LECTURE 14, 4.20.23

LINEAR-QUADRATIC CONTROL:

$$(S)\quad \begin{cases} \dot{y} = Ay + B\alpha \\ y(t) = x \end{cases}$$

$A \in M_{n\times n}$, $B \in B_{n\times m}$

$\alpha(\cdot) \in L^1_{loc}([0,T], \mathbb{R}^m)$

$R > 0$

$$(JQ)\quad J(x,t,\alpha(\cdot)) := \int_t^T \left[y(s)^T M y(s) + \alpha(s)^T R \alpha(s) \right] ds + y(T)^T Q y(T).$$

$S = BR^{-1}B^T$ $\quad \boxed{w(x,t) = x^T K(t) x}$ $\quad K(\cdot) \in S(n)$

$$(RT)\quad \begin{cases} \dot{K} = KSK - A^T K + K^T A - M \\ K(T) = Q \end{cases}$$

RICCATI MATRIX O.D.E.

<u>Theorem</u>. If $K \in C^1((t_0,T), Sym(n))$ cont. at $t=T$ & $t=t_0$ sol of (RT), then $w(t,x) = x^T K(t) x$ is a C^1 sol. of (CT), the feedback $\boxed{\Phi(y,s) = -R^{-1}B^T K(s) y}$ is ADMISSIBLE & OPTIMAL $\forall (x,t)$ &

$$w(x,t) = \text{Value function} := \inf_{\alpha \in L^1_{loc}([0,T],\mathbb{R}^n)} J(x,t,\alpha(\cdot))$$

<u>Proof</u>. $\operatorname{argmin}_a \mathcal{H}(p,x,a) = \operatorname{argmin}_a \tilde{\mathcal{H}}(p,a) = \left\{ -\frac{R^{-1}B^T p}{2} \right\}$

$p = D_x w = 2Kx$ $\qquad \Phi(y,s) = -\frac{R^{-1}B^T}{2} 2Ky = -R^{-1}B^T K(s) y$

Use verification Thm. : if Φ is admissible then it's OPTIMAL:

$$(S\Phi)\quad \begin{cases} \dot{y} = Ay - \underbrace{BR^{-1}B^T}_{S} K(s)y & t < s \leq T \\ y(t) = x \end{cases}$$

this is a LINEAR HOMOGENEOUS SYSTEM of ODE

$\Rightarrow \forall x, t \in [t_0, T[\quad \exists$ UNIQUE sol. $y \in C^1([t_0, T), \mathbb{R}^n)$

$\Rightarrow \Phi$ ADMISS. $\Rightarrow \Phi$ optimal. ∎

<u>Rmk.</u> Can be proved $\Phi = -R^{-1}B^T K y$ is the UNIQUE OPT. FEEDBACK [Engw. p. 178].

<u>Q</u>: $\exists$? sol. of (RT)?

<u>Prop.</u> (local $\exists$). In our ass. $(R > 0)$ $\exists\, t_0 < T$: (RT) has a unique sol. $K \in C^1((t_0, T), S(n))$.

<u>Pf.</u> R.H.S. of ODE is C^1, so local $\exists$ thm. $\Rightarrow \exists$ sol. $K \in C^1((t_0, T), M_{n \times n})$. Remains to prove $K(t) \in Sym(n)\ \forall t$.

Take the transpose of RICCATI EQ: $\dot{K} = KSK - A^T K - K^T A - M$

$$\begin{cases} \dot{K}^T = K^T S K^T - K^T A - A^T K - M \\ K^T(T) = Q \end{cases}$$

$\Rightarrow K^T$ solves the same ODE and term. cond. as K!

$\Rightarrow$ by UNIQ. of ODE we get $K = K^T\ \forall t. \Rightarrow K \in Sym(n)$

∎

<u>Recall.</u> $\dot{y} = y^2 \quad y(0) > 0$ blows up in finite time!

Thm: (global $\exists$) Supp. $R>0$, $Q, M \geq 0$ (pos. semidef.)

$\Rightarrow$ (RT) has a sol $K \in C^1((-\infty, T), Sym(n))$ cont. up to $t=T$.

Pf [Fl.-Rish.; my notes] Know that $K:]t_0, T] \to Sym(n)$

& if $t_0 > -\infty$ $\lim_{t \to t_0+} \|K(t)\| = +\infty$ by the properties of the maximal interval of $\exists$ of sol. of ODE.

Use "trace norm" of $X \in Sym(n)$:

$$\|X\| := \sum_{i=1}^{n} |\lambda_i| \quad , \lambda_i \text{ eigenvalues of } X.$$

Ex HW: $\|\cdot\|$ is a norm on $Sym(n)$.

Notation $\|X\|_2$ = euclidean norm.

Goal: $\forall \tau < T$ $\exists \bar{C}_\tau$: $\|K(t)\| \leq \bar{C}_\tau$ $\forall \tau < t \leq T$.

(Verif. thm)

Know that $w(x,t) = x^T K(t) x = \inf_{\alpha(\cdot)} J(x, t, \alpha(\cdot))$

$$J = \int_t^T (\underbrace{y^T M y}_{\geq 0} + \underbrace{\alpha^T R \alpha}_{\geq 0})\, ds + \underbrace{y(T)^T Q\, y(T)}_{\geq 0} \geq 0$$

$\Rightarrow$ $w(x,t) \geq 0$ $\forall (x,t)$ $\Rightarrow$ $K(t) \geq 0$ semidef. $\forall t$

$\Rightarrow$ $\lambda_{min}(K(t)) \geq 0$ $\forall t$

Estimate from above of $\|K(t)\|$, use $\alpha \equiv 0$

$$\begin{cases} \dot{y} = Ay \\ y(t) = x \end{cases}$$ the sol $y_x^0(s) = y_x(s; t, 0)$ sats.

$y_0^0(s) \equiv 0$ $\qquad L = \|A\|_2$

$$|y_x^0(s) - 0| \le e^{L(s-t)}|x-0| \le e^{L(T-\tau)}|x| \quad \forall \tau \le t \le s \le T.$$

$$\Rightarrow \boxed{|y_x^0(s)| \le C_\tau |x|} \qquad \text{For } t \in [\tau, T]$$

$$\boxed{x^T K(t)\, x \le J(x,t,0)} = \int_t^T y_x^0(s)^T M\, y_x^0(s)\, ds + y_x^0(T)^T Q\, y_x^0(T)$$

$$\le (T-t)\, \|M\|_2 \sup_{t\le s\le T} |y_x^0(s)|^2 + \|Q\|_2 |y_x^0(T)|^2$$

$$\le C_\tau^2 |x|^2 \left((T-\tau)\|M\|_2 + \|Q\|_2 \right) =: \boxed{\tilde{C}_\tau |x|^2} \quad \forall x$$

$$\Rightarrow K(t) - \tilde{C}_\tau Id_n \le 0 \Rightarrow d_{max} K(t) \le \tilde{C}_\tau$$

$$\Rightarrow \|K(t)\| \le n \max_i |d_i| = n \max_i d_i =$$

$$= n\, d_{max}(K(t)) \le n\, \tilde{C}_\tau := \bar{C}_\tau . \qquad \blacksquare$$

Rmk Conds. of Thm are not necessary: see HW at the end of the lecture: example where the sol. of (RT) $\exists$ in $[0,T]$ even if $M \not\ge 0$.

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How to solve explicitly systems of Riccati ODEs.

1. By reduction to LINEAR ODES.

Thm. If $U, V \in C^1([t_0,T], M_{n\times n})$ sol. of

$$\begin{cases} \dot{U} = AU - SV \\ \dot{V} = -MU - A^T V \\ U(T) = I, \quad V(T) = Q \end{cases} \qquad \text{linear homog syst.}$$

where A, M, Q, $S = BR^{-1}B^T$ are known. Then t:

det $V(t) \neq 0$, $K(t) = V(t)v^{-1}(t)$ is a sol. of (RT)
(Riccati terminal value pb.).

Pf. No: HW at least for $n=1$. $\square$

2. Other method for $n=1$ all data are scalar.

$$\begin{cases} \dot{K} = SK^2 - 2AK - M \\ K(T) = Q \end{cases}$$

$$\int \frac{dK}{SK^2 - 2AK - M} = \int 1\,dt = t + \text{const}.$$

can be solved explicitly! $\square$.

Next time: results & comments on the Ex. n. 1 on H-L formula:

$$u_t + \frac{(u_x)^2}{2} = 0, \qquad (a) \quad g(x) = -x^-$$

(b) $g(x) = x^+$ draw graph of sol. u.

(c) $v = u_x$ $\quad v_t + \left(\frac{v^2}{2}\right)_x = 0$ draw graph of v.

Try to do it yourself!

INTRODUCTION TO GAME THEORY.

Refs:
- A. Bressan on Milan J. Math.
- E.N. Barron. Game theory.
- L.C. Evans & P. Souganidis: Indiana U. Math. J.

• [BCD] chp. VIII.

PLAN: ▶ • 2-person 0-sum } "STATIC" or ONE SHOT GAMES.
• " non 0 sum.
(N ≥ 2 players is similar).

▶ DIFFERENTIAL games. → ≠ sum.
→ 0-sum.
→ "$N \to \infty$" MEAN-FIELD GAMES

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ZERO-SUM GAMES

DATA are: two sets A & B

A = decisions or strategies of pl. 1

B = " " " " " 2.

$\Phi : A \times B \to \mathbb{R}$ $\Phi(a,b)$ = payoff of pl. 1
= cost of pl. 2.

Goals: 1st player wants to MAXIMISE Φ
2nd " " " MINIMISE Φ.

Examples. Most simple: $A = \{1, \dots, m\}$
$B = \{1, \dots, n\}$ MATRIX GAMES. $\Phi(i,j) = \phi_{ij}$

$$M = \begin{pmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & & \\ \phi_{m1} & \cdots & \phi_{mn} \end{pmatrix}$$

A chooses row
B " column.

Ex. 1 "Head or tail". Each player chooses H or T
if choices are = A wins, else B wins.

$$\begin{array}{c|cc} & T & H \\ \hline T & 1 & -1 \\ H & -1 & 1 \end{array}$$

<u>Ex. 2</u>. "splitting a cake" 2 children must split the cake.

A cuts, B chooses

$$\begin{array}{ccc|cc} & & & \multicolumn{2}{c}{B} \\ & & & \text{small} & \text{big} \\ \hline A & \tfrac{1}{2} \text{ as exact as possible} & & \tfrac{1}{2}+\varepsilon & \tfrac{1}{2}-\varepsilon \\ & \text{big piece \& small piece} & & 3/4 & 1/4 \end{array}$$

Hypothesis I : $\begin{cases} A, B \text{ compact metric spaces} \\ \Phi \in C(A \times B) \end{cases}$

<u>Def</u>. MARGINAL FUNCTIONS.

$$\Phi^{max}(b) = \max_a \Phi(a,b) \quad , \quad \Phi^{min}(a) = \min_b \Phi(a,b)$$

BEST RESPONSE MAPS :

$$R^A(b) := \operatorname*{argmax}_a \Phi(a,b) = \{ \bar{a} \in A : \Phi(\bar{a}, b) = \max_a \Phi(a,b) \}$$

$$R^B(a) := \operatorname*{argmin}_b \Phi(a,b) .$$

<u>Lemma</u> $\Phi^{max} \in C(B)$, $\Phi^{min} \in C(A)$.

<u>Pf</u>. $b, \bar{b} \in B$ choose $\bar{a}$: $\Phi^{max}(\bar{b}) = \Phi(\bar{a}, \bar{b})$

$$\Phi^{max}(\bar{b}) - \Phi^{max}(b) \le \Phi(\bar{a}, \bar{b}) - \Phi(\bar{a}, b) \le \omega_\Phi(\operatorname{dist}_B(\bar{b}, b))$$

$\longrightarrow 0 \quad$ as $b \to \bar{b}$

Can invert the roles of b & $\bar{b}$ &

$$\Phi^{max}(b) - \Phi^{max}(\bar{b}) \longrightarrow 0 \quad \text{as } b \to \bar{b} \quad . \quad \blacksquare$$

<u>Def</u>: <u>Upper</u> value of game $= V^+ := \min_{b \in B} \Phi^{max}(b) =$

$$= \min_b \max_a \Phi(a, b).$$

<u>Lower</u> value $= V^- := \max_{a \in A} \Phi^{min}(a) = \max_a \min_b \Phi(a, b)$

<u>Prop.</u> $V^- \leq V^+$

<u>Pf</u> $\forall b' \in B \quad \Phi(a, b') \geq \min_b \Phi(a, b) \qquad \forall a$

$\Rightarrow \quad \max_a \Phi(a, b') \geq \max_a \min_b \Phi(a, b) = V^- \qquad \forall b'$

$\Rightarrow \quad \min_{b'} \max_a \Phi(a, b') \geq V^-$

$\quad \parallel$

$\quad V^+ \qquad \blacksquare$.

<u>Def</u>: If $V^+ = V^-$ then $V = V^+ = V^-$ is THE VALUE of the GAME.

<u>HW</u>. ADDED AFTER THE LECTURE.

Example of LQ control. $n = 1$, $m = 1$

$y(t) =$ invested capital of a firm

$\alpha(t) =$ investment at time t

Dynamics: $\dot{y}(t) = -a\, y(t) + \alpha(t)$, $a > 0$ given.

$0 < p$ = price of the product/unit of capital

Problem: $\max\limits_{\alpha(\cdot)} \int_0^T \left(p y^2(s) - r\, \alpha^2(s) \right) ds$, $r > 0$ given.

i.e. $\min\limits_{\alpha(\cdot)} \rightsquigarrow J(x,t,\alpha) = \int_t^T \left(r\, \alpha^2(s) - p\, y^2(s) \right) ds$.

N.B.: In the previous notations $M = -p < 0$, so it is NOT known if $\exists$ sol. of (RT) in $[0,T]$, and not just locally in $(t_0, T]$.

Q: Find values of the parameters s.t. the sol $K(t)$ of (RT) is defined in $[0, T]$.

(see [Engwerda, p. 184-5]).