LECTURE 16, May 2, 2023

Solutions to some exercises

1. $u_t + \frac{(u_x)^2}{2} = 0$ in $\mathbb{R}^n \times (0,\infty)$

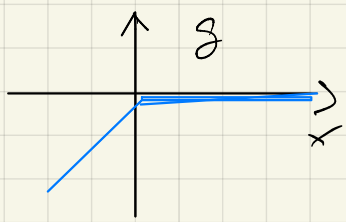
a) compute u by H-L if $u(x,0) = g(x) = -x^-$

b) same for $g(x) = x^+$

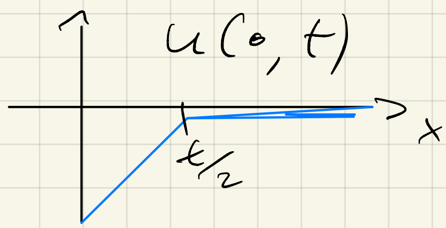
c) $v = u_x$ $\qquad v_t + \left(\frac{v^2}{2}\right)_x = 0$

Draw the graphs!

Sol. a)

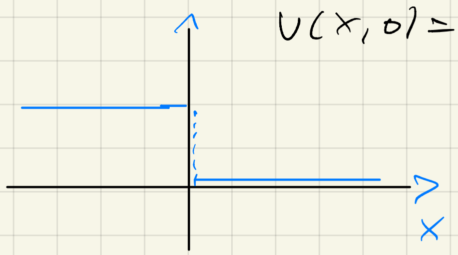


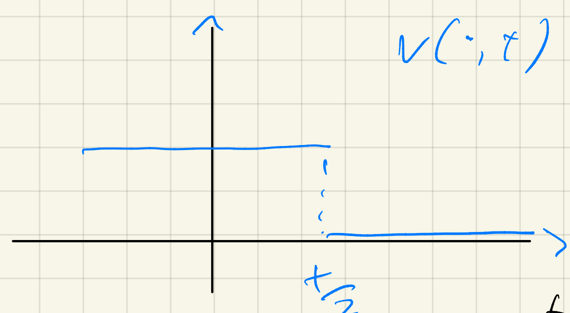
H-L $\quad u(x,t) = \begin{cases} x - \frac{t}{2}, & x < \frac{t}{2} \\ 0, & x > \frac{t}{2} \end{cases}$, $t > 0$



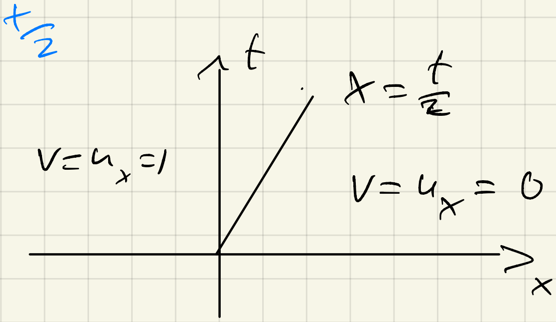
Comment & c)

$v(x,0) = g'(x)$

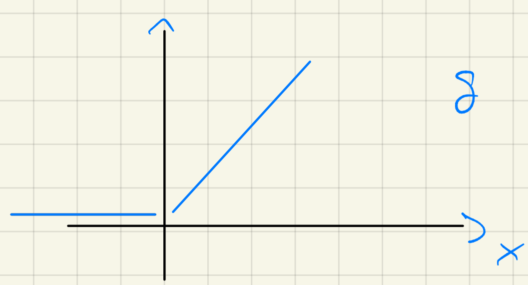




$v(x,t) := u_x(x,t) = \begin{cases} 1, & x < \frac{t}{2} \\ 0, & x > \frac{t}{2} \end{cases}$

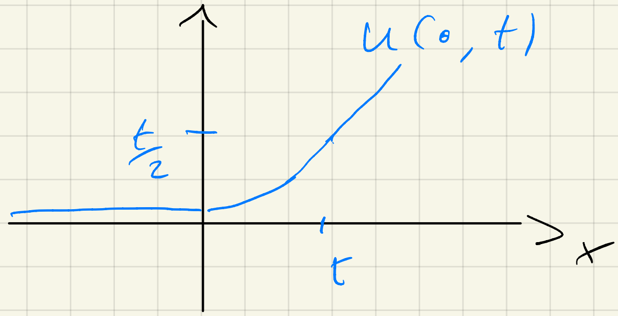


b) $u(x,0) = x^+$
$\quad\parallel$
$\quad g(x)$



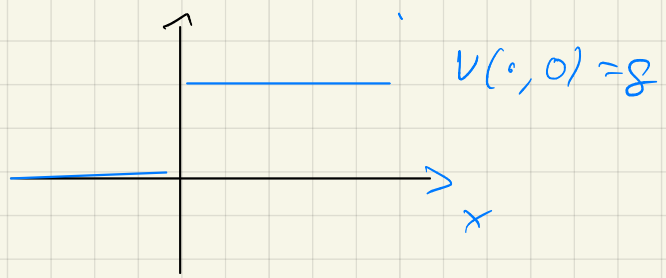
H-L $\Rightarrow$

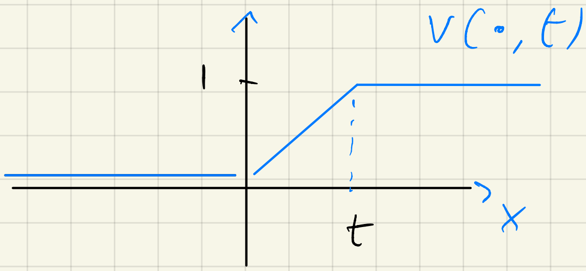
$$u(x,t) = \begin{cases} 0 & x<0 \\ x^2/2t, & 0<x<t \\ x-\frac{t}{2} & x>t \end{cases}$$



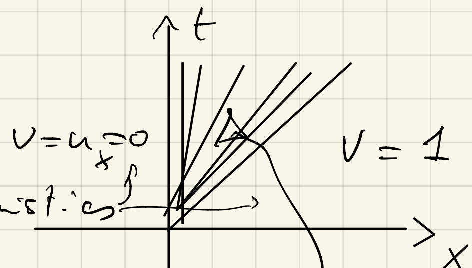
b-c) look at $v = u_x$

$$v(x,t) = \begin{cases} 0 & x<0 \\ x/t & 0<x<t \\ 1 & x>t \end{cases}$$





$t>0$ RAREFACTION WAVE



N.B.: from characteristics we had NO INFO. on v in

MIXED STRATEGIES for 0-SUM GAMES

A, B compact $\quad \Phi \in C(A\times B)$

$\mu \in P(A) \longleftrightarrow X$ Random variable with distrib. μ, in A

$\nu \in P(B) \longleftrightarrow Y$ " " in B

i.e. $P(X\in S) = \mu(S)$, $P(Y\in T) = \nu(B)$

S Borel set $\subseteq A \qquad T\subseteq B$

RMK: $\tilde{\Phi}(\mu,\nu) = \iint_{A\times B} \Phi(a,b)\, d\mu(a)\, d\nu(b) =$

if X & Y are independent. $= \mathbb{E}[\Phi(X,Y)]$ = the expectation of $\Phi(X,Y)$

$\tilde{\Phi} : \mathbb{P}(A) \times \mathbb{P}(B) \to \mathbb{R}$.

IMPORTANT EXAMPLE $\quad A = \{1, \dots, m\}$, $B = \{1, \dots, n\}$.

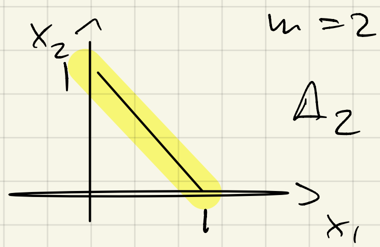
$\mu \in \mathbb{P}(A) \qquad \mu \longleftrightarrow x = (x_1, \dots, x_m) \qquad 0 \le x_i \le 1$

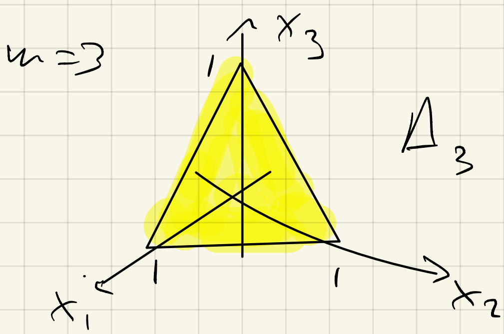
x = discrete density of $\mu \qquad \mu(\{i\}) = x_i \qquad \sum_{i=1}^{m} x_i = 1$

$\nu \longleftrightarrow y = (y_1, \dots, y_n)$

$\mathbb{P}(A) \longleftrightarrow \Delta_m$ = m-dimensional simplex =

$= \{ x \in [0,1]^m : \sum_{i=1}^{m} x_i = 1 \}$.





Matrix game $\longleftrightarrow \quad M = (\phi_{ij}) \qquad \phi_{ij} = \Phi(i,j)$

$$\tilde{\Phi} : \Delta_m \times \Delta_n \to \mathbb{R}. \qquad \tilde{\Phi}(x,y) = \sum_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \phi_{ij} x_i y_j = x^T M y$$

<u>Cor</u> (Von Neumann) In 2-person 0-sum MATRIX GAMES, the value in MIXED STRATEGIES EXISTS.

(i.e. the game $(\mathbb{P}(A), \mathbb{P}(B), \tilde{\Phi})$ has a value)

<u>Pf</u>. Δ_m, Δ_n are compact convex, $\tilde{\Phi}(x,y) = x^T M y$ is cont. & CONCAVE-CONVEX. V.N. Thm $\Rightarrow \tilde{v}^+ = \tilde{v}^-$

& $\exists$ saddle point $(x^*, y^*) \longleftrightarrow (\mu^*, \nu^*)$. ∎

COMPUTATION of SADDLE POINTS in MIXED STRATEGIES

Ex HW : If (A, B, Φ) has a value v in pure strat.

$\Rightarrow \quad \tilde{v}^{+} = v = \tilde{v}^{-} \quad \Rightarrow \quad \tilde{v} = v$

Look for $(x^*, y^*) \in \Delta_m \times \Delta_n$ saddle for $x^T M y = \tilde{\Phi}$,

i.e. x^* is a SECUR. STRAT. for Δ_m, y^* is S.S. for Δ_n, i.e.

$$\tilde{\Phi}^{min}(x^*) = \max_{x \in \Delta_m} \tilde{\Phi}^{min}(x) = \max_{x} \min_{y \in \Delta_n} x^T M y = \tilde{v}^{-}$$

N.B. $\tilde{\Phi}$ is LINEAR in y & Δ_n is POLYHEDRON $\Rightarrow$

$\min_{y \in \Delta_n} x^T M y$ is attained at a vertex, i.e. at $(0, \dots, 0, \overset{j\text{-th}}{1}, 0, \dots, 0)$

i.e. pure strategies. $\Rightarrow$

$$\tilde{\Phi}^{min}(x^*) = \max_{x \in \Delta_m} \underbrace{\min_{j=1,\dots,n} (x^T M)_j}_{\tilde{\Phi}^{min}(x)}$$

similarly y^* is S.S. $\iff$

$$\tilde{\Phi}^{max}(y^*) = \min_{y} \max_{i=1,\dots,m} (My)_i$$

$x^T M y$

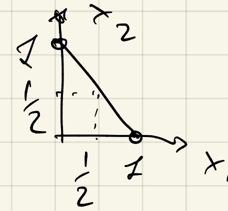
$x = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$

Ex. 1 Head or Tail game OR EVEN - ODDS with 2 fingers.

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad (x_1 \ \ x_2) M = (\underbrace{x_1 - x_2}, \underbrace{-x_1 + x_2})$$

$$\tilde{\Phi}^{min}(x) = \min \{ \ \ , \ \ \} = -|x_1 - x_2|$$

$$\max_{x \in \Delta_2} \tilde{\Phi}^{\min}(x) = \max_{\Delta_2} \big(\underbrace{-|x_1 - x_2|}_{\leq 0} \big)$$



$$\& = 0 \text{ iff } x_1 = x_2$$

$x^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ is the only sec. strat.

$$\tilde{\Phi}^{\max}(y) \overset{HW}{=} |y_1 - y_2| \geq 0 \qquad \& = 0 \iff y_1 = y_2 = \frac{1}{2}$$

So $y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ is SEC. STRAT.

Conclusion: $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ is the UNIQUE saddle pt. in mixed strategies & $\tilde{v} = 0$. $\boxtimes$

Example ROCK – PAPER – SCISSOR.

	R	P	S	min $\rightarrow$
R	0	−1	1	−1
P	1	0	−1	−1
S	−1	1	0	−1
max C_j	1	1	1	

$\downarrow$ max: $v^- = -1$

$\xrightarrow{\min} v^+ = 1$

$v \not\exists$, in pure strategies.

Look for $\tilde{v}$ in mixed strategies in $\Delta_3 \times \Delta_3$

$$\begin{cases} \tilde{\Phi}^{\min}(x^*) = \max\limits_{x \in \Delta_3} \tilde{\Phi}^{\min}(x) = \max\limits_{x} \min\limits_{j=1,\dots,3} (x^T M)_j \\[2ex] \tilde{\Phi}^{\max}(y^*) = \min\limits_{y} \tilde{\Phi}^{\max}(y) = \min\limits_{y \in \Delta_3} \max\limits_{i=1,\dots,3} (My)_i \end{cases}$$

$$x^T M = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \left(x_2 - x_3 \,,\, -x_1 + x_3 \,,\, x_1 - x_2 \right)$$

$$\tilde{\Phi}^{\min}(x) = \min_{\Delta_3} \{ x_2 - x_3, \; x_3 - x_1, \; x_1 - x_2 \}$$

Note that $\tilde{\Phi}^{\min}(x) \le 0 \quad \forall x$ because, if not,

$$\begin{cases} x_2 > x_3 \\ x_3 > x_1 \\ x_1 > x_2 \end{cases}$$ IMPOSSIBLE! Then search x^*:

$\tilde{\Phi}^{\min}(x^*) = 0$ I get: $x_2 = x_3 = x_1$

$x_2 + x_3 + x_1 = 1$
$\Rightarrow$ $x_1 = \frac{1}{3} = x_2 = x_3$ $\quad x^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is SEC. STRAT. (UNIQUE).

HW Similarly $y^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is the only S.S. for y.

CONCLUSION: (x^*, y^*) is the only SADDLE point of RPS game in MIXED STRATEGIES.

HW Ex [Bancn Thm. 2.11 p.56-58]

Every 2×2 game $M = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$ has a saddle point

EITHER in PURE STRAT.S or in MIXED STRT. COMPUTABLE explicitely by solving 2 linear equations.

Value in mixed strategies for general 2-p 0-sum games. (just main ideas, details in [Bressan]).

$\Phi : A \times B \to \mathbb{R}$ cont., $A, B \in \mathbb{R}^k$ compact

$\mathbb{P}(A)$, $\mathbb{P}(B)$ subsets of vector space of Radon measures.

<u>Lemma</u>. $\mathbb{P}(A)$, $\mathbb{P}(B)$ are convex.

<u>Pf</u>. $\lambda \in [0,1]$, $\bar{\mu}, \mu \in \mathbb{P}(A)$, $\lambda \bar{\mu} + (1-\lambda)\mu \geq 0$

$$\left(\lambda \bar{\mu} + (1-\lambda)\mu\right)(A) = \lambda \underbrace{\bar{\mu}(A)}_{=1} + (1-\lambda)\underbrace{\mu(A)}_{=1} = 1 . \quad \blacksquare$$

Topology on $\mathbb{P}(A)$ & $\mathbb{P}(B)$ that makes them metrizable & compact: weak $*$ topology on $\mathbb{P}(A)$ as a dual of $C(A)$ with uniform convergence.

<u>Def</u>. $(\mu_k)_{k\in\mathbb{N}}$ in $\mathbb{P}(A)$ conv. weak $*$ to $\mu \in \mathbb{P}(A)$ if

$$\forall f \in C(A) \qquad \lim_{k\to\infty} \int_A f \, d\mu_k = \int_A f \, d\mu .$$

& we write $\mu_k \xrightarrow{*} \mu$.

<u>Rmk</u>. $\tilde{\Phi} : \mathbb{P}(A) \times \mathbb{P}(B) \to \mathbb{R}$ (if Φ is cont) is cont. w.r.t. the product weak $*$ convergence topol., i.e., $\mu_k \xrightarrow{*} \mu$, $\nu_k \xrightarrow{*} \nu$

$$\lim_{k\to\infty} \tilde{\Phi}(\mu_k, \nu_k) = \lim_k \iint_{A\times B} \Phi(a,b)\, d\mu_k(a)\, d\nu_k(b) =$$

$$= \iint_{A\times B} \Phi(a,b)\, d\mu(a)\, d\nu(b) = \tilde{\Phi}(\mu,\nu) . \qquad \text{(NO PROOF)}$$

Thm A compact $\Rightarrow$ $\mathbb{P}(A)$ with $\overset{*}{\rightharpoonup}$ topology is METRIZABLE & SEQUENTIALLY COMPACT, i.e., $\forall (\mu_k)_{k\in\mathbb{N}}$ in $\mathbb{P}(A)$, $\exists\ \mu_{k_n}$, $k_n \to \infty$: $\mu_{k_n} \underset{n\to\infty}{\overset{*}{\rightharpoonup}} \mu \in \mathbb{P}(A)$.

Pf NO

Cor (of V. Neumann MIN-MAX THM.). $A, B \subseteq \mathbb{R}^k$ compact, $\Phi \in C(A\times B)$ $\Rightarrow$ the game has a value in mixed strategies.

Pf. Goal $(\mathbb{P}(A), \mathbb{P}(B), \tilde{\Phi})$ sats ass. of V. Neu. Thm.

Remains only: $\tilde{\Phi}$ is CONCAVE–CONVEX.

$$\tilde{\Phi}(\lambda\mu + (1-\lambda)\bar{\mu}, \nu) = \iint_{A\times B} \Phi(a,b)\big(\lambda\, d\mu(a) + (1-\lambda)\, d\bar{\mu}(a)\big)\, d\nu(b) =$$

$$= \lambda \iint_{A\times B} \Phi(a,b)\, d\mu(a)\, d\nu(b) + (1-\lambda) \iint \Phi(a,b)\, d\bar{\mu}(a)\, d\nu(b) =$$

$$= \lambda\, \tilde{\Phi}(\mu,\nu) + (1-\lambda)\, \tilde{\Phi}(\bar{\mu},\nu) \quad \Rightarrow \mu \mapsto \tilde{\Phi}(\mu,\nu) \text{ is LINEAR}$$

$\Rightarrow$ it is CONCAVE.

Similarly $\nu \mapsto \tilde{\Phi}(\mu,\nu)$ is LINEAR $\Rightarrow$ CONVEX.

V. Neu. Thm $\Rightarrow$ $(\mathbb{P}(A), \mathbb{P}(B), \tilde{\Phi})$ has a saddle. $\blacksquare$

NON-ZERO SUM GAMES. For simplicity just 2 players

Given A, B sets of decisions or strategies.

$\Phi^A : A\times B \to \mathbb{R}$ payoff of 1st player, want. to MAX it

$\Phi^B : A\times B \to \mathbb{R}$. " 2nd " " , " " "

- ZERO SUM : $\Phi^A + \Phi^B = 0 \quad \forall e, b \implies \Phi^B = -\Phi^A$
- if $\Phi^A - \Phi^B = 0$ leads to OPTIMIZ. PB. in $A \times B$