(c) deduce from the above that u is a viscosity subsolution of $(\partial u)/(\partial x_1)(x) \le 0$, $x \in \mathbb{R}^N$, if and only if

$$\eta'(x_1) \le 0, \qquad x_1 \in \mathbb{R}$$

in the viscosity sense, for any $(x_2, \dots, x_N) \in \mathbb{R}^{N-1}$.

2.9. Consider the distance function d from a set S. Under which conditions does $u := -d$ solve $|Du| = 1$ in the viscosity sense?

2.10. Prove that ν is a (generalized) exterior normal to $\overline{S}$ at z if and only if there exists $\varepsilon > 0$ such that

$$B(z + \varepsilon\nu, \varepsilon) \cap \overline{S} = \emptyset .$$

2.11. Compute the (generalized) exterior normal vectors to S in the following cases:

(a) $S = \{0\}$;

(b) $S = \{x \in \mathbb{R}^N : x = tx_0 + (1-t)x_1,\ t \in [0,1]\}$;

(c) $S = \{x \in \mathbb{R}^N : g(x) \le 0\} \cap \{x \in \mathbb{R}^N : f(x) \le 0\}$;

(d) $S = \{x \in \mathbb{R}^N : g(x) \le 0\} \cup \{x \in \mathbb{R}^N : f(x) \le 0\}$.

Here, f and g are C^1 functions. Finally, let

$$S = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_2| \le |x_1|^{3/2}\} .$$

Show that $N(0) = \emptyset$ although the exterior normal at $x = 0$ exists (namely, $n(0) = (0,1)$). Observe that ∂S is C^1 but not C^2.

2.12. Define a proximal normal to a closed set $S \subseteq \mathbb{R}^N$ at $x \in \partial S$ as any vector $p \in \mathbb{R}^N$ such that $d(x + rp, S) = r\,|p|$, for some $r > 0$. Prove the following characterization of generalized exterior normals to S:

$$N(x) = \{\nu = p/|p| : p \text{ proximal normal to } S \text{ at } x\} .$$

3. Some comparison and uniqueness results

In this section we address the problem of comparison and uniqueness of viscosity solutions. This is a major issue in the theory, also in view of its relevance in connection with sufficient conditions in optimal control problems. The results presented here are not the most general; in fact they are selected to show the main ideas involved in the proofs with as few technicalities as possible. In Chapters III, IV and V we prove many other comparison theorems especially designed for the Hamilton-Jacobi-Bellman equations arising in optimal control theory, and we refer also to the literature quoted in the bibliographical notes for more general results.

As an introduction to the subject, suppose that $u_1, u_2 \in C(\overline{\Omega}) \cap C^1(\Omega)$, satisfy the inequalities

$$\begin{aligned} &u_1(x) + H(x, Du_1(x)) \le 0 \\ &u_2(x) + H(x, Du_2(x)) \ge 0 \end{aligned} \tag{3.1}$$

for $x \in \Omega$ and

$$u_1 \le u_2 \qquad \text{on } \partial\Omega . \tag{3.2}$$

Suppose Ω bounded and let x_0 be a maximum point for $u_1 - u_2$ on $\overline{\Omega}$. If $x_0 \in \Omega$, then $Du_1(x_0) = Du_2(x_0)$ and (3.1) gives by subtraction

$$u_1(x) - u_2(x) \le u_1(x_0) - u_2(x_0) \le 0 \qquad \forall x \in \overline{\Omega} .$$

If, on the other hand, $x_0 \in \partial\Omega$, then

$$u_1(x) - u_2(x) \le u_1(x_0) - u_2(x_0) \le 0 \qquad \forall x \in \overline{\Omega},$$

as a consequence of (3.2). Hence, $u_1 \le u_2$ in $\overline{\Omega}$. Reversing the role of u_1 and u_2 in (3.1), (3.2), we get, of course, the uniqueness of a classical solution of the Dirichlet problem

$$\begin{cases} u(x) + H(x, Du(x)) = 0, & x \in \Omega, \\ u = \varphi, & \text{on } \partial\Omega . \end{cases}$$

The preceding elementary proof fails if u_1, u_2 are continuous functions satisfying the inequalities (3.1) in the viscosity sense since Du_i may not exist at x_0. However, the information contained in the notion of viscosity sub- and supersolution is strong enough to allow the extension of comparison and uniqueness results to continuous viscosity solutions of equation (HJ) with rather general F.

In the following we present some comparison theorems between viscosity sub- and supersolutions in the cases Ω bounded, $\Omega = \mathbb{R}^N$, and for the Cauchy problem. As a simple corollary, each comparison result produces a uniqueness theorem, as indicated in the remarks. For simplicity we restrict our attention to the case $F(x, r, p) = r + H(x, p)$; the results hold, however, for a general F provided $r \mapsto F(x, r, p)$ is strictly increasing (see Exercises 3.6 and 3.7), and for some special F independent of r, see Theorem 3.7, §5.3 and Chapter IV, §2.

THEOREM 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$. Assume that $u_1, u_2 \in C(\overline{\Omega})$ are, respectively, viscosity sub- and supersolution of*

$$u(x) + H(x, Du(x)) = 0, \qquad x \in \Omega \tag{3.3}$$

and

$$u_1 \le u_2 \qquad \text{on } \partial\Omega . \tag{3.4}$$

Assume also that H satisfies Es. $H = H(p) + f(x)$, H, f continue

(H$_1$) $$|H(x,p) - H(y,p)| \le \omega_1(|x-y|(1+|p|)),$$

for $x, y \in \Omega$, $p \in \mathbb{R}^N$, where $\omega_1 : [0,+\infty[\to [0,+\infty[$ is continuous nondecreasing with $\omega_1(0) = 0$. Then $u_1 \le u_2$ in $\overline{\Omega}$.

PROOF. Define, for $\varepsilon > 0$, a continuous function Φ_ε on $\overline{\Omega} \times \overline{\Omega}$ by setting

$$\Phi_\varepsilon(x,y) = u_1(x) - u_2(y) - \frac{|x-y|^2}{2\varepsilon}$$

and let $(x_\varepsilon, y_\varepsilon)$ be a maximum point for Φ_ε on $\overline{\Omega} \times \overline{\Omega}$. Then, for any $\varepsilon > 0$

(3.5) $$\max_{x\in\overline{\Omega}}(u_1 - u_2)(x) = \max_{x\in\overline{\Omega}} \Phi_\varepsilon(x,x) \le \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \Phi_\varepsilon(x,y) = \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)\,.$$

We claim that

(3.6) $$\liminf \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \le 0 \qquad \text{as } \varepsilon \to 0\,.$$

This, together with (3.5), proves the theorem.

In order to prove (3.6), let us observe first that the inequality

$$\Phi_\varepsilon(x_\varepsilon, x_\varepsilon) \le \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)$$

amounts to

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \le u_2(x_\varepsilon) - u_2(y_\varepsilon)\,.$$

This implies

$$|x_\varepsilon - y_\varepsilon| \le (C\varepsilon)^{1/2},$$

where C depends only on the maximum of $|u_2|$ in $\overline{\Omega}$. Therefore,

(3.7) $$|x_\varepsilon - y_\varepsilon| \longrightarrow 0 \qquad \text{as } \varepsilon \to 0;$$

and, by continuity of u_2,

(3.8) $$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \longrightarrow 0 \qquad \text{as } \varepsilon \to 0\,.$$

non serve se $H = H(p) + f(x)$

Now there are two possible cases:

(i) $(x_{\varepsilon_n}, y_{\varepsilon_n}) \in \partial(\Omega \times \Omega)$ for some sequence $\varepsilon_n \to 0^+$;

(ii) $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ for all $\varepsilon \in]0, \overline{\varepsilon}[$.

In case (i) either $x_{\varepsilon_n} \in \partial\Omega$, and then, by (3.4),

$$u_1(x_{\varepsilon_n}) - u_2(y_{\varepsilon_n}) \le u_2(x_{\varepsilon_n}) - u_2(y_{\varepsilon_n}),$$

or $y_{\varepsilon_n} \in \partial\Omega$ and then

$$u_1(x_{\varepsilon_n}) - u_2(y_{\varepsilon_n}) \le u_1(x_{\varepsilon_n}) - u_1(y_{\varepsilon_n})\,.$$

Note that the right-hand sides of both these inequalities tend to 0 as $n \to \infty$ by (3.7) and the uniform continuity of u_1 and u_2. Therefore

$$\Phi_{\varepsilon_n}(x_{\varepsilon_n}, y_{\varepsilon_n}) \le u_1(x_{\varepsilon_n}) - u_2(y_{\varepsilon_n}) \longrightarrow 0 \qquad \text{as } n \to \infty,$$

and the claim (3.6) is proved in this case.

Assume now that $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ and set

$$\varphi_2(x) = u_2(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon}\,, \qquad \varphi_1(y) = u_1(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon}\,.$$

It is immediate to check that $\varphi_i \in C^1(\Omega)$ $(i = 1,2)$ and x_ε is a local maximum point for $u_1 - \varphi_2$, whereas y_ε is a local minimum point for $u_2 - \varphi_1$. Moreover,

$$D\varphi_1(y_\varepsilon) = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} = D\varphi_2(x_\varepsilon)\,.$$

By the definition of viscosity sub- and supersolution, then,

$$u_1(x_\varepsilon) + H\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big) \le 0, \qquad -u_2(y_\varepsilon) - H\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big) \le 0\,.$$

By (H$_1$), this implies

$$u_1(x_\varepsilon) - u_2(y_\varepsilon) \le \omega_1\Big(|x_\varepsilon - y_\varepsilon|\Big(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\Big)\Big)$$

and, a fortiori

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \le \omega_1\Big(|x_\varepsilon - y_\varepsilon|\Big(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\Big)\Big)\,.$$

Taking (3.7) and (3.8) into account, (3.6) follows and the proof is complete. ◀

REMARK 3.2. If u_1, u_2 are both viscosity solutions of (3.3) with $u_1 = u_2$ on $\partial\Omega$, from Theorem 3.1 it follows that $u_1 = u_2$ in $\overline{\Omega}$. ◁

REMARK 3.3. The statement of Theorem 3.1 is true also for the equation

$$\lambda u(x) + H(x, Du(x)) = 0 \qquad x \in \Omega$$

with $\lambda > 0$. On the other hand, for the equation $H(x, Du(x)) = 0$ non-uniqueness phenomena may appear. An extreme case is $H(x,p) = 0$ for all x and p: in this case any $u \in C(\Omega)$ is a viscosity solution. ◁