# LECTURE 3, March 7, 2023

## METHOD OF CHARACTERISTICS

<u>Example 1</u> (LE) $u_t + b \cdot D_x u + cu = e$ in $\mathbb{R}^n \times ]0,T[$
$u(x,0) = g(x)$

$b \in \mathbb{R}^n$ const., $c, e$ cont. fns. of $x$ & $t$.

(C) $\Rightarrow$ $\bar{X}(y,s) = (y+bs, s)$ $\Rightarrow$ $\bar{Y}(x,t) = (x-bt, t)$

Candidate sol of (LE) is $u(x,t) = z(\bar{Y}(x,t))$

Who is $z$? (b) $\dot{z} = p \cdot F_p = \bar{p} \cdot G_{\bar{p}} + 1 p_{n+1} =$

$"= D_x u \cdot b + u_t = e - cu \overset{"}{=} (e - cz)(p, z, \bar{X})$

$$\begin{cases} \dot{z}(s) = -c(\overbrace{y+bs, s}^{\bar{X}})z(s) + e(y+bs, s) \\ z(0) = g(y) \end{cases}$$ linear non-homo ODE scalar.

"Variat. of cost." formula $\Rightarrow$

$$z(y,s) = g(y) + \int_0^s e(\overset{HW}{\dots}) e^{-\int_\tau^s c(\overset{HW}{\dots})dr} d\tau$$

$\Rightarrow$ candidate sol. of (LE) $+ u|_{t=0} = g$ is

$$u(x,t) = z(x-bt, t)$$

HW: complete the formula & verify it solves (LE).

<u>Rmk</u>. 1. I didn't use (a) for $p(\cdot)$

2. Sol. is def $\forall t$ s.t. $c$ & $\ell$ are defined,

$\Rightarrow$ Sol. is GLOBAL.

3. TE is a special case if $b \equiv$ const.

## Ex. 2 Scalar conservation law in $\mathbb{R}^n$

$$\begin{cases} u_t + \operatorname{div}_x f(u) = 0 \\ u(x,0) = g(x) \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}^n$ at least $C^1$

QUASILINEAR!

$$u_t + \underbrace{f'(u) \cdot D_x u}_{G(u, D_x u)} = 0 \qquad F = p_{n+1} + \underbrace{f'(u)\,\bar{p}}_{G(\bar{p}, z)}$$

$$G_{\bar{p}} = f'(z), \quad G_x = 0 \quad G_{p_{n+1}} = 1$$

(b) $\dot{z}(s) = p \cdot F_p = p_{n+1} \cdot 1 + f'(z) \cdot \bar{p} \;\text{"}=$

$\text{"}= u_t + D_x u\, f'(u) = 0\text{"}$ $\qquad \dot{z}(s) = 0$

$z(0) = g(y) \Rightarrow z(s) = g(y) \quad \forall s$

(c) $\dot{X}_i(s) = f_i'(z(s)) \qquad i = 1, \dots, n$

$X_{n+1}(s) = s \qquad X_i(0) = y$

$$\Rightarrow \quad X_i(s) = y_i + \int_0^s f_i'(z(\tau))\,d\tau =$$

$$= y_i + s\, f_i'(g(y)) \overset{?}{=} x$$

Try to invert it

$$y + s f'(g(y)) = x \iff y = \underbrace{(I + s f' \circ g)^{-1}(x)}_{Y(x,s)}$$

$s = t$

Candidate soln.

$$u(x,t) = g\left((I + t f' \circ g)^{-1}(x)\right)$$

Rmk: don't need eq. (Q) for $p$.

HW: $n=1$ verify $u$ solves $u_t + f(u)_x = 0$.

Q: • what is the maximal interval of existence for $u$?

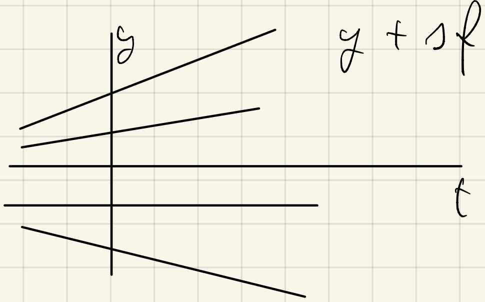
• Are there $f, g$ s.t. $u$ $\exists$ $\forall t > 0$ ?

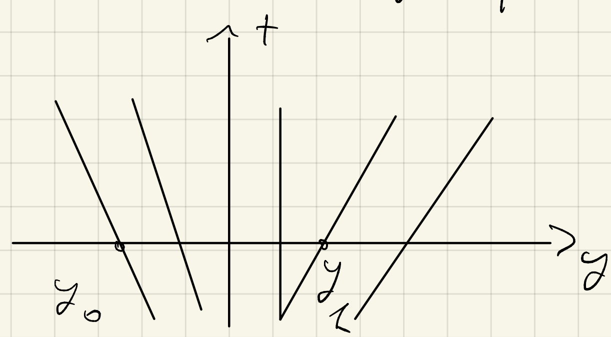
Rmk. • $u$ is const. on $\tilde{X}(x,t)$

• proj. charact. are STRAIGHT LINES.

Answers for $n=1$.

Case 1 $f' \circ g$ is NONDECREASING. (e.g. $f'' \geq 0$, $g' \geq 0$)



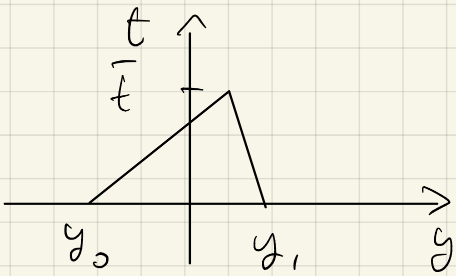




$x \mapsto \tilde{X}(x,t)$ INJective $\Rightarrow$ BIJECTIVE

$\Rightarrow$ sol. $u$ is def. $\forall t > 0$ $\Rightarrow$ $\exists$ GLOBAL SOL.n.

Case 2. $f' \circ g$ decreasing (e.g. $f'' \geq 0$, $g' < 0$)

$$y_0 < y_1 \Rightarrow f' \circ g(y_0) \overset{(*)}{>} f' \circ g(y_1)$$




Who is $\bar{t}$?

$$y_0 + \bar t f'(g(y_0)) \overset{?}{=} y_1 + \bar t f'(g(y_1))$$

$$y_0 - y_1 = \bar t \left( f'(g(y_1)) - f'(g(y_0)) \right)$$

$$\bar t = \frac{\overbrace{y_0 - y_1}^{<0}}{\underbrace{f'(g(y_1)) - f'(g(y_0))}_{<0 \ (*)}} > 0$$

if $g(y_0) \neq g(y_1)$

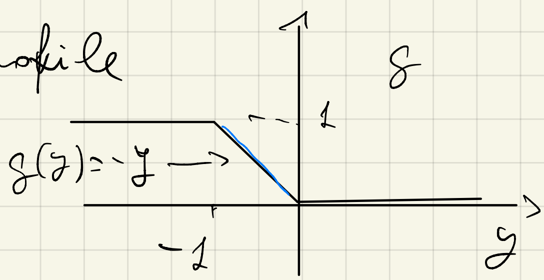
In $\bar t$ there is a JUMP DISCONTINUITY of $u$

$\Rightarrow$ a SHOCK.

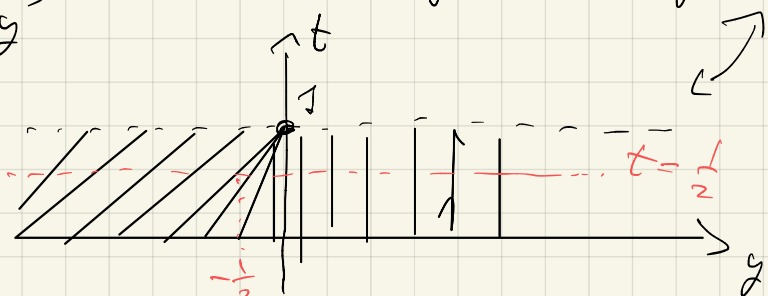
Ex. BURGER: $f(u) = \dfrac{u^2}{2}$ $\quad u_t + u u_x = 0 \quad f'(u) = u$

$X(t,y) = y + t g(y)$. Draw graph of $u$ for $g$ with

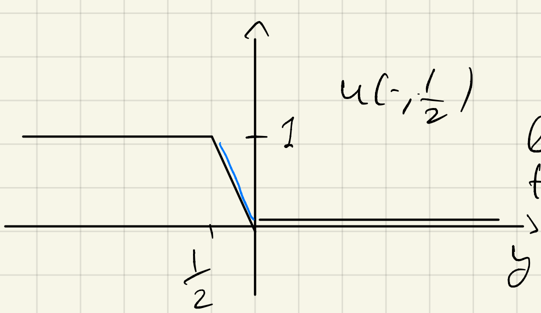
profile



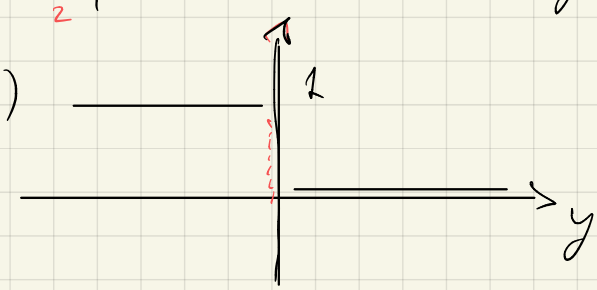

$$X(t,y) = \begin{cases} y + t & y < -1 \\ y - ty = (1-t)y & -1 < y < 0 \\ y & y > 0 \end{cases}$$




Profile of $u(\cdot, \frac{1}{2})$




$\lim_{t \to 1^-} (u(x,t))$



$\underline{Q}$ : How to construct a "solution" after the shock, i.e., for $t > 1$?

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$\underline{\text{Ex. 3}}$ Hamilton-Jacobi eqs.

$$(CP) \begin{cases} u_t + H(D_x u, x) = 0 & \text{in } \mathbb{R}^n \times ]0,T[ \\ u(x,0) = g(x) & x \in \mathbb{R}^n \end{cases}$$

$H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ at least $C^2$ $\qquad F = p_{n+1} + G \qquad G = H$

$G_z = 0, \; G_t = 0, \; G_x = H_x = D_x H, \; G_p = H_p := D_p H$

$G_{x_{n+1}} = 0$

Char. syst.:

$$\begin{aligned} &(a) \begin{cases} \dot{p}_i = -H_{x_i}(\bar{p}, \bar{X}) & i = 1, \dots, n \quad p_i(0,y) = Dg(y) \\ \dot{p}_{n+1} = 0 & p_{n+1}(s) = p_{n+1}(0) = -H(Dg(y), y) \end{cases} \\ &(b) \quad \dot{z} = p_{n+1} + \bar{p} \cdot H_{\bar{p}}(\bar{p}, \bar{X}), \quad z(0,y) = g(y) \\ &(c) \begin{cases} \dot{X}_i = H_{p_i}(\bar{p}, \bar{X}) & i = 1, \dots, n \quad X_i(0) = y_i \\ \dot{X}_{n+1} = 1 & \dots \quad X_{n+1}(s) = s \end{cases} \end{aligned}$$

$\dot{z} = \dots$ no $z$ $\dots \qquad z(s) = g(y) + \int_0^s \dots$

IMPORTANT PART of CHARACT. SYSTEM is

HAMILTONIAN SYSTEM
$$\begin{cases} \dot{\bar{p}} = -D_x H(\bar{p}, \bar{X}) \\ \dot{\bar{X}} = D_{\bar{p}} H(\bar{p}, \bar{X}) \end{cases} \quad 2n\text{-equations}$$

Ex. 3 bis SPECIAL CASE : $H = H(\bar{p}) \Rightarrow H_x = 0$

$\Rightarrow \dot{\bar{p}}(s) = 0 \quad \bar{p}(y,s) = Dg(y) \quad \forall s$

$\dot{\bar{X}}(s) = H_p(Dg(y)) \Rightarrow \bar{X}(y,s) = y + s H_p(Dg(y))$

if $\exists$ inverse $y \mapsto \bar{X}(y,s)$, $\bar{Y}(x,t) \Rightarrow$ candidate

sol. of (CP) $u$ is $\qquad p_{n+1} + \bar{p} \cdot D_{\bar{p}} H$

$$z(s) = g(y) + s\left[-H(Dg(y)) + Dg(y) \cdot DH(Dg(y))\right]$$

$$(D) \quad u(x,t) := g(\bar{Y}) + t\left[DH(Dg(\bar{Y})) \cdot Dg(\bar{Y}) - H(Dg(\bar{Y}))\right]$$

$\bar{Y} = \bar{Y}(x,t)$

Thm. Ass. $H \in C^2(\mathbb{R}^n)$, $g \in C^2(\mathbb{R}^n)$, $y : \bar{X}(y,t)$ has an inverse $\forall t \in [0,T[ \Rightarrow u$ def. by (D) solves (CP)

$$\begin{cases} u_t + H(D_x u) = 0 & \text{in } \mathbb{R}^n \times ]0,T[ \\ u(x,0) = g(x) & \text{in } \mathbb{R}^n . \end{cases}$$

Pf. • $u(x,0) = g(x)$ trivial. • [P.L. Lions 1983 ...]

Claim 1. $D_x u(x,t) = Dg(\bar{Y}(x,t))$

Claim 2. $\frac{\partial}{\partial t} u(x,t) = -H(Dg(\bar{Y}(x,t)))$

C.1 + C2. $\Rightarrow \frac{\partial u}{\partial t} = -H(D_x u)$. ∎

Pf of C1. Fix $x_k$.

$$\frac{\partial u}{\partial x_k} = \sum_{i=1}^n \frac{\partial g}{\partial x_i} \frac{\partial \bar{Y}_i}{\partial x_k} + t\left[-\cancel{\sum_{i=1}^n H_{p_i}(Dg) \frac{\partial}{\partial x_k}\left(\frac{\partial g}{\partial y_i}(\bar{Y})\right)} + \right.$$

$$+ \sum_{i=1}^{n} \frac{\partial}{\partial x_k}\Big(H_{p_i}(Dg(Y))\, g_{y_i}(Y)\Big) + \cancel{\sum_{i=1}^{n} H_{p_i}(Dg(Y)) \frac{\partial}{\partial x_k}\Big(\frac{\partial g}{\partial y_i}(Y)\Big)}\Big] =$$

$$= \sum_{i=1}^{n} \frac{\partial g}{\partial y_i}(Y)\Big[\frac{\partial Y_i}{\partial x_k} + t \sum_{j,e=1}^{n} H_{p_i p_j}\, g_{y_j y_e} \frac{\partial Y}{\partial x_k}\Big] \qquad k = 1, \dots, n$$

$$(D_x u)^T = Dg(Y)^T \big[D_x Y + t\, D^2 H\, D^2 g\, D_x Y\big] \overset{?}{=} (Dg(Y))^T$$

cl 1.

cl 1 $\iff [+] = Id$

Inv. Fn. Thm.

$$[+] = \big[I + t D^2 H D^2 g\big] D_x Y \qquad D_x Y = (D_y X(Y))^{-1}$$

$$X(y,t) = y + t\, DH(Dg(y)) \Rightarrow D_y X \overset{(*)}{=} I + t D^2 H D^2 g$$

$$\Rightarrow [+] = (D_x Y)^{-1} D_x Y = Id. \qquad \boxtimes \text{ cl. 1.}$$

Pf. of Claim 2

$$\frac{\partial u}{\partial t} = Dg \cdot \frac{\partial Y}{\partial t} + \big[DH(Dg)\cdot Dg - H(Dg)\big] + t\Big[Dg\cdot D^2 H D^2 g \frac{\partial Y}{\partial t}$$

$$+ \cancel{DH(Dg)\cdot \frac{\partial}{\partial t}(Dg(Y))} - \cancel{DH(Dg)\cdot \frac{\partial}{\partial t}(Dg(Y))}\Big] =$$

$$= Dg \cdot \big[I + t D^2 H D^2 g\big] \frac{\partial Y}{\partial t} + DH(Dg)\cdot Dg - H(Dg) \quad \dots (Y) \dots$$

Claim 3 $\frac{\partial Y}{\partial t} = -\big[I + t D^2 H(Dg) D^2 g\big]^{-1}(Y)\, DH(Dg(Y))$. Then

$$\frac{\partial u}{\partial t} = \cancel{-Dg\cdot DH(Dg(Y))} + \cancel{DH(Dg)\cdot Dg} - H(Dg) = -H(Dg(Y))$$

$\Rightarrow$ Claim 2.

Pf of claim 3

$$X(Y(x,t),t) = x$$

$$\frac{\partial}{\partial t} \Downarrow \qquad = 0$$

$$\Rightarrow 0 = D_y X(Y,t) \frac{\partial Y}{\partial t} + \frac{\partial X}{\partial t}(Y,t) \Rightarrow$$

$$\Rightarrow \quad \frac{\partial \underline{Y}}{\partial t} = -\left(D_y \underline{X}(\underline{Y},t)\right)^{-1} \frac{\partial \underline{X}}{\partial t}(\underline{Y},t) \overset{(*)}{=} -\left(I + tD^2H\, D^2g\right)^{-1} DH(Dg(\underline{Y}))$$

$$\underline{X} = y + tDH(Dg(y)) \Rightarrow \frac{\partial \underline{X}}{\partial t} = DH(Dg(y))$$

which is Claim 2 $\blacksquare$

Next time: Q: GLOBAL SOLUTIONS for H-J?