Some properties of the sub- and superdifferential are collected in Lemma 1.8.

LEMMA 1.8. *Let $u \in C(\Omega)$ and $x \in \Omega$. Then,*

(a) *$D^+u(x)$ and $D^-u(x)$ are closed convex (possibly empty) subsets of $\mathbb{R}^N$;*

(b) *if u is differentiable at x, then $\{Du(x)\} = D^+u(x) = D^-u(x)$;*

(c) *if for some x both $D^+u(x)$ and $D^-u(x)$ are nonempty, then*

$$D^+u(x) = D^-u(x) = \{Du(x)\};$$

(d) *the sets $A^+ = \{x \in \Omega : D^+u(x) \neq \emptyset\}$, $A^- = \{x \in \Omega : D^-u(x) \neq \emptyset\}$ are dense.*

PROOF. The convexity of $D^+u(x)$ and $D^-u(x)$ is a straightforward consequence of the definition of lim sup and lim inf.

To prove that $D^+u(x)$ is closed we take a sequence $p_n \to p$ such that $p_n \in D^+u(x)$ for all n, and assume by contradiction that

$$\lim_n \frac{u(y_n) - u(x) - p \cdot (y_n - x)}{|y_n - x|} = \alpha > 0 \tag{1.4}$$

for a sequence $y_n \to x$. For k large enough we have $|p_k - p| \leq \alpha/2$. Then, by adding and subtracting $p_k \cdot (y_n - x)/|y_n - x|$ to (1.4) we get

$$\limsup_n \frac{u(y_n) - u(x) - p_k \cdot (y_n - x)}{|y_n - x|} \geq \frac{\alpha}{2},$$

a contradiction to $p_k \in D^+u(x)$.

To proceed in the proof observe that for any $x, y \in \Omega$ and $p, q \in \mathbb{R}^N$ we have

$$(p - q) \cdot \frac{y - x}{|y - x|} = \frac{u(y) - u(x) - q \cdot (y - x)}{|y - x|} - \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|}. \tag{1.5}$$

For any $n \in \mathbb{N}$, set $y_n := x + (1/n)(p - q)$ and take $y = y_n$ in (1.5) to obtain

$$|p - q| = \frac{u(y_n) - u(x) - q \cdot (y_n - x)}{|y_n - x|} - \frac{u(y_n) - u(x) - p \cdot (y_n - x)}{|y_n - x|};$$

by definition of lim sup and lim inf this yields

(1.6)
$$|p - q| \leq \limsup_{y \to x,\, y \in \Omega} \frac{u(y) - u(x) - q \cdot (y - x)}{|y - x|} - \liminf_{y \to x,\, y \in \Omega} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|}.$$

If u is differentiable at x, then $D^+u(x) \cap D^-u(x) \neq \emptyset$ since it contains $Du(x)$. In this case $D^+u(x)$ and $D^-u(x)$ reduce to singletons as a consequence of (1.6). Conversely, if for some x one has $D^+u(x) \neq \emptyset$, $D^-u(x) \neq \emptyset$, then by (1.6) $D^+u(x) = D^-u(x)$ is a singleton. This means that u is differentiable at x with $\{Du(x)\} = D^+u(x) = D^-u(x)$.

In order to prove (d), let $\overline{x} \in \Omega$ and consider the smooth function $\varphi_\varepsilon(x) = |x - \overline{x}|^2/\varepsilon$. For any $\varepsilon > 0$, $u - \varphi_\varepsilon$ attains its maximum over $\overline{B} = \overline{B}(\overline{x}, R)$ at some point x_ε. From the inequality

$$(u - \varphi_\varepsilon)(x_\varepsilon) \geq (u - \varphi_\varepsilon)(\overline{x}) = u(\overline{x})$$

we get, for all $\varepsilon > 0$,

$$|x_\varepsilon - \overline{x}|^2 \leq 2\varepsilon \sup_{x \in \overline{B}} |u(x)|.$$

Thus x_ε is not on the boundary of $\overline{B}$ for ε small enough, and by Lemma 1.7 (a), $D\varphi_\varepsilon(x_\varepsilon) = 2(x_\varepsilon - \overline{x})/\varepsilon$ belongs to $D^+u(x_\varepsilon)$. This proves that A^+ is dense, and similar arguments show that A^- is dense too. ◀

As a direct consequence of Lemma 1.7 the following new definition of viscosity solution turns out to be equivalent to the initial one: a function $u \in C(\Omega)$ is a viscosity subsolution of (HJ) in Ω if

$$F(x, u(x), p) \leq 0 \qquad \forall x \in \Omega,\ \forall p \in D^+u(x); \tag{1.7}$$

a viscosity supersolution of (HJ) in Ω if

$$F(x, u(x), p) \geq 0 \qquad \forall x \in \Omega,\ \forall p \in D^-u(x). \tag{1.8}$$

Of course, u will be called a viscosity solution of (HJ) in Ω if (1.7) and (1.8) hold simultaneously.

The above definition, which is more in the spirit of nonsmooth analysis, is sometimes easier to handle than the previous one. We employ it in the proofs of some important properties of viscosity solutions.

As a first example we present a consistency result that improves Proposition 1.3.

PROPOSITION 1.9. (a) *If $u \in C(\Omega)$ is a viscosity solution of* (HJ), *then*

$$F(x, u(x), Du(x)) = 0$$

at any point $x \in \Omega$ where u is differentiable;

(b) *if u is locally Lipschitz continuous and it is a viscosity solution of* (HJ), *then*

$$F(x, u(x), Du(x)) = 0 \quad \textit{almost everywhere in } \Omega.$$

PROOF. If x is a point of differentiability for u then by Lemma 1.8 (b) $\{Du(x)\} = D^+u(x) = D^-u(x)$. Hence, by definitions (1.7), (1.8)

$$0 \geq F(x, u(x), Du(x)) \geq 0,$$

which proves (a). Statement (b) follows immediately from (a) and the Rademacher's theorem on the almost everywhere differentiability of Lipschitz continuous functions (see [EG92]). ◀

REMARK 1.10. Part (b) of Proposition 1.9 says that any viscosity solution of (HJ) is also a *generalized solution*, i.e., a locally Lipschitz continuous function u such that

$$F(x,u(x),Du(x))=0 \qquad \text{a.e. in } \Omega .$$

The converse is false in general: there are many generalized solutions which are not viscosity solutions. As an example, observe that $u(x)=|x|$ satisfies

$$|u'(x)|-1=0 \qquad \text{in} \quad]-1,1[\setminus\{0\},$$

but it is not a viscosity supersolution of the same equation in $]-1,1[$ (see Example 1.6 or, alternatively, just observe that $p=0$ belongs to $D^-u(0)$ and (1.8) is not satisfied at $x=0$). In Remark 2.3 we define infinitely many generalized solutions of this equation which are not viscosity solutions. We shall come back to this point in §5. ◁

Exercises

1.1. Check that

$$u(x)=\begin{cases} x, & 0<x\le 1/2\\ 1-x, & 1/2<x<1\end{cases}$$

is a viscosity solution of $|u'(x)|-1=0$, $x\in(0,1)$. Is u a viscosity solution of equation $-|u'(x)|+1=0$ in $]0,1[$?

1.2. Let

$$u(x)=\begin{cases} 0, & x\le 0\\ \frac{1}{2}bx^2+ax, & x>0\,.\end{cases}$$

Compute $D^+u(0)$.

1.3. If $u:\mathbb{R}^N\to\mathbb{R}$ is convex (i.e., $u(\lambda x+(1-\lambda)y)\le\lambda u(x)+(1-\lambda)u(y)$, for any x, y in $\mathbb{R}^N$, $\lambda\in[0,1]$), then its subdifferential at x in the sense of convex analysis is the set

$$\partial_c u(x):=\{\,p\in\mathbb{R}^N : u(y)\ge u(x)+p\cdot(y-x),\forall y\in\mathbb{R}^N\,\}\,.$$

Show that if u is convex then $\partial_c u(x)=D^-u(x)$ at any x.

1.4. Let $u\in C(\Omega)$. Prove that $D^-u(x_0)\ne\emptyset$ if $x_0\in\Omega$ is a local minimum of u and that $D^+u(x_0)\ne\emptyset$ if x_0 is a local maximum.

1.5. Show that $D^+u(0)=D^-u(0)=\emptyset$ where u is given by

$$u(x)=|x|^{1/2}\sin 1/x^2, \qquad x\ne 0, \qquad u(0)=0$$

while $D^+v(0)=\emptyset$, $D^-v(0)=\{\,0\,\}$ for

$$v(x)=|x\sin 1/x|, \qquad x\ne 0, \qquad v(0)=0\,.$$

1.6. Let $u\in C([a,b])$. Prove the mean value property: there exists $\xi\in(a,b)$ such that

$$u(b)-u(a)=p(b-a)$$

for some $p\in D^-u(\xi)\cup D^+u(\xi)$.

1.7. Check that both $u_1(t,x)\equiv 0$ and $u_2(t,x)=(t-|x|)^+$ are viscosity supersolutions of

$$\begin{aligned} u_t-|u'(x)|&=0 \qquad &&\text{in } [0,+\infty[\,\times\mathbb{R}\\ u(0,x)&=0, &&x\in\mathbb{R}.\end{aligned}$$

Is u_2 a subsolution? [Hint: look at $D^+u_2(1,0)$.]

1.8. Consider for $x\in\mathbb{R}$ the function $u(x):=Ce^{-E|x|}$ with $C,E>0$. Check that

$$D^-u(x)=\begin{cases} -CEe^{-E|x|}x/|x| & \text{if } x\ne 0\\ \emptyset & \text{if } x=0\,.\end{cases}$$

1.9. Let $F(x,r,p):=r+H(p)-\ell(x)$ with H,ℓ such that, for some constants L,A,B,

$$|H(p)|\le L\,|p|, \qquad |\ell(x)|\le Ae^{-B|x|}, \qquad \forall x,p\in\mathbb{R}^N\,.$$

Show that if $E<\min\{\,B;1/L\,\}$ and $C=A/(1-LE)$, then u given in Exercise 1.8 is a viscosity supersolution of

$$u(x)+H(Du(x))-\ell(x)=0, \qquad x\in\mathbb{R}^N\,.$$

1.10. Assume that $u_n\in C(\Omega)$, $u_n\to u$ as $n\to+\infty$ locally uniformly in Ω. Show that for any $x_0\in\Omega$ the following holds

$$D^+u(x_0)\subseteq\limsup_{\substack{n\to+\infty\\ x\to x_0}} D^+u_n(x)$$

(i.e., for any $p\in D^+u(x_0)$ there exist $x_n\in\Omega$, $p_n\in D^+u_n(x_n)$ such that $x_n\to x_0$, $p_n\to p$ as $n\to+\infty$).

1.11. Show that $u\in C(\Omega)$ is a viscosity subsolution of (HJ) if and only if the following holds: for all $\varphi\in C_0^1(\Omega)$, $\varphi\ge 0$ and $k\in\mathbb{R}$, if $\max_\Omega\varphi(u-k)>0$ then there exists $x_0\in\Omega$ satisfying $\varphi(u-k)(x_0)=\max_\Omega\varphi(u-k)$ and

$$F\Big(x_0,u(x_0),-\frac{D\varphi(x_0)}{\varphi(x_0)}(u(x_0)-k)\Big)\le 0\,.$$

Also find the corresponding property of supersolutions.

1.12. Let u be a Lipschitz continuous function in Ω with Lipschitz constant L. Prove that $D^+u(x)$ and $D^-u(x)$ are contained in $\overline{B}(0,L)$ for all $x\in\Omega$, and that u is a (viscosity) subsolution of $|Du|-L=0$ and supersolution of $-|Du|+L=0$ in Ω.

2. Some calculus and further properties of viscosity solutions

In the first part of this section we collect some important stability properties of viscosity solutions and basic rules of calculus (change of unknown in (HJ), chain rule, ...). In the second part we establish some useful formulas for the semidifferentials of continuous functions of the form $u(x) = \inf_{b \in B} g(x, b)$. This is an important class of nonsmooth functions, sometimes called *marginal functions* which includes the *distance function* and is closely related to Hamilton-Jacobi equations.

The first result is on the stability with respect to the lattice operations in $C(\Omega)$:

$$(u \vee v)(x) = \max\{ u(x), v(x) \}$$
$$(u \wedge v)(x) = \min\{ u(x), v(x) \} .$$

PROPOSITION 2.1.

(a) *Let $u, v \in C(\Omega)$ be viscosity subsolutions of* (HJ)*; then $u \vee v$ is a viscosity subsolution of* (HJ).

(b) *Let $u, v \in C(\Omega)$ be viscosity supersolutions of* (HJ)*; then $u \wedge v$ is a viscosity supersolution of* (HJ).

(c) *Let $u \in C(\Omega)$ be a viscosity subsolution of* (HJ) *such that $u \geq v$ for any viscosity subsolution $v \in C(\Omega)$ of* (HJ)*; then u is a viscosity supersolution and therefore a viscosity solution of* (HJ).

PROOF. Let x_0 be a local maximum for $u \vee v - \varphi$ with $\varphi \in C^1(\Omega)$ and assume without loss of generality that $(u \vee v)(x_0) = u(x_0)$. Then x_0 is a local maximum for $u - \varphi$; so

$$F(x_0, u(x_0), D\varphi(x_0)) \leq 0$$

and (a) is proved. An analogous argument can be used for (b).

In order to prove (c), let us suppose by contradiction that

$$h := F(x_0, u(x_0), D\varphi(x_0)) < 0$$

for some $\varphi \in C^1(\Omega)$ and $x_0 \in \Omega$ such that

$$u(x_0) - \varphi(x_0) \leq u(x) - \varphi(x) \qquad \forall x \in \overline{B}(x_0, \delta_0) \subseteq \Omega,$$

for some $\delta_0 > 0$. Consider next the function $w \in C^1(\Omega)$ defined by

$$w(x) = \varphi(x) - |x - x_0|^2 + u(x_0) - \varphi(x_0) + \frac{1}{2}\delta^2$$

for $0 < \delta < \delta_0$. It is immediate to check that

$$(u - w)(x_0) < (u - w)(x), \qquad \forall x \text{ such that } |x - x_0| = \delta . \tag{2.1}$$

Let us prove now that, for sufficiently small δ,

$$F(x, w(x), Dw(x)) \leq 0 \qquad \forall x \in B(x_0, \delta) . \tag{2.2}$$

For this purpose, a local uniform continuity argument shows that, for $0 < \delta < \delta_0$,

$$\begin{cases} |\varphi(x) - \varphi(x_0)| \leq \omega_1(\delta), \\ |D\varphi(x) - 2(x - x_0) - D\varphi(x_0)| \leq \omega_2(\delta) + 2\delta, \end{cases} \tag{2.3}$$

for any $x \in \overline{B}(x_0, \delta)$, where ω_i $(i = 1, 2)$ are the moduli of continuity of φ and $D\varphi$. Hence

$$|w(x) - u(x_0)| \leq \omega_1(\delta) + \delta^2 \qquad \forall x \in \overline{B}(x_0, \delta) .$$

Now,

$$\begin{aligned} F(x, w(x), Dw(x)) & \\ &= h + F(x, w(x), D\varphi(x) - 2(x - x_0)) - F(x_0, u(x_0), D\varphi(x_0)) . \end{aligned} \tag{2.4}$$

If ω is a modulus of continuity for F, then

$$F(x, w(x), Dw(x)) \leq h + \omega(\delta, \omega_1(\delta) + \delta^2, \omega_2(\delta) + 2\delta),$$

for all $x \in \overline{B}(x_0, \delta)$. Since $h < 0$, the preceding proves the validity of (2.2) for small enough $\delta > 0$. Fix any such δ and set

$$\widehat{v}(x) = \begin{cases} u \vee w & \text{on } B(x_0, \delta) \\ u & \text{on } \Omega \setminus B(x_0, \delta) . \end{cases}$$

It is easy to check that $\widehat{v} \in C(\Omega)$ (see (2.1)) and, by Propositions 1.3 (a) and 2.1 (a), $\widehat{v}$ is a subsolution of (HJ) in Ω. Since $\widehat{v}(x_0) > u(x_0)$, the statement is proved. ◀

The next result is a stability result in the uniform topology of $C(\Omega)$.

PROPOSITION 2.2. *Let $u_n \in C(\Omega)$ $(n \in \mathbb{N})$ be a viscosity solution of*

$$F_n(x, u_n(x), Du_n(x)) = 0 \qquad \text{in } \Omega . \tag{HJ$_n$}$$

Assume that

$$\begin{aligned} u_n &\longrightarrow u && \text{locally uniformly in } \Omega \\ F_n &\longrightarrow F && \text{locally uniformly in } \Omega \times \mathbb{R} \times \mathbb{R}^N . \end{aligned}$$

Then u is a viscosity solution of (HJ) *in Ω.*

PROOF. Let $\varphi \in C^1(\Omega)$ and x_0 be a local maximum point of $u - \varphi$. As observed before, it is not restrictive to assume that

$$u(x_0) - \varphi(x_0) > u(x) - \varphi(x)$$

for $x \neq x_0$ in a neighborhood of x_0. By uniform convergence, $u_n - \varphi$ attains, for large n, a local maximum at a point x_n close to x_0 (see Lemma 2.4). Then,

$$F_n(x_n, u_n(x_n), D\varphi(x_n)) \leq 0 .$$

Since $x_n \to x_0$, passing to the limit as $n \to +\infty$ in the above yields

$$F(x_0, u(x_0), D\varphi(x_0)) \leq 0 .$$

A similar argument proves that u is also a viscosity supersolution. ◀

REMARK 2.3. Proposition 2.2 does not hold for generalized solutions of $(\text{HJ})_n$. As an example, consider the *saw-tooth* like functions u_n defined by $u_1(x) = 1 - x$ and for $n \geq 2$ by

$$u_n(x) = \begin{cases} x - \dfrac{2j}{2^n} & x \in]2j/2^n, (2j+1)/2^n[\\ \dfrac{2j+2}{2^n} - x & x \in](2j+1)/2^n, (2j+2)/2^n[\end{cases} \qquad j = 0, 1, \dots, 2^{n-1} - 1$$

for $x \in]0,1[$. It is evident that $|u_n'(x)| - 1 = 0$ almost everywhere in $]0,1[$, but although u_1 is a classical (and therefore a viscosity) solution, u_n is not a viscosity solution for $n \geq 2$. The uniform limit of the sequence $\{u_n\}$ is identically zero and does not satisfy the equation at any point. ◁

In the proof of Proposition 2.2 we used the following elementary fact which is useful in many situations.

LEMMA 2.4. *Let $v \in C(\Omega)$ and suppose that $x_0 \in \Omega$ is a strict maximum point for v in $\overline{B}(x_0, \delta) \subseteq \Omega$. If $v_n \in C(\Omega)$ converges locally uniformly to v in Ω, then there exists a sequence $\{x_n\}$ such that*

$$x_n \longrightarrow x_0, \qquad v_n(x_n) \geq v_n(x) \quad \forall x \in \overline{B}(x_0, \delta) . \tag{2.5}$$

PROOF. Let x_n be a maximum point for v_n on $\overline{B}(x_0, \delta)$ and let $\{x_{n_k}\}$, $k \in \mathbb{N}$, be any converging subsequence of $\{x_n\}$, $n \in \mathbb{N}$. By uniform convergence,

$$v_{n_k}(x_{n_k}) \longrightarrow v(\overline{x}) \qquad \text{as } k \to +\infty,$$

where $\overline{x} = \lim x_{n_k}$ as $k \to +\infty$. The choice of $\{x_n\}$ yields

$$v(\overline{x}) \geq v(x) \qquad \forall x \in \overline{B}(x_0, \delta),$$

so that, in particular,

$$v(\overline{x}) \geq v(x_0) .$$

This implies $\overline{x} = x_0$ and the convergence of the whole sequence. ◀

The next result is on the change of unknown in (HJ).

PROPOSITION 2.5. *Let $u \in C(\Omega)$ be a viscosity solution of* (HJ) *and $\Phi \in C^1(\mathbb{R})$ be such that $\Phi'(t) > 0$. Then $v = \Phi(u)$ is a viscosity solution of*

$$F(x, \Psi(v(x)), \Psi'(v(x))Dv(x)) = 0 \qquad x \in \Omega, \tag{2.6}$$

where $\Psi = \Phi^{-1}$.

PROOF. Since $G(x,r,p) := F(x, \Psi(r), \Psi'(r)p)$ is defined only for $r \in \Phi(\mathbb{R})$, here by viscosity solution of (2.6) we mean a function taking its values in $\Phi(\mathbb{R})$ and satisfying the properties of Definition 1.1.

Let $x \in \Omega$ and $p \in D^+v(x)$. Then

$$v(y) \leq v(x) + p \cdot (y - x) + o(|y - x|) \quad \text{as } y \to x .$$

Since Ψ is increasing,

$$\begin{aligned} \Psi(v(y)) &\leq \Psi(v(x) + p \cdot (y - x) + o(|y - x|)) \\ &= \Psi(v(x)) + \Psi'(v(x))p \cdot (y - x) + o(|y - x|) . \end{aligned}$$

By definition of v, this amounts to saying that

$$\Psi'(v(x))p \in D^+u(x) .$$

Therefore,

$$F(x, u(x), \Psi'(v(x))p) \leq 0,$$

showing that v is a viscosity subsolution of (2.6).

In a completely similar way one can prove that v is also a viscosity supersolution of (2.6). ◀

A slight generalization which is useful when dealing with evolution equations (see Exercise 2.3) is as follows.

PROPOSITION 2.6. *Let $u \in C(\Omega)$ be a viscosity solution of* (HJ) *and $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ a C^1 function such that*

$$\Phi_r(x,r) > 0 \qquad \forall (x,r) \in \Omega \times \mathbb{R} .$$

Then the function $v \in C(\Omega)$ defined implicitly by

$$\Phi(x, v(x)) = u(x),$$

is a viscosity solution of

$$\widetilde{F}(x, v(x), Dv(x)) = 0 \qquad \text{in } \Omega, \tag{2.7}$$

where

$$\widetilde{F}(x,r,p) = F(x, \Phi(x,r), D_x\Phi(x,r) + \Phi_r(x,r)p) .$$