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CHAPTER II

Continuous viscosity solutions of Hamilton-Jacobi equations

This chapter is devoted to the basic theory of continuous viscosity solutions of the Hamilton-Jacobi equation

$$\text{(HJ)} \qquad F(x, u(x), Du(x)) = 0 \qquad x \in \Omega,$$

where Ω is an open domain of $\mathbb{R}^N$ and the Hamiltonian $F = F(x, r, p)$ is a continuous real valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$.

Special attention will be dedicated in §4,5 to the case where $p \mapsto F(x, r, p)$ is convex and, more particularly, of the form

$$\text{(0.1)} \qquad F(x, r, p) = r + H(x, p) = r + \sup_{a \in A}\{ -f(x, a) \cdot p - \ell(x, a) \} .$$

Hamiltonian functions of this form arise naturally in connection with optimal control problems as indicated in Chapter I.

1. Definitions and basic properties

In this section we recall the two equivalent definitions of viscosity solutions of (HJ) introduced in Chapter I and discuss their relations with a comparison principle as well as some connections with classical notions of solutions of (HJ).

DEFINITION 1.1. A function $u \in C(\Omega)$ is a *viscosity subsolution* of (HJ) if, for any $\varphi \in C^1(\Omega)$,

$$\text{(1.1)} \qquad F(x_0, u(x_0), D\varphi(x_0)) \leq 0$$

at any local maximum point $x_0 \in \Omega$ of $u - \varphi$. Similarly, $u \in C(\Omega)$ is a *viscosity supersolution* of (HJ) if, for any $\varphi \in C^1(\Omega)$,

$$\text{(1.2)} \qquad F(x_1, u(x_1), D\varphi(x_1)) \geq 0$$

at any local minimum point $x_1 \in \Omega$ of $u - \varphi$. Finally, u is a *viscosity solution* of (HJ) if it is simultaneously a viscosity sub- and supersolution. ◁

Let us mention explicitly that the definition applies to evolutionary Hamilton-Jacobi equation of the form

$$u_t(t,y) + F(t,y,u(t,y),D_y u(t,y)) = 0, \qquad (t,y) \in]0,T[\times D .$$

Indeed, the equation above is reduced to the form (HJ) by the positions

$$x = (t,y) \in \Omega =]0,T[\times D \subseteq \mathbb{R}^{N+1}, \qquad \widetilde{F}(x,r,q) = q_{N+1} + F(x,r,q_1,\dots,q_N)$$

with

$$q = (q_1,\dots,q_N,q_{N+1}) \in \mathbb{R}^{N+1} .$$

REMARK 1.2. In the definition of subsolution we can always assume that x_0 is a local strict maximum point for $u-\varphi$ (otherwise, replace $\varphi(x)$ by $\varphi(x)+|x-x_0|^2$). Moreover, since (1.1) depends only on the value of $D\varphi$ at x_0, it is not restrictive to assume that $u(x_0)=\varphi(x_0)$. Similar remarks apply of course to the definition of supersolution. Geometrically, this means that the validity of the subsolution condition (1.1) for u is tested on smooth functions "touching from above" the graph of u at x_0.

We note also that the space $C^1(\Omega)$ of test functions in Definition 1.1 can be replaced by $C^\infty(\Omega)$, see Exercise 2.1. ◁

The following proposition explains the local character of the notion of viscosity solution and its consistency with the classical pointwise definition.

PROPOSITION 1.3. (a) *If $u \in C(\Omega)$ is a viscosity solution of* (HJ) *in Ω, then u is a viscosity solution of* (HJ) *in Ω', for any open set $\Omega' \subset \Omega$;*

(b) *if $u \in C(\Omega)$ is a classical solution of* (HJ), *that is, u is differentiable at any $x \in \Omega$ and*

$$F(x,u(x),Du(x)) = 0 \qquad \forall x \in \Omega, \tag{1.3}$$

then u is a viscosity solution of (HJ);

(c) *if $u \in C^1(\Omega)$ is a viscosity solution of* (HJ), *then u is a classical solution of* (HJ).

PROOF. (a) If x_0 is a local maximum (on Ω') for $u-\varphi$, $\varphi \in C^1(\Omega')$, then x_0 is a local maximum (on Ω) for $u-\widetilde{\varphi}$, for any $\widetilde{\varphi} \in C^1(\Omega)$ such that $\widetilde{\varphi} \equiv \varphi$ on $\overline{B}(x_0,r)$ for some $r>0$. By (1.1)

$$0 \geq F(x_0,u(x_0),D\widetilde{\varphi}(x_0)) = F(x_0,u(x_0),D\varphi(x_0))$$

showing that u is a viscosity subsolution of (HJ) on Ω'. The same argument applies to prove that u is also a supersolution on Ω'.

(b) Take any $\varphi \in C^1(\Omega)$. By the differentiability of u, at any local maximum or minimum $x \in \Omega$ of $u-\varphi$ we have $Du(x)=D\varphi(x)$. Hence (1.3) yields

$$0 = F(x_0,u(x_0),D\varphi(x_0)) \leq 0$$

if x_0 is a local maximum for $u-\varphi$ and

$$0 = F(x_1,u(x_1),D\varphi(x_1)) \geq 0$$

if x_1 is a local minimum for $u-\varphi$.

(c) If $u \in C^1(\Omega)$, then $\varphi \equiv u$ is a feasible choice in the definition of viscosity solution. With this choice, any $x \in \Omega$ is simultaneously a local maximum and minimum for $u-\varphi$. Hence, by (1.1) and (1.2),

$$F(x,u(x),Du(x)) = 0 \qquad \forall x \in \Omega . \quad ◀$$

Statement (a) says that the notion of viscosity solution is a local one. Consequently, one can take the test functions in (1.1) and (1.2) in $C^1(\mathbb{R}^N)$ or in any sufficiently small ball $B(x,r)$ centered at $x \in \Omega$.

The definition of viscosity solution is closely related to two properties that are typical in the theory of elliptic and parabolic equations, namely the *maximum principle* (MP) and the *comparison principle* (CP). For equation (HJ) these properties can be respectively formulated as follows.

DEFINITION 1.4. A function $u \in C(\Omega)$ satisfies the comparison principle with smooth strict supersolutions, briefly (CP), if for any $\varphi \in C^1(\Omega)$ and $\mathcal{O}$ open subset of Ω,

$$F(x,\varphi(x),D\varphi(x)) > 0 \quad \text{in } \mathcal{O}, \qquad u \leq \varphi \quad \text{on } \partial\mathcal{O}$$

implies $u \leq \varphi$ in $\mathcal{O}$.

We say that $u \in C(\Omega)$ satisfies the maximum principle (MP) if for any $\varphi \in C^1(\Omega)$ and $\mathcal{O}$ open subset of Ω the inequality

$$F(x,\varphi(x),D\varphi(x)) > 0 \qquad \text{in } \mathcal{O},$$

implies that $u-\varphi$ cannot have a nonnegative maximum in $\mathcal{O}$. ◁

It is quite clear that (MP) implies (CP). The connections with the notion of viscosity subsolution of (HJ) are expressed by the next result.

PROPOSITION 1.5. *If $u \in C(\Omega)$ satisfies* (CP), *then u is a viscosity subsolution of* (HJ). *Conversely, if u is a viscosity subsolution of* (HJ) *and $r \mapsto F(x,r,p)$ is nondecreasing for all x,p, then u satisfies* (MP) *and* (CP).

PROOF. Assume that $u \in C(\Omega)$ satisfies (CP). If, by contradiction, u is not a subsolution of (HJ) there exist $x_0 \in \Omega$, $\varphi \in C^1(\Omega)$, such that x_0 is a strict maximum point for $u-\varphi$, $(u-\varphi)(x_0)=0$, and

$$F(x_0,u(x_0),D\varphi(x_0)) > 0 .$$

For n large enough we have

$$a_n := \sup_{\partial B(x_0,1/n)} (u-\varphi) < 0 .$$

Observe also that

$$u - (\varphi + a_n) \le 0 \qquad \text{on } \partial B(x_0, 1/n),$$
$$u(x_0) - \varphi(x_0) - a_n > 0 .$$

By (CP) for any n there exists $x_n \in \mathcal{O}_n := B(x_0, 1/n)$ such that

$$F(x_n, \varphi(x_n) + a_n, D\varphi(x_n)) \le 0 .$$

Since $a_n \to 0$ and $x_n \to x_0$ we obtain the contradiction

$$F(x_0, u(x_0), D\varphi(x_0)) \le 0 .$$

Conversely, let u be a viscosity subsolution of (HJ) and take any $\varphi \in C^1(\Omega)$ such that

$$F(x, \varphi(x), D\varphi(x)) > 0 \qquad \text{for all } x \in \mathcal{O} .$$

If $u - \varphi$ attains a local maximum at some $x_0 \in \mathcal{O}$ with $u(x_0) - \varphi(x_0) \ge 0$, then the monotonicity assumption on F implies the contradiction

$$0 < F(x_0, \varphi(x_0), D\varphi(x_0)) \le F(x_0, u(x_0), D\varphi(x_0)) \le 0 .$$

Therefore, u satisfies (MP) and, a fortiori, (CP). ◀

A similar result holds for viscosity supersolutions, provided all inequalities are reversed in (CP), (MP) and nonnegative maximum is replaced by nonpositive minimum.

A perhaps striking fact to be stressed here is that viscosity solutions are not preserved by change of sign in the equation. Indeed, since any local maximum of $u - \varphi$ is a local minimum of $-u - (-\varphi)$, u is a viscosity subsolution of (HJ) if and only if $v = -u$ is a viscosity supersolution of $-F(x, -v(x), -Dv(x)) = 0$ in Ω; similarly, u is a viscosity supersolution of (HJ) if and only $v = -u$ is a viscosity subsolution of $-F(x, -v(x), -Dv(x)) = 0$. An explicit example is as follows.

EXAMPLE 1.6. The function $u(x) = |x|$ is a viscosity solution of the 1-dimensional equation

$$-|u'(x)| + 1 = 0 \qquad x \in]-1, 1[.$$

To check this, notice first that if $x \neq 0$ is a local extremum for $u - \varphi$, then $u'(x) = \varphi'(x)$. Therefore, at those points both the supersolution and the subsolution conditions are trivially satisfied. Also, if 0 is a local minimum for $u - \varphi$, a simple calculation shows that $|\varphi'(0)| \le 1$ and the supersolution condition holds. To conclude it is enough to observe that 0 cannot be a local maximum for $u - \varphi$ with $\varphi \in C^1(]-1, 1[)$ (this would imply $-1 \ge \varphi'(0) \ge 1$).

On the other hand, $u(x) = |x|$ is *not* a viscosity solution of

$$|u'(x)| - 1 = 0, \qquad x \in]-1, 1[.$$

Actually, the supersolution condition is not fulfilled at $x_0 = 0$ which is a local minimum for $|x| - (-x^2)$. ◁

We describe now an alternative way of defining viscosity solutions of equation (HJ) and prove the equivalence of the new definition with the one given previously (see Exercise 1.11 for another equivalent definition). Let us associate with a function $u \in C(\Omega)$ and $x \in \Omega$ the sets

$$D^+u(x) := \{ p \in \mathbb{R}^N : \limsup_{y \to x,\, y \in \Omega} \frac{u(y) - u(x) - p \cdot (y - x)}{|x - y|} \le 0 \}$$
$$D^-u(x) := \{ p \in \mathbb{R}^N : \liminf_{y \to x,\, y \in \Omega} \frac{u(y) - u(x) - p \cdot (y - x)}{|x - y|} \ge 0 \} .$$

These sets are called, respectively, the *super-* and the *subdifferential* (or *semidifferentials*) of u at x.

The next lemma provides a description of $D^+u(x)$, $D^-u(x)$ in terms of test functions.

LEMMA 1.7. *Let $u \in C(\Omega)$. Then,*

(a) *$p \in D^+u(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local maximum at x;*

(b) *$p \in D^-u(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local minimum at x.*

PROOF. Let $p \in D^+u(x)$. Then, for some $\delta > 0$,

$$u(y) \le u(x) + p \cdot (y - x) + \sigma(|y - x|)|y - x| \qquad \forall y \in B(x, \delta),$$

where σ is a continuous increasing function on $[0, +\infty[$ such that $\sigma(0) = 0$. Now define a C^1 function ϱ by

$$\varrho(r) = \int_0^r \sigma(t)\, dt .$$

The following properties of ϱ

$$\varrho(0) = \varrho'(0) = 0, \qquad \varrho(2r) \ge \sigma(r) r$$

imply, as it is easy to check, that the function φ defined by

$$\varphi(y) = u(x) + p \cdot (y - x) + \varrho(2|y - x|)$$

belongs to $C^1(\mathbb{R}^N)$ and $D\varphi(x) = p$. Moreover, for $y \in B(x, \delta)$,

$$(u - \varphi)(y) \le \sigma(|y - x|)|y - x| - \varrho(2|y - x|) \le 0 = (u - \varphi)(x) .$$

For the opposite implication it is enough to observe that

$$u(y) - u(x) - D\varphi(x) \cdot (y - x) \le \varphi(y) - \varphi(x) - D\varphi(x) \cdot (y - x)$$

for $y \in B(x, \delta)$ and the proof of (a) is complete.

Since $D^-u(x) = -(D^+(-u)(x))$, the proof of (b) follows from the above argument when applied to $-u$. ◀