## LECTURE 15, April 27, 23

GAME THEORY: $\Phi \in C(A \times B)$, $A, B$ compact metric spaces.

MARGINAL FNS.: $\Phi^{\max}(b) := \max_a \Phi(a,b)$, $\Phi^{\min}(a) := \min_b \Phi(a,b)$

Best Response: $R^A(b) := \arg\max_a \Phi(a,b)$, $R^B(a) = \arg\min_b \Phi(a,b)$

UPPER VALUE: $v^+ := \min_b \Phi^{\max}(b) = \min_b \max_a \Phi(a,b)$

LOWER VALUE: $V^- := \max_a \Phi^{\min}(b) = \max_a \min_b \Phi(a,b)$

<u>Prop.</u> $V^- \leq V^+$. If $v^- = v^+$ GAME has a VALUE.

<u>Examples</u> of $v^- \neq v^+$.

<u>Ex</u>. MATRIX GAMES $\Phi(i,j) = \phi_{ij}$

$$\begin{pmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & & \\ \phi_{m1} & \cdots & \phi_{mn} \end{pmatrix} \begin{array}{l} \xrightarrow{\min} \min_j \phi_{1j} = \Phi^{\min}(1) \\ \\ \longrightarrow \min_j \phi_{mj} = \Phi^{\min}(m) \end{array}$$

$\xrightarrow{\max}$ $V^- = \max_i \Phi^{\min}(i)$

$\xrightarrow{\max}$ $\Phi^{\max}(1) = \max_i \phi_{i1}$ $\quad$ $\max_i \phi_{in}$

$\xrightarrow{\min}$ $V^+ = \min_j \Phi^{\max}(j)$

**Ex. 2** "Cake"

$0 \le \varepsilon < \frac{1}{4}$

$$\begin{array}{cc|c} & & \xrightarrow{\min} \\ \multicolumn{2}{c|}{\begin{pmatrix} \frac{1}{2}+\varepsilon & \frac{1}{2}-\varepsilon \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}} & \begin{matrix} \frac{1}{2}-\varepsilon \\ \frac{1}{4} \end{matrix} \\ \max \downarrow & & \downarrow \max \\ \frac{3}{4} & \frac{1}{2}-\varepsilon & v^- = \frac{1}{2}-\varepsilon \\ & \xrightarrow{\min} \frac{1}{2}-\varepsilon = v^+ & \end{array}$$

$\Rightarrow \quad v = \frac{1}{2} - \varepsilon$ is THE VALUE of the game.

**Ex 3** "Head & Tail"

$$\begin{array}{cc|c} & & \xrightarrow{\min} \\ \multicolumn{2}{c|}{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} & \begin{matrix} -1 \\ -1 \end{matrix} \\ \max \downarrow & & \downarrow \max \\ 1 & 1 & v^- = -1 \\ & \xrightarrow{\min} v^+ = 1 & \end{array}$$

$v^- = -1 < v^+ = 1$

The VALUE does not exist!

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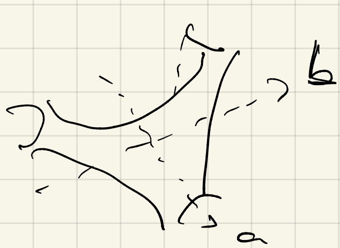
**Def.** A SADDLE POINT of the game is $(a^*, b^*) \in A \times B$:

$$\forall a \in A \qquad \Phi(a, b^*) \le \Phi(a^*, b^*) \le \Phi(a^*, b) \qquad \forall b \in B$$

**Ex.** $A = B = [-1, 1] \qquad \Phi(a, b) = b^2 - a^2$




$b^* = a^* = 0$ is a SADDLE

Rmk : $(a^*, b^*)$ is a saddle $\iff$

(S) $$\max_a \Phi(a, b^*) \leq \Phi(a^*, b^*) \leq \min_b \Phi(a^*, b)$$

$a^* \in R^A(b^*)$ $\qquad$ $b^* \in R^B(a^*)$

(S) $\iff$ (S') $$\max_a \Phi(a, b^*) = \Phi(a^*, b^*) = \min_b \Phi(a^*, b).$$

Rmk. Suppose $R^A$ & $R^B$ are functions (single-valued)

$R^A : B \to A$, $R^B : A \to B$. $(a^*, b^*)$ saddle $\Rightarrow$

$a^*$ is a fixed point of $R^A \circ R^B : A \to A$ because

$R^A \circ R^B(a^*) = R^A(b^*) = a^*$.

& $b^*$ is a fixed point of $R^B \circ R^A : B \to B$. $\blacksquare$

Def. SECURITY STRATEGIES : $v^+ = \min_b \Phi^{\max}(b)$

is $b^*$ : $v^+ = \Phi^{\max}(b^*)$, i.e., $b^* \in \arg\min_b \Phi^{\max}(b)$

for $2^{nd}$ player.

$a^*$ is S.S. for $1^{st}$ player if $v^- = \Phi^{\min}(a^*)$

i.e., $a^* \in \arg\max_a \Phi^{\min}(a)$

Thm. The game has a value $\iff$ $\exists$ a saddle point.

Pf "$\Leftarrow$" Ass.: $(a^*, b^*)$ saddle pt. Goal: $v^+ \leq v^-$

$$v^- = \max_a \min_b \Phi(a, b) \geq \min_b \Phi(a^*, b) = \max_a \Phi(a, b^*)$$

$$\geq \min_b \max_a \Phi(a, b) = v^+ \qquad \blacksquare$$

Rmk $a^*$ is a SEC. STR. for $1^{st}$ player. & $b^*$ is S. S. for $2^{nd}$

"$\Rightarrow$" Ass $v^+ = v^-$. Take $a^*$ a SEC. STR. for A.

$$v^- = \Phi^{\min}(a^*) = \min_b \Phi(a^*, b).$$

Take $b^*$ a SEC. STR. for B: $v^+ = \Phi^{\max}(b^*) = \max_a \Phi(a, b^*)$

$$\forall a \in A \quad \Phi(a, b^*) \le \max_a \Phi(a, b^*) = v^+ \overset{Ass}{=} v^- = \min_b \Phi(a^*, b)$$
$$\le \Phi(a^*, b) \quad \forall b$$

For $a = a^*$, $b = b^*$ $\Rightarrow$ $\Phi(a^*, b^*) = v^+ = v^- \le \Phi(a^*, b^*)$

$\Rightarrow$ "$\le$" are "$=$" $\Rightarrow$ $\Phi(a^*, b^*) = v$ & (S) hold,

$\Rightarrow$ $(a^*, b^*)$ a SADDLE PT. ∎

Corollary If game has a value $\Rightarrow$

(i) $(a^*, b^*)$ is a Saddle $\Leftrightarrow$ $a^*$ is SEC. STR. for $1^{st}$
$b^*$ " " " for $2^{nd}$

(ii) (EXCHANGEABILITY): If $(\bar{a}, \bar{b})$ is also a Saddle.

$\Rightarrow$ $(a^*, \bar{b})$, $(\bar{a}, b^*)$ are Saddles.

Pf (i) See pf. of thm. (ii) from (i). ∎

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THE MINIMAX THEOREM of Von Neumann (1924)

Thm. $A, B \subseteq$ vector spaces, COMPACT & CONVEX, $\Phi \in C(A \times B)$
$\Phi$ CONCAVE-CONVEX, i.e.,

$$\begin{cases} \forall b, & a \mapsto \Phi(a,b) \text{ is CONCAVE} \\ \forall a, & b \mapsto \Phi(a,b) \text{ is CONVEX} \end{cases}$$

$\Rightarrow$ $V^+ = V^-$, i.e., $(A, B, \Phi)$ has a value & at least one SADDLE POINT.

<u>PROOF</u>: will be done if $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$.

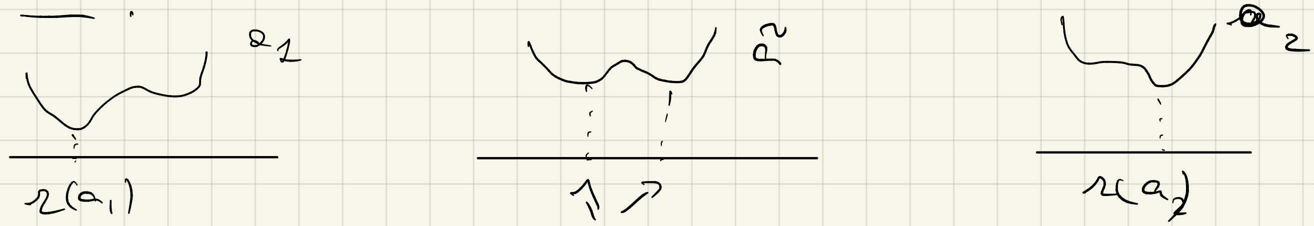
<u>Rmk.</u> Supp. $\Phi(a, \cdot)$ STRICTLY CONVEX $\Rightarrow \exists$ UNIQUE

$r(a) \in B$ : $\Phi(a, r(a)) = \min_b \Phi(a, b)$

$\Rightarrow$ $R^B(a) = \{r(a)\}$.

<u>Lemma</u> If $\Phi(a, \cdot)$ is strictly convex $\forall a \in A$, $\Phi \in C$, $A, B$ comp. CONVEX $\Rightarrow$ $r : A \to B$ is continuous.

<u>Rmk</u> Not true without convexity.




here $r$ jumps to right at $\tilde{a}$! $R^B(\tilde{a})$

<u>Pf Lemma</u>. $\bar{a} \in A$, $a_n \to \bar{a}$ GOAL: $r(a_n) \xrightarrow[n\to\infty]{} r(\bar{a})$.

Extract $a_{n_k}$ : $r(a_{n_k}) \xrightarrow[k\to\infty]{} \bar{b} \in B$ (B compact.)

$\Rightarrow$ $\Phi(a_{n_k}, r(a_{n_k})) \to \Phi(\bar{a}, \bar{b})$ $k \to \infty$

$\forall b$ $\quad \leq \Phi(a_{n_k}, b) \longrightarrow \Phi(\bar{a}, b)$ ..

$\Rightarrow \quad \Phi(\bar{a}, b) \geq \Phi(\bar{a}, \bar{b}) \qquad \forall\, b \in B$

$\Rightarrow \quad \bar{b} = r(\bar{a}) \quad \Rightarrow \quad r(a_{n_k}) \xrightarrow[k \to \infty]{} r(\bar{a})$

By the arbitrariness of $a_{n_k} \Rightarrow r(a_n) \xrightarrow[n \to \infty]{} r(\bar{a})$. ∎

<u>Pf of V.N. Theorem</u>: <u>Step 1</u> Supp. $\forall a \in A \quad b \mapsto \Phi(a, b)$ is strictly convex. Then $\forall a \ \exists\, r(a)$ cont.

$$\Phi(a, r(a)) = \min_b \Phi(a, b) .$$

<u>Step 2</u> Take $a^+$ SEC. STRAT. for 1st pl., i.e.,

$$v^- = \max_a \Phi^{\min}(a) = \Phi^{\min}(a^+) . \qquad b^* := r(a^+)$$

Goal: $(a^+, b^+)$ is a saddle pt.

Note: $\underbrace{\Phi(a^+, b^+)}_{\uparrow\, r(a^+)} = \min_b \Phi(a^+, b) \leq \Phi(a^+, b) \quad \forall b$.

Remains the goal. $\Phi(a^+, b^+) \geq \Phi(a, b^+) \quad \forall a$.

<u>Step 3</u>: Idea: approximate $a^*$ with $a_d := d a + \underbrace{(1-d)}_{=\mu} a^+ \quad (\to a^* \text{ if } d \to 0)$

Fix $a \in A$, $d \in [0, 1]$, $\mu = 1 - d$

$$\Phi(a^*, b^*) \geq \Phi^{\min}(a^+) \underset{a^+ \text{ is SEC. STRAT.}}{\geq} \Phi^{\min}(a_d) =$$

$$= \Phi(a_d, r(a_d)) \underset{\Phi \text{ conc. in } a.}{\geq} d\,\Phi(a, r(a_d)) + \mu\, \Phi(a^+, r(a_d))$$

$$\geq d\,\Phi(a, r(a_d)) + (1-d)\,\Phi^{\min}(a^+)$$

$$\Rightarrow \quad \cancel{d}\,\Phi^{\min}(a^+) \geq \cancel{d}\,\Phi(a, r(a_d)) \qquad \downarrow \ d \to 0^+$$

$$\Rightarrow \quad \Phi^{\min}(a^*) \geq \Phi(a, \underbrace{z(a^*)}_{=b^*})$$ USE LEMMA)

$$\Rightarrow \quad \Phi^{\min}(a^*) \geq \Phi(a, b^*) \qquad \forall a$$

$\Phi(a^*, b^*) \overset{?}{\geq}$

which is the goal ▣ St. 3

<u>Step 4</u>. Remove the strict convexity of $b \mapsto \Phi(a,b)$.

For simplicity here $B \subseteq \mathbb{R}^k$. Fix $\varepsilon > 0$:

$$\Phi_\varepsilon(a,b) = \Phi(a,b) + \varepsilon |b|^2$$ is STRICTLY CONVEX in $b$ $\forall a$.

St. 1–3 $\Rightarrow$ $\Phi_\varepsilon$ has a saddle point $(a_\varepsilon, b_\varepsilon)$, i.e.

$$\forall a \qquad \Phi_\varepsilon(a, b_\varepsilon) \leq \Phi_\varepsilon(a_\varepsilon, b_\varepsilon) \leq \Phi_\varepsilon(a_\varepsilon, b) \qquad \forall b.$$

By compactness of $A, B$, construct $\varepsilon_n \searrow 0^+$:

$$a_{\varepsilon_n} \to a^* \in A, \qquad b_{\varepsilon_n} \to b^* \in B.$$

$$\Phi_{\varepsilon_n}(a, b_{\varepsilon_n}) \leq V_{\varepsilon_n} = \Phi_{\varepsilon_n}(a_{\varepsilon_n}, b_{\varepsilon_n}) = \Phi(a_{\varepsilon_n}, b_{\varepsilon_n}) + \varepsilon_n |b_{\varepsilon_n}|^2$$

$$\leq \Phi(a_{\varepsilon_n}, b) + \varepsilon_n |b|^2 \qquad \forall b$$

Let $\varepsilon_n \to 0$

$$\Phi(a, b^*) \leq \Phi(a^*, b^*) \leq \Phi(a^*, b) \qquad \forall b$$

$\Rightarrow$ $(a^*, b^*)$ is a SADDLE. ▣

<u>Examples</u>. 1: $\Phi(a,b) = \varphi_1(a) - \varphi_2(b)$

$\varphi_1, \varphi_2$ CONCAVE in $[1,1]^m$ $\Rightarrow$ Thm. applies.

2. IMPORTANT: $\Phi(a,b) = a^T M b$ $M \in \mathcal{M}_{m\times n}$

$a \in A \subseteq \mathbb{R}^m$, $b \in B \subseteq \mathbb{R}^n$. $\Phi$ is bilinear $\Rightarrow$

comp. & convex $\longrightarrow$ cont. & linear in a & in b

$\Rightarrow$ $\Phi$ CONC.-CONVEX THM. is OK.

3. MATRIX GAMES $A = \{1, \dots, m\}$, $B = \{1, \dots, n\}$.

are NOT CONVEX, V.N.T. does NOT apply & in fact we know examples without value.

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MIXED STRATEGIES. Idea: choose in a stochastic instead of deterministic way.

Def. A mixed strategy of 1st player is a $\mu \in \mathbb{P}(A) :=$ $= \{$probability measures on $A\}$. and for 2nd player it is $\nu \in \mathbb{P}(B) = \{$ ... ... on $B\}$.

Ex: $\delta_{\bar a}$ = Dirac measure concentrated in $\bar a \in A$. i.e.

$\forall S \subseteq A$ Borel $\delta_{\bar a}(S) = \begin{cases} 1 & \text{if } S \ni \bar a \\ 0 & \text{if } S \not\ni \bar a \end{cases}$

$\mathbb{P}(A) \supseteq$ "copy of $A$". $A$ = pure strategies

Def $\tilde\Phi(\mu,\nu) := \iint_{A\times B} \Phi(a,b)\, d\mu(a)\, d\nu(b)$

$\tilde\Phi : \mathbb{P}(A) \times \mathbb{P}(B) \to \mathbb{R}$.

N.B. $\tilde{\Phi}(\delta_{\bar a}, \delta_{\bar b}) = \iint_{A\times B} \Phi(a,b)\, d\delta_{\bar a}(a)\, d\delta_{\bar b}(b) =$

$= \Phi(\bar a, \bar b)$ $\qquad \boxed{\int_A f(a)\, d\delta_{\bar a}(a) = f(\bar a).}$

$\Rightarrow$ $\tilde{\Phi}$ "EXTENDS" $\Phi$ from $A\times B$ to $\mathbb{P}(A)\times\mathbb{P}(B)$

Def. If $\exists$ value of the game $(\mathbb{P}(A), \mathbb{P}(B), \tilde{\Phi})$ & a saddle pt., they are called value & saddle of $(A, B, \Phi)$ in mixed strategies.