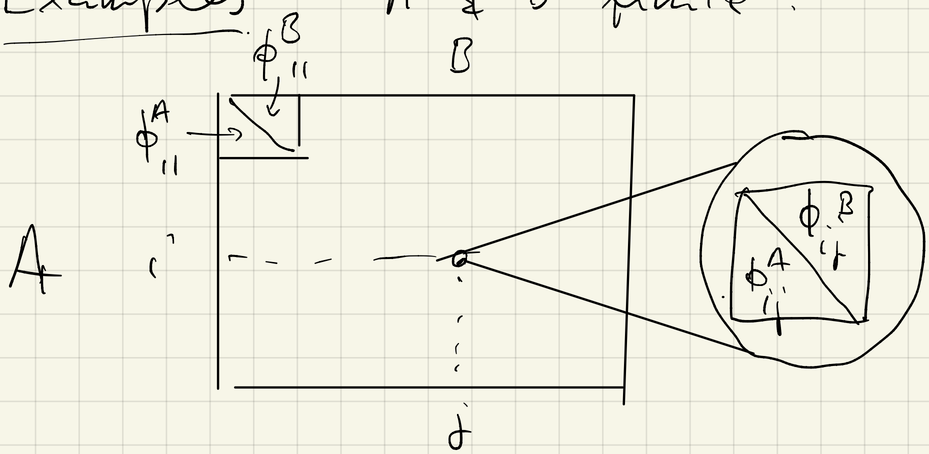
LECTURE 17, May 4, 2023

NON-ZERO SUM GAMES (2-person)

$\Phi^A : A \times B \to \mathbb{R}$ payoff of 1st player

$\Phi^B : A \times B \to \mathbb{R}$ " " 2nd "

Examples A & B finite. BIMATRIX GAMES.



Def $(a^*, b^*) \in A \times B$ is a Nash equilibrium (~1950-51)

if $\forall a \in A$ $\quad \Phi^A(a, b^*) \leq \Phi^A(a^*, b^*)$

$\forall b \in B$ $\quad \Phi^B(a^*, b) \leq \Phi^B(a^*, b^*)$

it is not convenient to deviate from (a^*, b^*) if the other player does not deviate.

N.B. $\Phi^A + \Phi^B = 0$ then (a^*, b^*) is a N. equil $\Leftrightarrow$ it is a saddle point for (A, B, Φ^A).

Another possible notion of "solution" of the game.

Uses MAXIMALITY w.r.t. partial order in $\mathbb{R}^2$

Def. $x \in \mathbb{R}^2$ is PREFERABLE to $y \in \mathbb{R}^2$ if $x > y \overset{\text{Def.}}{\iff}$

$x_1 \geq y_1$, $x_2 \geq y_2$ and at least one $\geq$ is $>$, STRICT

Def. PARETO OPTIMUM (1896) is (a^*, b^*) : there is NO (a,b) preferable to (a^*, b^*), i.e.,

$$(\Phi^A, \Phi^B)(a,b) > (\Phi^A, \Phi^B)(a^*, b^*), \quad \text{i.e.}$$

(a^*, b^*) s.t. $\nexists\ (a,b)$:

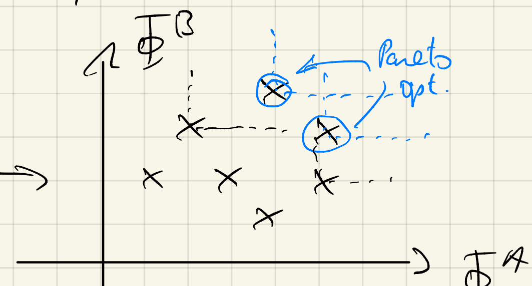
$$\begin{cases} \Phi^A(a,b) \geq \Phi^A(a^*, b^*) \\ \Phi^B(a,b) > \Phi^B(a^*, b^*) \end{cases} \quad \text{or} \quad \begin{cases} \ldots > \ldots \\ \ldots \geq \ldots \end{cases}$$

Rmk 1 $\Phi^A + \Phi^B = 0 \implies$ all (a,b) are Pareto optima!

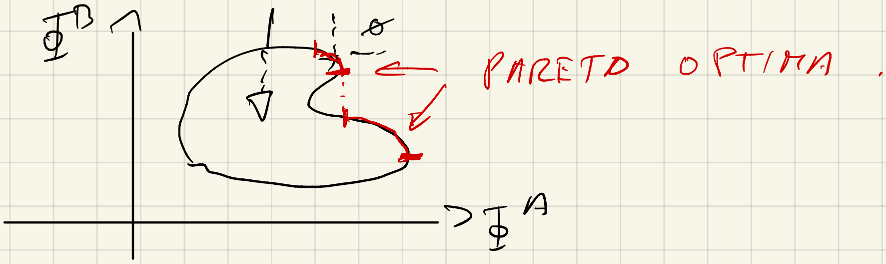
Rmk 2 $\forall\ \lambda \in]0,1[$ fixed, if $\lambda \Phi^A + (1-\lambda)\Phi^B$ has a max at $(a^*, b^*) \implies (a^*, b^*)$ is a Pareto optimum (then they exist if $\Phi^A, \Phi^B \in C(A \times B)$, A, B compact).

Graphically:

image of $(\Phi^A, \Phi^B) \longrightarrow$



discrete



Example 1 The PRISONER'S DILEMMA (RAND Corporation ~49 Flood ~52 Tucker)

2 thieves are arrested

Can collaborate or not

A \ B	C	N
C	-6, -6	0, -8
N	-8, 0	-1, -1

The UNIQUE NASH EQ. is

(C, C)

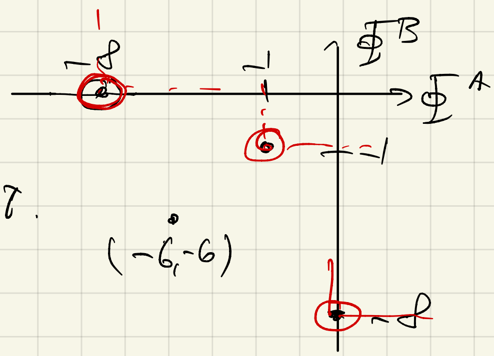
N.B. $(\Phi^A + \Phi^B)(C, C) = -12 = \min(\Phi^A + \Phi^B)$

Nash eq. MODELS STRONG NON-COOPERATION.

PARETO OPTIMA

$(N,C), (C,N), (N,N)$ are P. OPT.

(C,C) is NOT.



Example 2 (Arms race) 2 superpowers

C : keep a conventional arsenal.

N : build a nuclear arsenal.

A \ B	N	C
N	-1 , -1	6 , -5
C	-5 , 6	0 , 0

(N,N) is the unique NASH EQUIL.

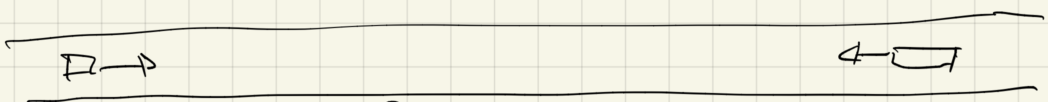
$\in \arg\min(\Phi^A + \Phi^B)$

$\Rightarrow$ arms race

MAD = MUTUAL ASSURED DISTRUCTION

<u>Example 3</u> "Chicken run"

American Graffiti 1973



A \ B	T	S
T	0 , 0	-2 , 3
S	3 , -2	-10 , -10

T = turn

S = straight.

$(T,S), (S,T)$ are 2 Nash equil.

- NOT UNIQUENESS.
- NO NOTION of VALUE. $(\Phi^A, \Phi^B)(T,S) \neq (\Phi^A, \Phi^B)(S,T)$
- NO EXCHANGEABILITY

<u>Example</u> Welfare game [Baron]

state \ unemployed		search	not search
	W	3, 2	−1, 3
	N	−1, 1	0, 0

HW: NO NASH EQUILIBRIA

Examples of games with a CONTINUUM of STRATEGIES

A, B closures of bounded open sets

$\Phi^A, \Phi^B \in C^1(A \times B)$. RMK: $A, B =$ compact INTERVALS

$(a^*, b^*) \in \mathring{A} \times \mathring{B}$ NASH EQ. $\Rightarrow$ $\boxed{\frac{\partial \Phi^A}{\partial a}(a^*, b^*) = 0}$

$\frac{\partial \Phi^B}{\partial b}(a^*, b^*) = 0$. Look at level sets of Φ^A & Φ^B

Supp. $\nabla \Phi^A(a^*, b^*) \neq (0,0) \neq \nabla \Phi^B(a^*, b^*)$

$\Rightarrow \{(a,b) : \Phi^A(a,b) = \Phi^A(a^*, b^*)\}$ is a C^1 curve locally,

e.g. $(x(t), y(t))$, $t \in]-\varepsilon, \varepsilon[$, $x(0) = a^*$, $y(0) = b^*$.

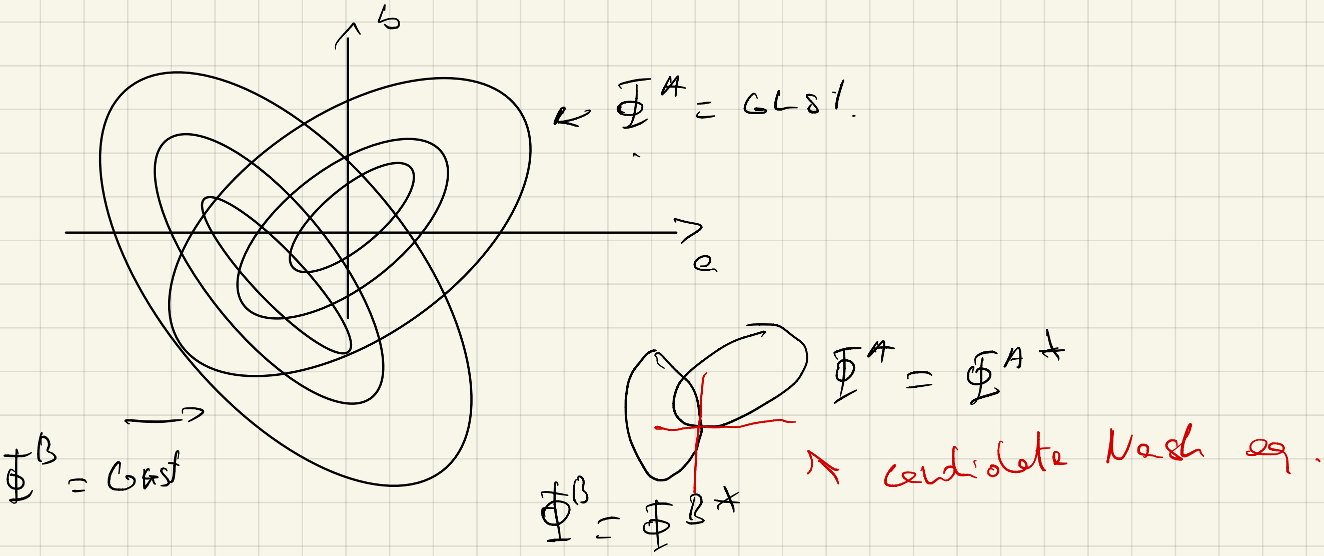
$\Phi^A(x(t), y(t))$ const. $\Rightarrow \underbrace{\frac{\partial \Phi^A}{\partial a}(x(t), y(t))}_{\text{at } t=0 \;=\; 0} \dot{x}(t) + \underbrace{\frac{\partial \Phi^A}{\partial b}(\;)}_{\neq 0} \dot{y}(t) = 0$

$\Rightarrow \dot{y}(0) = 0 \Rightarrow$ the level set of Φ^A at (a^*, b^*)

has horizontal tangent.

Similar for Φ^B (HW) $\Rightarrow \dot{x}(0) = 0$ $\Rightarrow$

the level sets of Φ^B at (a^*, b^*) has VERTICAL tangent.



<u>Example</u> Cournot duopoly model 1838

[Jorgensen - Zaccour: book dyn. games in marketing].

a, b = quantity of product of 2 firms.

$A = [0, M_1]$, $B = [0, M_2]$ p = price of the product.

Law of demand: $p = P - k(a+b)$, k in $\frac{\$}{\text{units of prod}} > 0$

$P > 0$

$$\Phi^A(a,b) = a(P - k(a+b)) = Pa - ka^2 - kab$$

$$\Phi^B(a,b) = b(P - k(a+b)) = Pb - kab - kb^2$$

look for $a^* \in]0, M_1[$, $b^* \in]0, M_2[$ Nash eq.

$$\begin{cases} \dfrac{\partial \Phi^A}{\partial a} = 0 = P - 2ka - kb \\[2ex] \dfrac{\partial \Phi^B}{\partial b} = 0 = P - ka - kb \end{cases} \qquad \begin{cases} P = k(2a+b) \\[2ex] P = k(a+2b) \end{cases} \iff a = b$$

$$\Rightarrow \quad P = 3ka \qquad a^* \overset{\iff}{=} \frac{P}{3k} = b^*$$

$\Rightarrow \left(\frac{P}{3k}, \frac{P}{3h}\right)$ is the unique available NASH EQ. in $\mathring{A} \times \mathring{B}$, if $\frac{P}{3k} < r_1$, $\frac{P}{3k} < r_2$.

It can be checked that if $\longrightarrow$ $\longrightarrow$ hold then $\left(\frac{P}{3k}, \frac{P}{3k}\right)$ is Nash eq. ∎

EXISTENCE of NASH EQUILIBRIA.

Hyp.: A, B compact, $\Phi^A, \Phi^B \in C(A \times B)$

Best REPLY MAPS: $R^A(b) = \arg\max_a \Phi^A(a,b)$ $(\neq \emptyset)$

$R^B(a) = \arg\max_b \Phi^B(a,b)$

N.B. (a^*, b^*) is N. Eq. $\iff \begin{cases} a^* \in R^A(b^*) \\ b^* \in R^B(a^*) \end{cases} \quad (*)$

A MULTIFUNCTION or (set-valued map) is

$F : X \rightsquigarrow X$
$x \mapsto F(x) \subseteq X$

A FIXED point of F is $x^* \in F(x^*)$

Then (a^*, b^*) is a N. Eq. $\iff$ Fixed point of

$(a,b) \mapsto R^A(b) \times R^B(a) \subseteq A \times B \qquad A \times B \to \mathcal{P}(A \times B)$

Fixed pt. is (a^*, b^*): $a^* \in R^A(b^*)$, $b^* \in R^B(a^*)$ $\iff (*)$

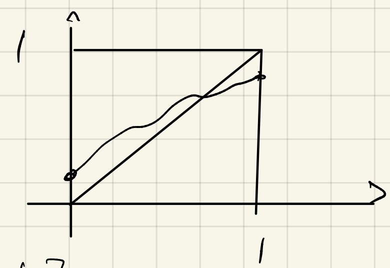
For existence of N. Eq. [Nash 51], [Bre] use a Fixed point. for multif. (Kakutani)

Other proof: use just a fixed pt. thm. for SINGLE-VALUED functions.

Brouwer Thm. K (metric) compact space & CONVEX, $f: K \to K$ CONTINUOUS $\Rightarrow$ $\exists$ fixed pt. $\bar{x}$ of f, i.e.

$$\bar{x} = f(\bar{x}).$$

Pf NO $\blacksquare$ Ex. $n=1$



$K = [0,1]$

HW: Prove it for $K = [a,b]$.

Thm (Nash 50-51) A, B compact convex, $\Phi^A, \Phi^B \in C(A \times B)$ &

$\forall b \in B \quad a \mapsto \Phi^A(a,b)$ CONCAVE

$\forall a \in A \quad b \mapsto \Phi^B(a,b)$ "

$\Rightarrow$ $\exists$ a Nash EQUILIBRIUM.

Rmk. If $\Phi^A = -\Phi^B$ get exactly Von Neumann thm.

Proof For simplicity $A, B \subseteq \mathbb{R}^k$

Step 1 Ass. $a \mapsto \Phi^A(a,b)$, $b \mapsto \Phi^B(a,b)$ STRICTLY CONCAVE $\Rightarrow$ $R^A(b) = \{r^A(b)\}$, $R^B(a) = \{r^B(a)\}$.

Lemma. $r^A: B \to A$, $r^B: A \to B$ are continuous

Pf seen before V.Neu.Thm. $\blacksquare$

Step 2 $F(a,b) = (r^A(b), r^B(a))$, $K = A \times B$ compact convex

$F: K \to K$ cont. : Brouwer Fix pt. thm. $\Rightarrow$

$\exists (a^*, b^*) \in K$: $a^* = r^A(b^*)$, $b^* = r^B(a^*)$ $\Leftrightarrow$

(a^*, b^*) Nash. equil. $\blacksquare$ St 1 + 2.

<u>Step. 3</u> : General case, $\varepsilon > 0$

$\Phi^A_\varepsilon(a,b) = \Phi^A(a,b) - \varepsilon |a|^2$ strict. concave in a

$\Phi^B_\varepsilon(a,b) = \Phi^B(a,b) - \varepsilon |b|^2$ " " in b.

St 2 $\Rightarrow$ $\exists (a_\varepsilon, b_\varepsilon)$ N. eq. for $\Phi^A_\varepsilon, \Phi^B_\varepsilon$.

$$\Phi^A(a_\varepsilon, b_\varepsilon) \geq \Phi^A_\varepsilon(a_\varepsilon, b_\varepsilon) \geq \Phi^A(a, b_\varepsilon) - \varepsilon |a|^2 \quad \forall a$$

$$\Phi^B(a_\varepsilon, b_\varepsilon) \geq \Phi^B_\varepsilon(a_\varepsilon, b_\varepsilon) \geq \Phi^B(a_\varepsilon, b) - \varepsilon |b|^2 \quad \forall b$$

Compactness of $A \times B$ $\Rightarrow$ $\exists$ $\varepsilon_n \to 0+$: $a_{\varepsilon_n} \to a^* \in A$, $b_{\varepsilon_n} \to b^* \in B$

as $n \to \infty$. $\Rightarrow$

$$\Phi^A(a^*, b^*) \geq \Phi^A(a, b^*) \quad \forall a$$

$$\Phi^B(a^*, b^*) \geq \Phi^b(a^*, b) \quad \forall b$$

$\Rightarrow$ (a^*, b^*) is N. Eq. $\blacksquare$