LECTURE 11, April 4, 2023

<u>Thm.</u> (Comp. Princ. #1) $u, v : \bar{\Omega} \to \mathbb{R}$ Lip & bdd, resp. sub- and supersol. of

(E) $\quad u_t + H(D_x u) + f(x,t) = 0 \quad$ in $\Omega = \mathbb{R}^n \times]0,T[$,

$H \in C(\mathbb{R}^n)$, $f \in UC(\bar{\Omega})$, $u(x,0) \le v(x,0)\ \forall x \implies u \le v$ in $\bar{\Omega}$.

<u>Proof.</u> $\boxed{\Phi(x,y,t,s) := u(x,t) - v(y,s) - \frac{|x-y|^2}{2\varepsilon} - \frac{|t-s|^2}{2\varepsilon} - \beta(\ell(x)+\ell(y)) - \eta(t+s)}$

$\varepsilon, \beta, \eta > 0$, $\ell(x) = \log(1+|x|^2)$.

<u>Step 1</u> By contradict. $\exists (x_0,t_0)$, $\delta > 0$:

$(u-v)(x_0,t_0) = \delta$.

$\Phi(x_0,x_0,t_0,t_0) = \delta - 2\beta\,\ell(x_0) - 2\eta t_0 > \frac{\delta}{2}$

for all $\beta, \eta \le \bar{\beta}$, $\bar{\beta} > 0$.

<u>Step 2</u> $\exists (\bar{x},\bar{y},\bar{t},\bar{s})$ max point of Φ

depends on ε, β, η.

$\Phi(\bar{x},\bar{y},\bar{t},\bar{s}) > \frac{\delta}{2} > 0$

<u>Step 3</u> Estimates at max pt.:

$\Phi(\bar{x},\bar{x},\bar{t},\bar{t}) + \Phi(\bar{y},\bar{y},\bar{s},\bar{s}) \le 2\,\Phi(\bar{x},\bar{y},\bar{t},\bar{s}) \implies$

$\cancel{u(\bar{x},\bar{t})} - v(\bar{x},\bar{t}) - \cancel{2\beta\,\ell(\bar{x})} - \cancel{2\eta\bar{t}} + u(\bar{y},\bar{s}) - \cancel{v(\bar{y},\bar{s})} - \cancel{2\beta\,\ell(\bar{y})}$

$\cancel{-2\eta\bar{s}} \le \cancel{2}u(\bar{x},\bar{t}) - \cancel{2}v(\bar{y},\bar{s}) - \frac{|\bar{x}-\bar{y}|^2}{\varepsilon} - \frac{(\bar{t}-\bar{s})^2}{\varepsilon} - \cancel{2\beta}(\cancel{\ell(\bar{x})} + \cancel{\ell(\bar{y})})$

$-2\eta(\cancel{\bar t}+\cancel{\bar s})$. $\Longrightarrow$

$$\frac{|\bar x-\bar y|^2}{\varepsilon}+\frac{|\bar t-\bar s|^2}{\varepsilon}\le u(\bar x,\bar t)-u(\bar y,\bar s)+v(\bar x,\bar t)-v(\bar y,\bar s)$$

$u,v \in Lip$

$(a+b)^2\le 2(a^2+b^2)$

$$\le 2L\big(|\bar x-\bar y|+|\bar t-\bar s|\big)$$

$$\frac12\,\frac{(|\bar x-\bar y|+|\bar t-\bar s|)^2}{2\varepsilon}\le$$

$$\Longrightarrow \quad \frac{|\bar x-\bar y|+|\bar t-\bar s|}{\varepsilon}\le 8L \qquad (S)$$

$$\Longrightarrow \quad |\bar x-\bar y|,\ |\bar t-\bar s|\to 0 \quad \text{as } \varepsilon\to 0+$$

<u>Step 4</u> <u>Case 1</u> Supp. $\bar t=0 \Rightarrow u(\bar x,0)-v(\bar y,\bar s)\le$

Init cond.

$\le v(\bar x,0)-v(\bar y,\bar s)\longrightarrow 0$ by $v\in Lip$ & (S)

Contradiction because $u(\bar x,\bar t)-v(\bar y,\bar s)\ge \Phi(\bar x,\bar y,\bar t,\bar s)\ge\frac{\delta}{2}>0$.

<u>Case 2</u> Supp $\bar s=0$ ... analogous ... HW

<u>Step 5.</u> $0<\bar t,\bar s\le T$ use (E) + Lemma of last lect. if $\bar s=T$ or $\bar t=T$.

Freeze $y=\bar y$, $s=\bar s$

$$\varphi(x,t)=\frac{|x-\bar y|^2}{2\varepsilon}+\frac{|t-\bar s|^2}{2\varepsilon}+\beta\,\ell(x)+\eta t \quad \in C^\infty$$

$u-\varphi$ has a max at $(\bar x,\bar t)$. $D_x\varphi=\frac{x-\bar y}{\varepsilon}+\beta D\ell(x)$

$$\varphi_t=\frac{t-\bar s}{\varepsilon}+\eta$$

$\in C^\infty$

Freeze $x=\bar x$, $t=\bar t$: $\psi(y,s)=-\frac{|\bar x-y|^2}{2\varepsilon}-\frac{|\bar t-s|^2}{2\varepsilon}-\beta\,\ell(y)-\eta s$

$-v+\psi$ has a max at $\bar y,\bar s \Rightarrow v-\psi$ has a min at $\bar y,\bar s$.

$$D_x\psi = \frac{\bar{x}-y}{\varepsilon} - \beta Dl(y), \quad \psi_s = \frac{\bar{t}-s}{\varepsilon} - \eta$$

Step 6 u, v sub - supersol. of (E) $\Rightarrow$

$$\cancel{\frac{\bar{t}-\bar{s}}{\varepsilon}} + \eta + H\Big(\underbrace{\frac{\bar{x}-\bar{y}}{\varepsilon}}_{=P_\varepsilon} + \beta Dl(\bar{x})\Big) + f(\bar{x},\bar{t}) \le 0 \le$$

$$\cancel{\frac{\bar{t}-\bar{s}}{\varepsilon}} - \eta + H\Big(\underbrace{\frac{\bar{x}-\bar{y}}{\varepsilon}}_{=P_\varepsilon} - \beta Dl(\bar{y})\Big) + f(\bar{y},\bar{s})$$

$$0 < 2\eta \le -f(\bar{x},\bar{t}) + f(\bar{y},\bar{s}) + H(P_\varepsilon - \beta Dl(\bar{y})) - H(P_\varepsilon + \beta Dl(\bar{x}))$$

NB $(S) \Rightarrow |P_\varepsilon| = \left|\frac{\bar{x}-\bar{y}}{\varepsilon}\right| \le \delta L, \quad |Dl| \le 2$

$K = \bar{B}(0, \delta L+2)$, $H \in UC(K) \Rightarrow \exists$ modulus of cont. ω_H in K. $\Rightarrow$ use (S)

$$0 < 2\eta \le \omega_f(\delta L\varepsilon) + \omega_H\big(\beta\, \underbrace{|Dl(\bar{x}) - Dl(\bar{y})|}_{\le 4}\big)$$

Choose ε, β small & get a contradiction. $\boxtimes$

Rmk. HW: refine the proof so that it holds for

$$u_t + |Du|^2 + V(x) = 0 \qquad V \in C(\mathbb{R}^n).$$

Back to Hopf-Lax formula.

$$(CP) \quad \begin{cases} u_t + H(D_x u) = 0 & \text{in } \Omega \\ u(x,0) = g(x) & \text{in } \mathbb{R}^n \end{cases}$$

H convex & superlinear, $g \in Lip(\mathbb{R}^n)$.

Corollary H as above, g also bounded. Then $\forall\ 0 < T < +\infty$ $u_{HL}(x,t) =$ Hopf-Lax formula is the UNIQUE visco. solution of (CP) in $Lip(\bar\Omega) \cap L^\infty(\bar\Omega)$.

Proof Use Comparison Princ. 1 with $f \equiv 0$. Must check $|g(x)| \le C_g \ \forall x \Rightarrow \forall T > 0$ u_{HL} is bounded in $\mathbb{R}^N \times [0,T] = \bar\Omega$. Recall

$$|u_{HL}(x,t) - g(x)| \le Ct \le CT \Rightarrow |u_{HL}(x,t)| \le CT + C_g \quad \forall (x,t) \in \bar\Omega.$$

Comp. Princ. $\Rightarrow$ uniqueness. ∎.

More on Comparison Principles for evolutive equations. Many possible variants for general HJ.

(HJ) $\quad u_t + H(D_x u, x) = 0$.

Assumptions. $\exists\ \omega$ modulus, $M \ge 0$:

(RH) $\quad |H(p,x) - H(p,y)| \le \omega(|x-y|(1+|p|))$

(Lip H) $\quad |H(p,x) - H(q,x)| \le M|p-q|$ $\quad \forall x, y, p, q$

Thm (Comp. Princ #2): $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satg. (RH)(Lip H), $u, v \in BUC(\bar\Omega)$ $\quad \Omega = \mathbb{R}^n \times]0,T[$, resp. sub & supersol.

bounded unif. cont.

of (HJ) in Ω, $u(x,0) \le v(x,0)$ $\forall x$ $\Rightarrow u \le v$ in $\overline{\Omega}$.

Pf See Thm 1 p. 547 in [Ev].

Idea of pf. same as Comp. Princ. #1 with 2 changes.

Step 3 instead of (S) prove

(S1) $|\bar{x}-\bar{y}| + |\bar{t}-\bar{s}| \le c\sqrt{\varepsilon}$

(S2) $\dfrac{|\bar{x}-\bar{y}|^2}{\varepsilon} \le \omega_u(c\sqrt{\varepsilon}) + \omega_v(c\sqrt{\varepsilon})$

Step 6. Can be done with (S1)(S2) because H sats. (RH)(Lip H). .... Details .... HW. $\blacksquare$

INTRODUCTION TO (DETERMINISTIC) OPTIMAL CONTROL.

CONTROL SYSTEM.

$$(S) \quad \begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s > t = \text{initial time} \\ \phantom{\dot{y}(s) = f(y(s), } \uparrow \text{ control fn.} \\ y(t) = x \\ \uparrow \text{ initial state} \end{cases}$$

$y(s)$ = current state at time s

$\alpha(s)$ = control at time s

$f : \mathbb{R}^n \times A \to \mathbb{R}^n$ continuous, A top. space, often $A \subseteq \mathbb{R}^m$ compact

HYPOTHESES (as in [Ev.], more general in [BCD]).

$$\begin{cases} |f(x,a)| \le M & \forall x \in \mathbb{R}^n, \forall a \in A \\ |f(x,a) - f(z,a)| \le L|x-z| & \forall x, z \in \mathbb{R}^n \quad \forall a \in A. \end{cases}$$

Fix $\mathbb{R} \ni T > t \ge 0$ $\quad T$ = final time or t. horizon.

KNOWN FACTS on ODES.

$\forall \alpha : [t,T] \to A$ CONT. $\exists$ unique C^1 sol. of (S) in $[t,T]$., $y(s) := y_x(s) := y_x(s; \alpha, t)$.

Moreover:

(E1) $\quad |y_x(s;\alpha,t) - x| \leq M(s-t)$

(E2) $\quad |y_x(s;\alpha,t) - y_z(s;\alpha,t)| \leq e^{L(s-t)} |x-z|$.

Def. Admissible controls (OPEN LOOP CONTROLS).

$\mathcal{A} := \{\alpha : [0,T] \to A \text{ measurable}\}$.

Recall. (S) with $\alpha \in C$ is EQUIVALENT to

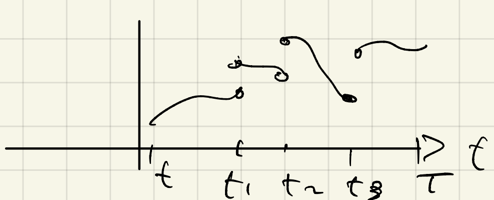
$$y(s) = x + \int_t^s f(y(\tau), \alpha(\tau))\, d\tau \qquad \forall s > t.$$

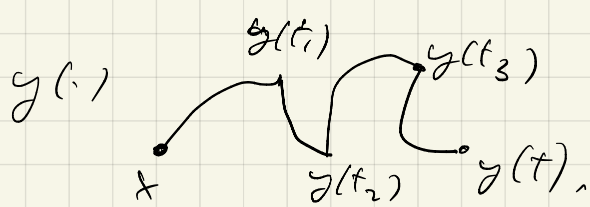
Fund. Th. Calc. $\quad \dot{y}(s) = f(y(s), \alpha(s)) \quad \forall s$

Def For $\alpha \in \mathcal{A}$, $0 \leq t \leq T$ a GENERALIZED sol. of (S) (or TRAJECTORY of (S)) is $y : [t,T] \to \mathbb{R}^n$ ABSOLUT. CONT. & s.t.

$$y(s) = x + \int_t^s f(y(\tau), \alpha(\tau))\, d\tau \qquad \forall s \in [t,T].$$

N.B. If at $\bar{s}$ α is cont., then F.T.C. $\Rightarrow$
$\exists\ \dot{y}(\bar{s}) = f(y(\bar{s}), \alpha(\bar{s}))$, but at points of discont. of $\alpha(\cdot)$ $\dot{y}$ may be discont.





Thm. Under the standing ass. on f $\forall \alpha \in \mathcal{A}$ $\exists$ UNIQUE GENERAL SOL. of (S) def. in $[t,T]$. $y(\cdot) = y_x(\cdot; \alpha, t)$ &

it satisfies:

(E1) $\quad |y_x(s;\alpha,t) - x| \leq M(s-t)$

(E2) $\quad |y_x(s;\alpha,t) - y_z(s;\alpha,t)| \leq e^{L(s-t)} |x-z|$.

& y is Lip in s. $\square$

Pf. Very similar to the case $\alpha \in C$, see Sec. III.5 [BCD]. $\square$

HW (easy) prove it for piecewise cont. α. $\square$.

FINITE HORIZON COST FUNCTIONAL.

Data: $\ell : \mathbb{R}^n \times A \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$, $T < +\infty$ horizon or final time

$$J(x,t,\alpha(\cdot)) = \int_t^T \underbrace{\ell(y(s), \alpha(s))\,ds}_{\text{running cost}} + \underbrace{g(y(T))}_{\text{terminal cost}}$$

where $y(s) = y_x(s;\alpha,t)$

GOAL: MINIMIZE $J(x,0,\alpha(\cdot))$ over $\alpha(\cdot) \in \mathcal{A}$.

Def. $\ell, g \not\equiv 0$ Bolza problem.

if $\ell \equiv 0$ Mayer pb., if $g \equiv 0$ Lagrange pb.

N.B. Connection Calc. of Var.: $J = \int_0^T L(y(s), \dot{y}(s))\,ds$

it is an opt. cont. pb. with $\dot{y}(s) = \alpha(s)$ $\quad A = \mathbb{R}^n$

"CoV $\subseteq$ Opt. Cont."

Rmk. Bolza pb. can be reduced to Mayer ($\ell \equiv 0$) by adding the state variable y_{n+1}:

$$\begin{cases} \dot{y}_{n+1} = \ell(\underbrace{y_1, \dots, y_n}_{y}, \alpha) \\ y_{n+1}(t) = 0 \end{cases} \Rightarrow y_{n+1}(T) = \int_0^T \ell(y(s), \alpha(s))\, ds$$

$\min \rightsquigarrow \int_t^T \ell\, ds + g(y(T))$ is equivalent to

$\min \rightsquigarrow \left(y_{n+1}(T) + g(y(T))\right)$ which is a Mayer pb.

We will make some proofs with $\ell \equiv 0$. ▨