LECTURE 13, 4.10.23

$$=: J(x,t,\alpha(\cdot))$$

Recall: $v(x,t) := \inf_{\alpha\in\mathcal{A}} \Big\{ \overbrace{\int_t^T \ell(y(s),\alpha(s))\,ds + g(y(T))}^{=:J(x,t,\alpha(\cdot))} \Big\}$ value function

$$(S)\begin{cases} \dot{y} = f(y,\alpha), & s>t \\ y(t) = x \end{cases} \qquad \alpha : [0,T] \to A \text{ measurable}$$

$$(H) \quad \boxed{H(p,x) := \max_{a\in A} \{ -f(x,a)\cdot p - \ell(x,a) \}.}$$

Theorem. Under the standing assumptions, v is the UNIQUE viscosity solution in $BUC(\bar{\Omega})$, $\Omega = \mathbb{R}^n \times]0,T[$ of the TERMINAL VALUE PROBLEM

$$(CT)\begin{cases} -u_t + H(D_x u, x) = 0 & \text{in } \Omega \\ u(x,T) = g(x) & \forall x\in\mathbb{R}^n \end{cases}$$

Proof. : PART 1 : v solves (CT) : DONE.

PART 2 : UNIQUENESS. : $w(x,t) := u(x, T-t)$

$\Rightarrow \quad w(x,0) = u(x,T) = g(x), \quad w_t = -u_t, \quad D_x w = D_x u$

"Then" w solves

$$(CI)\begin{cases} w_t + H(D_x w, x) = 0 & \text{in } \Omega \\ w(x,0) = g(x). \end{cases}$$

HW : check that w solves (CI) in visc. sense.

Same HW shows that any other sol. of (CT) $\to$ a soln. of (CI) by $t \mapsto T-t$.

$\Rightarrow$ UNIQUENESS for (CT) is EQUIVALENT to uniqueness for (CI): Can apply a known COMPARISON THM.:

I'll use Comp. Princ. #2. Ass. are (RH) & (Lip H) for (H). Recall ass. on data: A compact

$\boxed{|f|\le M}$, $|f(x,a)-f(z,a)|\le L|x-z| \quad \forall x,z,a$

$|\ell|\le C_1$, $|\ell(x,a)-\ell(z,a)|\le C_1|x-z|$

Goal 1: check (RH): $\exists$ ω modulus:

$$|H(p,x)-H(p,y)|\le \omega(|x-y|(1+|p|)).$$

$$H(p,x)-H(p,y)\underbrace{\le -f(x,\bar a)\cdot p-\ell(x,\bar a)+f(y,\bar a)\cdot p+\ell(y,\bar a)}_{\exists\,\bar a\,:\,H(p,x)=-f(x,\bar a)\cdot p-\ell(x,\bar a)}\le$$

$$\le |p|\,|f(x,\bar a)-f(y,\bar a)|+C_1|x-y|\le |p|L|x-y|+C_1|x-y|$$

$$\le \underbrace{\max\{L,C_1\}}_{L'}|x-y|(1+|p|)$$ choose $\omega(r)=L'r$ $\Rightarrow$

$$H(p,x)-H(p,y)\le \omega(|x-y|(1+|p|))$$ exchange roles of x & y

$\Rightarrow |\quad\quad| \le$ ✓. ∎ Goal 1

Goal 2: (Lip H): $\exists C\in\mathbb{R}$: $|H(p,x)-H(q,x)|\le C|p-q|$

$$H(p,x)-H(q,x)\le -f(x,\bar a)\cdot p-\cancel{\ell(x,\bar a)}+f(x,\bar a)\cdot q+\cancel{\ell(x,\bar a)}$$

same $\bar a$ as above

$$\le |f(x,\bar a)|\,|p-q|\le M|p-q|$$ Exchange x & y

$\Rightarrow \quad |H(p,x) - H(q,x)| \le M|p-q|$. ∎ Cond 2.

Conclusion: Comparison for (CI) $\Rightarrow$ UNIQUENESS for (CT) in $BUC(\bar{\Omega})$. ∎

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VERIFICATION THM. & SYNTHESIS of OPTIMAL FEEDBACK control.

So far
$$\begin{cases} \dot{y} = f(y, \alpha) \\ y(t) = x \end{cases}$$
$\alpha : [0,T] \to A$ meas.le

OPEN LOOP.

CLOSED LOOP or FEEDBACK controls:

<u>Def.</u> $\Phi : \mathbb{R}^n \times [0,T] \to A$ s.t.

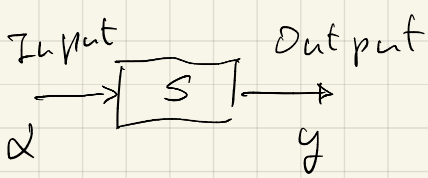
$$\begin{cases} \dot{y}(s) = f(y(s), \Phi(y(s), s)) & s > t \\ y(t) = x \end{cases}$$
has a UNIQUE sol. $\forall\, t, x$ & s.t.

$s \mapsto \Phi(y(s), s) =: \alpha_{\Phi}(s) \in \mathcal{Q}$ is measurable, is called an ADMISSIBLE FEEDBACK control.

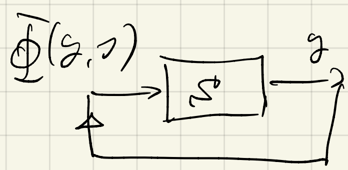
<u>Note</u> 1. we can compute $J(x, t, \alpha_{\Phi}(\cdot))$

2. If $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ is Lip in (x, a), then any $\Phi : \mathbb{R}^n \times [0,T] \to A$ Lip. is admissible.

3.



OPEN LOOP



CLOSED LOOP

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VERIFICATION THM. for OPT. CONTROL

here we don't need A compact, nor the usual ass. on f, g, ℓ : it's enough they are cont. & $\forall \alpha \in \mathcal{A}$ $\exists !$ traj. of (S) $y_x(\cdot\,; t, \alpha)$.

Pre-Hamiltonian : $\mathcal{H} := f(x,a)\cdot p + \ell(x,a)$

$H(p,x) = -\inf\limits_a \mathcal{H}(p,x,a) \qquad \mathcal{H} \geq -H \quad \forall a$.

<u>Thm.</u> Let $w \in C^1$ sol. (for $t_0 \geq 0$) of

$$(CT) \quad \begin{cases} w_t + \inf\limits_a \mathcal{H}(D_x w, x, a) = 0 & \text{in } \mathbb{R}^n \times]t_0, T[\\ w(T,x) = g(x). \end{cases}$$

Then $\forall x \in \mathbb{R}^n \quad \forall \; t_0 < t < T$:

(i) $\forall \alpha \in \mathcal{A}$, $y(s) = y_x(s; t, \alpha)$

$s \mapsto w(y(s), s) + \int_t^s \ell(y(\tau), \alpha(\tau))\, d\tau$ is INCREASING

(ii) $w(x,t) \leq v(x,t) = \inf\limits_{\alpha \in \mathcal{A}} J(x,t,\alpha)$

(iii) if α^* with traj y^* is s.t. $\forall \; s \in [t_0, T]$

$$\mathcal{H}(D_x w(y^*(s), s), y^*(s), \alpha^*(s)) \overset{(HJP)}{=} \inf_{a \in A} \mathcal{H}(D_x w(y^*(s), s), y^*(s), a)$$

$$= -H(D_x w(y^*(s), s), y^*(s))$$

$\Rightarrow$ α^* is OPTIMAL, i.e., $J(x, t, \alpha^*(\cdot)) = v(x,t)$

& $w(x,t) = v(x,t)$.

Pf. Just for $\ell \equiv 0$ (HW: general case).

(i) $\varphi(s) = w(y(s), s)$. Goal $\dot{\varphi} \geq 0$ a.e. s

$$\boxed{\dot{\varphi}(s) =} \; \underbrace{w_t(y(s), s) + D_x w(y(s), s) \cdot \underset{\substack{\parallel \\ f(y(s), \alpha(s))}}{\dot{y}(s)}}_{\mathcal{H}(D_x w(y(s),s), y(s), \alpha(s))} \quad \text{a.e. } s.$$

$$\overset{(\mathrm{In})}{\boxed{\geq}} \; w_t(y(s), s) - H(D_x w(y(s), s), y(s)) \boxed{= 0} \quad \forall \alpha$$

$=$ if (Hyp) holds.

∎ (i)

(ii) Goal: $w(x,t) \leq g(y_x(T; t, \alpha)) \quad \forall \alpha \in \mathcal{Q}$.

$$\text{(i)} \Rightarrow \quad w(x,t) \leq w(y(T), T) = g(y(T)) \quad \forall \alpha$$

↑ by terminal cond. on w ∎ (ii)

(iii) If (Hyp) holds $\Rightarrow$ $\dot{\varphi}(s) = 0$ a.e. s.

$\exists \alpha^*, \dots y^*$ $\qquad \overset{\parallel}{\frac{d}{ds}} w(y^*(s), s)$

$$\Rightarrow \quad w(x,t) = w(y^*(T), T) = g(y^*(T))$$

$$\Rightarrow \quad w(x,t) = v(x,t) = J(x,t,\alpha^*) \quad \text{\& } \alpha^* \text{ is OPTIMAL.}$$

∎

SYNTHESIS of OPTIMAL FEEDBACKS.

(a) Find a C^1 sol. of w of (CT) in $\Omega = \mathbb{R}^n \times]t_0, T[$.

(b) Look for a feedback $\Phi : \bar{\Omega} \to A$:

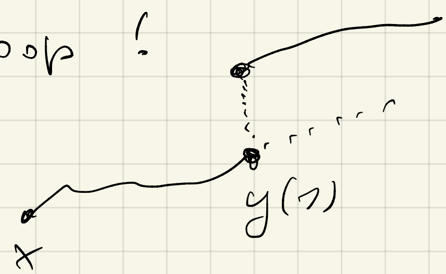
$$\Phi(y,s) \in \arg\min_a \mathcal{H}(D_x w(y,s), y, a) \quad \text{\& ADMISSIBLE.}$$

(c) then, by Verif. thm., if $y^*(s)$ solves

$$\begin{cases} \dot{y}^*(s) = f(y^*(s), \underbrace{\Phi(y^*(s), s)}_{\alpha^*}) & s > t \\ y^*(t) = x \end{cases}$$

then α^*, y^* is OPTIMAL!

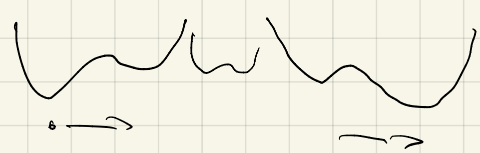
Rmk FEEDBACK controls have BETTER properties than open loop! they are ROBUST



BAD NEWS

In general $\nexists\, w \in C^1$ solving (CT).

& argmin $\notin C$



We will see! LINEAR-QUADRATIC problems are s.t. $\exists\, w \in C^1$ sol of (CT).

We will not see! If v is visco sol. of (CT), not C^1, one can approximate (CT) & find approximate feedback ε-optimal (see e.g. chap. VI of [BCD]).

LINEAR - QUADRATIC REGULATOR

Ref • Fleming - Rishel

• J. ENGWERDA : L-Q dyn. optim. & diff. games 2005.

The system is Linear: $f(x,a) = Ax + Ba$

$A \in M_{n\times n}$, $a \in \mathbb{R}^m$, $B \in M_{n\times m}$

$$(S) \quad \begin{cases} \dot{y} = Ay + B\alpha & s > t \\ y(t) = x & y, x \in \mathbb{R}^n \end{cases}$$

Admissible controls: $\alpha \in L^1_{loc}([0,T], \mathbb{R}^m)$ s.t. $\exists$ unique sol. of (S), i.e. $y(s) = x + \int_t^s [Ay(\tau) + B\alpha(\tau)]\, d\tau$.

The cost functional is quadratic:

$$(JQ) \quad J(x,t,\alpha(\cdot)) := \int_t^T \left[y(s)^T M y(s) + \alpha(s)^T R\, \alpha(s) \right] ds + y(T)^T Q y(T)$$

$M, Q \in Sym(n)$ Ip. 1 : $R > 0$ posit. def MATRIX.

$R \in Sym(m)$ $\left(\Leftrightarrow \alpha^T R \alpha \geq \lambda^R_{min} |\alpha|^2 \longrightarrow +\infty \text{ as } |\alpha| \to \infty, \quad \lambda^R_{min} > 0 \right)$.

The HJB equation is: $\quad -H = \inf_a \{ f \cdot p + \ell \}$

$$w_t + D_x w \cdot Ax + x^T M x + \inf_{a \in \mathbb{R}^m} \{ \underbrace{D_x w \cdot Ba + a^T R a}_{=: \tilde{H}} \} = 0$$

$$\tilde{H}(p,a) := p \cdot Ba + a^T R a = (Ba)^T p + a^T R a = a^T B^T p + a^T R a$$

$\tilde{H} \to +\infty$ as $|a| \to \infty \quad \forall p \Rightarrow \exists \min_{a \in \mathbb{R}^m} \tilde{H}$.

Compute $D_a \tilde{\mathcal{H}}$: $D_a\, a^T B^T p$: $\dfrac{\partial}{\partial a_j} \sum_{i,k} a_i b_{ki} p_k = \sum_k b_{kj} p_k = (B^T p)_j$

$\Rightarrow D_a(a^T B^T p) = B^T p$

$$D_a \tilde{\mathcal{H}}(p,a) = B^T p + 2Ra \overset{?}{=} 0 \iff a = -\frac{R^{-1}B^T p}{2}$$

$$\Rightarrow \operatorname*{argmin}_a \tilde{\mathcal{H}}(p,a) = \left\{ -\frac{R^{-1}B^T p}{2} \right\} = \{ \hat{a} \} \,.$$

$$\min_a \tilde{\mathcal{H}}(p,a) = \hat{a}^T B^T p + \hat{a}^T R\, \hat{a} = -\frac{1}{2} p^T \underbrace{BR^{-1}B^T}_{S} p + \frac{1}{4} p^T \underbrace{BR^{-1}RR^{-1}B^T}_{S} p$$

$S = BR^{-1}B^T$

$$= -\frac{p^T S p}{2} + \frac{p^T S p}{4} = -\frac{p^T S p}{4}$$

$$(CT) \begin{cases} w_t + D_x w \cdot Ax + x^T M x - \frac{1}{4}(D_x w)^T S D_x w = 0 & \text{in } \mathbb{R}^n \times (t_0, T) \\ w(x,T) = x^T Q x \,. \end{cases}$$

Look for $w \in C^1$ sol. of (CT) in (t_0,T) for the Verif. Thm.

ANSATZ (EDUCATED GUESS) $\quad w(x,t) = x^T K(t)\, x$

$K(\cdot) \in \mathrm{Sym}(n)$, $K \in C^1$, $K(T) = Q$

$w_t = x^T \dot{K} x$, $D_x w = 2Kx$

$$x^T \dot{K} x + 2 \underbrace{x^T A^T K x}_{x^T \frac{A^T K + K^T A}{2} x} + x^T M x - x^T K S K x = 0$$

$x^T(\dot{K} + A^T K + K^T A + M - KSK)x = 0 \quad \forall x \iff$

$$(RT) \begin{cases} \dot{K} = KSK - A^T K - K^T A - M \\ K(T) = Q \end{cases}$$

RICCATI MATRIX ODE