Chapter 10

HAMILTON–JACOBI EQUATIONS

10.1. INTRODUCTION, VISCOSITY SOLUTIONS

This chapter investigates the existence, uniqueness and other properties of appropriately defined weak solutions of the initial-value problem for the *Hamilton-Jacobi equation*:

$$(1) \qquad \begin{cases} u_t + H(Du, x) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given, as is the initial function $g : \mathbb{R}^n \to \mathbb{R}$. The unknown is $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$, $u = u(x,t)$, and $Du = D_x u = (u_{x_1}, \dots, u_{x_n})$. We will write $H = H(p,x)$, so that "p" is the name of the variable for which we substitute the gradient Du in the PDE.

We recall from our study of characteristics in §3.2 that in general there can be no smooth solution of (1) lasting for all times $t \geq 0$. We recall further that if H depends only on p and is convex, then the Hopf-Lax formula (expression (21) in §3.3.2) provides us with a type of generalized solution.

In this chapter we consider the general case that H depends also on x and, more importantly, is no longer necessarily convex in the variable p. We

will discover in these new circumstances a different way to define a weak solution of (1).

Our approach is to consider first this approximate problem:

$$(2) \qquad \begin{cases} u^\epsilon_t + H(Du^\epsilon, x) - \epsilon\Delta u^\epsilon = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\epsilon = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

for $\epsilon > 0$. The idea is that whereas (1) involves a fully nonlinear first-order PDE, (2) is an initial-value problem for a quasilinear parabolic PDE, which turns out to have a smooth solution. The term $\epsilon\Delta$ in (2) in effect regularizes the Hamilton–Jacobi equation. Then of course we hope that as $\epsilon \to 0$ the solutions u^ϵ of (2) will converge to some sort of weak solution of (1). This technique is the method of *vanishing viscosity*.

However, as $\epsilon \to 0$ we can expect to lose control over the various estimates of the function u^ϵ and its derivatives: these estimates depend strongly on the regularizing effect of $\epsilon\Delta$ and blow up as $\epsilon \to 0$. However, it turns out that we can often in practice at least be sure that the family $\{u^\epsilon\}_{\epsilon>0}$ is bounded and equicontinuous on compact subsets of $\mathbb{R}^n \times [0,\infty)$. Consequently the Arzela-Ascoli compactness criterion, §C.7, ensures that

$$(3) \qquad u^{\epsilon_j} \to u \quad \text{locally uniformly in } \mathbb{R}^n \times [0,\infty),$$

for some subsequence $\{u^{\epsilon_j}\}_{j=1}^\infty$ and some limit function

$$(4) \qquad u \in C(\mathbb{R}^n \times [0,\infty)).$$

Now we can surely expect that u is some kind of solution of our initial-value problem (1), but as we only know u is continuous, and have absolutely no information as to whether Du and u_t exist in any sense, such an interpretation is difficult.

Similar problems have arisen before in Chapters 8 and 9, where we had to deal with the weak convergence of various would-be approximate solutions to other nonlinear partial differential equations. Remember in particular that in §9.1 we solved a divergence structure quasilinear elliptic PDE by passing to limits using the method of Browder and Minty. Roughly speaking, we there integrated by parts to throw "hard-to-control" derivatives onto a fixed test function, and only then tried to go to limits to discover a solution. We will for the Hamilton-Jacobi equation (1) attempt something similar. We will fix a smooth test function v and will pass from (2) to (1) as $\epsilon \to 0$ by first "putting the derivatives onto v".

But since (1) is fully nonlinear, and in particular is not of divergence structure, we cannot just integrate by parts, as we did in §9.1, to switch to differentiations on v. Instead we will exploit the maximum principle to accomplish this transition, at least at certain points.

We will call the solution we build a *viscosity solution*, in honor of the vanishing viscosity technique. Our main goal will then be to discover an intrinsic characterization of such generalized solutions of (1).

10.1.1. Definitions.

Motivation for definition of viscosity solution. We henceforth assume that H, g are continuous and will as necessary later add further hypotheses.

The technique alluded to above works as follows. Fix any smooth test function $v \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and suppose

$$\begin{cases} u - v \text{ has a } \textit{strict} \text{ local maximum at some point} \\ (x_0, t_0) \in \mathbb{R}^n \times (0,\infty). \end{cases} \tag{5}$$

This means

$$(u-v)(x_0,t_0) > (u-v)(x,t)$$

for all points (x,t) sufficiently close to (x_0,t_0), with $(x,t) \neq (x_0,t_0)$.

Now recall (3). We claim for each sufficiently small $\epsilon_j > 0$, there exists a point $(x_{\epsilon_j}, t_{\epsilon_j})$ such that

$$u^{\epsilon_j} - v \text{ has a local maximum at } (x_{\epsilon_j}, t_{\epsilon_j}) \tag{6}$$

and

$$(x_{\epsilon_j}, t_{\epsilon_j}) \to (x_0, t_0) \quad \text{as } j \to \infty. \tag{7}$$

To confirm this, note that for each sufficiently small $r > 0$, (5) implies $\max_{\partial B}(u-v) < (u-v)(x_0,t_0)$, B denoting the closed ball in $\mathbb{R}^{n+1}$ with center (x_0,t_0) and radius r. In view of (3), $u^{\epsilon_j} \to u$ uniformly on B, and so $\max_{\partial B}(u^{\epsilon_j} - v) < (u^{\epsilon_j} - v)(x_0,t_0)$ provided ϵ_j is small enough. Consequently $u^\epsilon - v$ attains a local maximum at some point in the interior of B. We can next replace r by a sequence of radii tending to zero to obtain (6), (7).

Now owing to (6) we see that the equations

$$Du^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) = Dv(x_{\epsilon_j}, t_{\epsilon_j}), \tag{8}$$

$$u_t^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) = v_t(x_{\epsilon_j}, t_{\epsilon_j}), \tag{9}$$

and the inequality

$$-\Delta u^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) \geq -\Delta v(x_{\epsilon_j}, t_{\epsilon_j}) \tag{10}$$

hold. We consequently can calculate

$$\begin{aligned} &v_t(x_{\epsilon_j}, t_{\epsilon_j}) + H(Dv(x_{\epsilon_j}, t_{\epsilon_j}), x_{\epsilon_j}) \\ &\quad = u_t^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) + H(Du^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}), x_{\epsilon_j}) \quad \text{by (8),(9)} \\ &\quad = \epsilon_j \Delta u^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) \quad \text{by (2)} \\ &\quad \leq \epsilon_j \Delta v(x_{\epsilon_j}, t_{\epsilon_j}) \quad \text{by (10).} \end{aligned} \tag{11}$$

Now let $\epsilon_j \to 0$ and remember (7). Since v is smooth and H is continuous, we deduce

$$v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) \leq 0. \tag{12}$$

We have established this inequality assuming (5). Suppose now instead that

$$u - v \text{ has a local maximum at } (x_0,t_0), \tag{13}$$

but that this maximum is not necessarily strict. Then $u - \tilde{v}$ has a strict local maximum at (x_0,t_0), for $\tilde{v}(x,t) := v(x,t) + \delta(|x-x_0|^2 + (t-t_0)^2)$ $(\delta > 0)$. We thus conclude as above that $\tilde{v}_t(x_0,t_0) + H(D\tilde{v}(x_0,t_0), x_0) \leq 0$; whereupon (12) again follows.

Consequently (13) implies inequality (12). Similarly, we deduce the reverse inequality

$$v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) \geq 0, \tag{14}$$

provided

$$u - v \text{ has a local minimum at } (x_0,t_0). \tag{15}$$

The proof is exactly like that above, except that the inequalities in (10), and thus in (11), are reversed.

In summary, we have discovered for any smooth function v that inequality (12) follows from (13), and (14) from (15). We have in effect put the derivatives onto v, at the expense of certain inequalities holding. □

Our intention now is to *define* a weak solution of (1) in terms of (12), (13) and (14), (15).

DEFINITION. *A bounded, uniformly continuous function u is called a* viscosity solution *of the initial-value problem* (1) *for the Hamilton–Jacobi equation provided:*

(i) $u = g$ *on* $\mathbb{R}^n \times \{t = 0\}$, *and*

(ii) *for each* $v \in C^\infty(\mathbb{R}^n \times (0,\infty))$,

$$(16)\quad \begin{cases} \text{if } u - v \text{ has a local maximum at a point } (x_0,t_0) \in \mathbb{R}^n \times (0,\infty), \\ \text{then} \\ \qquad v_t(x_0,t_0) + H(Dv(x_0,t_0),x_0) \le 0, \end{cases}$$

and

$$(17)\quad \begin{cases} \text{if } u - v \text{ has a local minimum at a point } (x_0,t_0) \in \mathbb{R}^n \times (0,\infty), \\ \text{then} \\ \qquad v_t(x_0,t_0) + H(Dv(x_0,t_0),x_0) \ge 0. \end{cases}$$

Remark. Note carefully that by definition a viscosity solution satisfies (16), (17), and so all subsequent deductions must be based on these inequalities. The previous discussion was purely motivational.

For emphasis, we repeat the same point, which has caused some confusion among students. To verify that a given function u is a viscosity solution of the Hamilton–Jacobi equation $u_t + H(Du,x) = 0$, we must confirm that (16), (17) hold for all smooth functions v. Now the argument above shows that *if* u is constructed using the vanishing viscosity method, it is indeed a viscosity solution. But we will also see later in §10.3 that viscosity solutions can be built in entirely different ways, which have nothing whatsoever to do with vanishing viscosity.

The point is that the inequalities (16), (17) provide an intrinsic characterization, and indeed the very definition, of our generalized solutions.

□

We devote the rest of this chapter to demonstrating that viscosity solutions provide an appropriate and useful notion of weak solutions for our Hamilton–Jacobi PDE.

10.1.2. Consistency.

Let us begin by checking that the notion of viscosity solution is consistent with that of a classical solution. First of all, note that if $u \in C^1(\mathbb{R}^n \times [0,\infty))$ solves (1) and if u is bounded and uniformly continuous, then u is a viscosity solution. That is, we assert that any *classical solution of* $u_t + H(Du,x) = 0$ *is also a viscosity solution.* The proof is easy. If v is smooth and $u - v$ obtains a local maximum at (x_0,t_0), then

$$\begin{cases} Du(x_0,t_0) = Dv(x_0,t_0) \\ u_t(x_0,t_0) = v_t(x_0,t_0). \end{cases}$$

Consequently

$$\begin{aligned} &v_t(x_0,t_0) + H(Dv(x_0,t_0),x_0) \\ &\qquad = u_t(x_0,t_0) + H(Du(x_0,t_0),x_0) = 0, \end{aligned}$$

since u solves (1). A similar equality holds at any point (x_0,t_0) where $u - v$ has a local minimum.

Next we assert that *any sufficiently smooth viscosity solution is a classical solution*, and, even more, that if a viscosity solution is differentiable at some point, then it solves the Hamilton–Jacobi PDE there. We will need the following calculus fact:

LEMMA (Touching by a C^1 function). *Assume* $u : \mathbb{R}^n \to \mathbb{R}$ *is continuous and is also differentiable at some point* x_0. *Then there exists a function* $v \in C^1(\mathbb{R}^n)$ *such that*

$$(18)\qquad u(x_0) = v(x_0)$$

and

$$(19)\qquad u - v \text{ has a strict local maximum at } x_0.$$

Proof. 1. We may as well assume

$$(20)\qquad x_0 = 0,\ u(0) = Du(0) = 0;$$

for otherwise we could consider $\tilde{u}(x) := u(x + x_0) - u(x_0) - Du(x_0)\cdot x$ in place of u.

2. In view of (20) and our hypothesis, we have

$$(21)\qquad u(x) = |x|\rho_1(x),$$

where

$$(22)\qquad \rho_1 : \mathbb{R}^n \to \mathbb{R} \text{ is continuous, } \rho_1(0) = 0.$$

Set

$$(23)\qquad \rho_2(r) := \max_{x \in B(r)}\{|\rho_1(x)|\} \quad (r \ge 0).$$

Then

$$(24)\qquad \rho_2 : [0,\infty) \to [0,\infty) \text{ is continuous, } \rho_2(0) = 0,$$

and

(25) ρ_2 is nondecreasing.

3. Now write

$$v(x) := \int_{|x|}^{2|x|} \rho_2(r)\,dr + |x|^2 \quad (x \in \mathbb{R}^n).$$

Since $|v(x)| \leq |x|\rho_2(2|x|) + |x|^2$, we observe

(26) $$v(0) = Dv(0) = 0.$$

Furthermore if $x \neq 0$, we have

$$Dv(x) = \frac{2x}{|x|}\rho_2(2|x|) - \frac{x}{|x|}\rho_2(|x|) + 2x,$$

and so $v \in C^1(\mathbb{R}^n)$.

4. Finally note that if $x \neq 0$,

$$\begin{aligned} u(x) - v(x) &= |x|\rho_1(x) - \int_{|x|}^{2|x|} \rho_2(r)\,dr - |x|^2 \\ &\leq |x|\rho_2(|x|) - \int_{|x|}^{2|x|} \rho_2(r)\,dr - |x|^2 \\ &\leq -|x|^2 \quad \text{by (25)} \\ &< 0 = u(0) - v(0). \end{aligned}$$

Thus $u - v$ has a strict local maximum at 0, as required. □

THEOREM 1 (Consistency of viscosity solutions). *Let u be a viscosity solution of* (1), *and suppose u is differentiable at some point* $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. *Then*

$$u_t(x_0, t_0) + H(Du(x_0, t_0), x_0) = 0.$$

Proof. 1. Applying the lemma above to u, with $\mathbb{R}^{n+1}$ replacing $\mathbb{R}^n$ and (x_0, t_0) replacing x_0, we deduce there exists a C^1 function v such that

(27) $u - v$ has a strict maximum at (x_0, t_0).

2. Now set $v^\epsilon := \eta_\epsilon * v$, η_ϵ denoting the usual mollifier in the $n+1$ variables (x, t). Then

(28) $$\begin{cases} v^\epsilon \to v \\ Dv^\epsilon \to Dv & \text{uniformly near } (x_0, t_0) \\ v_t^\epsilon \to v_t; \end{cases}$$

and so (27) implies

(29) $u - v^\epsilon$ has a maximum at some point (x_ϵ, t_ϵ),

with

(30) $$(x_\epsilon, t_\epsilon) \to (x_0, t_0) \quad \text{as } \epsilon \to 0.$$

Applying then the definition of viscosity solution, we see

$$v_t^\epsilon(x_\epsilon, t_\epsilon) + H(Dv^\epsilon(x_\epsilon, t_\epsilon), x_\epsilon) \leq 0.$$

Let $\epsilon \to 0$ and use (28), (30) to deduce

(31) $$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \leq 0.$$

But in view of (27), we see that since u is differentiable at (x_0, t_0),

$$Du(x_0, t_0) = Dv(x_0, t_0), \; u_t(x_0, t_0) = v_t(x_0, t_0).$$

Substitute above, to conclude from (31) that

(32) $$u_t(x_0, t_0) + H(Du(x_0, t_0), x_0) \leq 0.$$

3. Now apply the lemma above to $-u$ in $\mathbb{R}^{n+1}$, to find a C^1 function v such that $u - v$ has a strict minimum at (x_0, t_0). Then, arguing as above, we likewise deduce

$$u_t(x_0, t_0) + H(Du(x_0, t_0), x_0) \geq 0.$$

This inequality and (32) complete the proof. □

10.2. UNIQUENESS

Our goal now is to establish the uniqueness of a viscosity solution of our initial-value problem for Hamilton–Jacobi PDE. To be slightly more general, let us fix a time $T > 0$ and consider the problem

(1) $$\begin{cases} u_t + H(Du, x) = 0 & \text{in } \mathbb{R}^n \times (0, T] \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

We say that a bounded, uniformly continuous function u is a viscosity solution of (1) provided $u = g$ on $\mathbb{R}^n \times \{t = 0\}$, and the inequalities in (16) (or (17)) from §10.1.1 hold if $u - v$ has a local maximum (or minimum) at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$.

LEMMA (Extrema at a terminal time). *Assume u is a viscosity solution of* (1) *and $u - v$ has a local maximum (minimum) at a point* $(x_0, t_0) \in \mathbb{R}^n \times (0, T]$. *Then*

(2) $$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \leq 0 \quad (\geq 0).$$

The point is that we are now allowing for $t_0 = T$.

Proof. Assume $u - v$ has a local maximum at the point (x_0, T); as before we may assume that this is a strict local maximum. Write

$$\tilde{v}(x,t) := v(x,t) + \frac{\epsilon}{T-t} \quad (x \in \mathbb{R}^n,\ 0 < t < T).$$

Then for $\epsilon > 0$ small enough, $u - \tilde{v}$ has a local maximum at a point (x_ϵ, t_ϵ), where $0 < t_\epsilon < T$ and $(x_\epsilon, t_\epsilon) \to (x_0, T)$. Consequently

$$\tilde{v}_t(x_\epsilon, t_\epsilon) + H(D\tilde{v}(x_\epsilon, t_\epsilon), x_\epsilon) \leq 0,$$

and so

$$v_t(x_\epsilon, t_\epsilon) + \frac{\epsilon}{(T-t_\epsilon)^2} + H(Dv(x_\epsilon, t_\epsilon), x_\epsilon) \leq 0.$$

Letting $\epsilon \to 0$, we find

$$v_t(x_0, T) + H(Dv(x_0, T), x_0) \leq 0.$$

This proves (2) if $u - v$ has a maximum at (x_0, T). A similar proof gives the reverse inequality should $u - v$ have a minimum at (x_0, T). □

To go further, let us hereafter suppose the Hamiltonian H to satisfy these conditions of Lipschitz continuity:

$$\begin{cases} |H(p,x) - H(q,x)| \leq C|p-q| \\ |H(p,x) - H(p,y)| \leq C|x-y|(1+|p|) \end{cases} \tag{3}$$

for $x, y, p, q \in \mathbb{R}^n$ and some constant $C \geq 0$.

We come next to the central fact concerning viscosity solutions of the initial-value problem (1), namely uniqueness. This important assertion justifies our taking the inequalities (16) and (17) from §10.1.1 as the foundation of our theory.

THEOREM 1 (Uniqueness of viscosity solution). *Under assumption* (3) *there exists at most one viscosity solution of* (1).

Remark. The following proof is based upon an unusual idea of "doubling the number of variables". See the proof of Theorem 3 in §11.4.3 for a related technique. □

Proof*. 1. Assume u and $\tilde{u}$ are both viscosity solutions with the same initial conditions, but

$$\sup_{\mathbb{R}^n \times [0,T]} (u - \tilde{u}) =: \sigma > 0. \tag{4}$$

Choose $0 < \epsilon,\ \lambda < 1$ and set

$$\begin{aligned} \Phi(x,y,t,s) :=& u(x,t) - \tilde{u}(y,s) - \lambda(t+s) \\ & - \frac{1}{\epsilon^2}(|x-y|^2 + (t-s)^2) - \epsilon(|x|^2 + |y|^2), \end{aligned} \tag{5}$$

for $x, y \in \mathbb{R}^n$, $t, s \geq 0$. Then there exists a point $(x_0, y_0, t_0, s_0) \in \mathbb{R}^{2n} \times [0,T]^2$ such that

$$\Phi(x_0, y_0, t_0, s_0) = \max_{\mathbb{R}^{2n} \times [0,T]^2} \Phi(x,y,t,s). \tag{6}$$

2. We may fix $0 < \epsilon,\ \lambda < 1$ so small that (4) implies

$$\Phi(x_0, y_0, t_0, s_0) \geq \sup_{\mathbb{R}^n \times [0,T]} \Phi(x,x,t,t) \geq \frac{\sigma}{2}. \tag{7}$$

In addition, $\Phi(x_0, y_0, t_0, s_0) \geq \Phi(0,0,0,0)$; and therefore

$$\begin{aligned} &\lambda(t_0 + s_0) + \frac{1}{\epsilon^2}(|x_0 - y_0|^2 + (t_0 - s_0)^2) + \epsilon(|x_0|^2 + |y_0|^2) \\ &\quad \leq u(x_0, t_0) - \tilde{u}(y_0, s_0) - u(0,0) + \tilde{u}(0,0). \end{aligned} \tag{8}$$

Since u and $\tilde{u}$ are bounded, we deduce

$$|x_0 - y_0|, |t_0 - s_0| = O(\epsilon) \quad \text{as } \epsilon \to 0. \tag{9}$$

Furthermore (8) implies $\epsilon(|x_0|^2 + |y_0|^2) = O(1)$, and consequently

$$\begin{aligned} \epsilon(|x_0| + |y_0|) &= \epsilon^{1/4}\epsilon^{3/4}(|x_0| + |y_0|) \\ &\leq \epsilon^{1/2} + C\epsilon^{3/2}(|x_0|^2 + |y_0|^2) \\ &\leq C\epsilon^{1/2}. \end{aligned}$$

Thus

$$\epsilon(|x_0| + |y_0|) = O(\epsilon^{1/2}). \tag{10}$$

*Omit on first reading.

3. Since $\Phi(x_0, y_0, t_0, s_0) \geq \Phi(x_0, x_0, t_0, t_0)$, we also have

$$u(x_0,t_0) - \tilde{u}(y_0,s_0) - \lambda(t_0+s_0) - \frac{1}{\epsilon^2}(|x_0-y_0|^2 + (t_0-s_0)^2) - \epsilon(|x_0|^2+|y_0|^2) \geq u(x_0,t_0) - \tilde{u}(x_0,t_0) - 2\lambda t_0 - 2\epsilon|x_0|^2.$$

Hence

$$\frac{1}{\epsilon^2}(|x_0-y_0|^2 + (t_0-s_0)^2) \leq \tilde{u}(x_0,t_0) - \tilde{u}(y_0,s_0) + \lambda(t_0-s_0) + \epsilon(x_0+y_0)\cdot(x_0-y_0).$$

In view of (9), (10) and the uniform continuity of $\tilde{u}$, we deduce

$$|x_0-y_0|, |t_0-s_0| = o(\epsilon). \tag{11}$$

4. Now write $\omega(\cdot)$ to denote the modulus of continuity of u; that is,

$$|u(x,t) - u(y,s)| \leq \omega(|x-y| + |t-s|)$$

for all $x, y \in \mathbb{R}^n$, $0 \leq t, s \leq T$, and $\omega(r) \to 0$ as $r \to 0$. Similarly, $\tilde{\omega}(\cdot)$ will denote the modulus of continuity of $\tilde{u}$.

Then (7) implies

$$\begin{aligned}\frac{\sigma}{2} \leq u(x_0,t_0) - \tilde{u}(y_0,s_0) &= u(x_0,t_0) - u(x_0,0) + u(x_0,0) - \tilde{u}(x_0,0) \\ &+ \tilde{u}(x_0,0) - \tilde{u}(x_0,t_0) + \tilde{u}(x_0,t_0) - \tilde{u}(y_0,s_0) \\ &\leq \omega(t_0) + \tilde{\omega}(t_0) + \tilde{\omega}(o(\epsilon)),\end{aligned}$$

by (9), (11) and the initial condition. We can now take $\epsilon > 0$ to be so small that the foregoing implies $\frac{\sigma}{4} \leq \omega(t_0) + \tilde{\omega}(t_0)$; and this in turn implies $t_0 \geq \mu > 0$ for some constant $\mu > 0$. Similarly we have $s_0 \geq \mu > 0$.

5. Now observe in light of (6) that the mapping $(x,t) \mapsto \Phi(x, y_0, t, s_0)$ has a maximum at the point (x_0, t_0). In view of (5) then,

$$u - v \text{ has a maximum at } (x_0, t_0)$$

for

$$v(x,t) := \tilde{u}(y_0,s_0) + \lambda(t+s_0) + \frac{1}{\epsilon^2}(|x-y_0|^2 + (t-s_0)^2) + \epsilon(|x|^2+|y_0|^2).$$

Since u is a viscosity solution of (1) we conclude, using the lemma if necessary, that

$$v_t(x_0,t_0) + H(D_x v(x_0,t_0), x_0) \leq 0.$$

Therefore

$$\lambda + \frac{2(t_0-s_0)}{\epsilon^2} + H\left(\frac{2}{\epsilon^2}(x_0-y_0) + 2\epsilon x_0, x_0\right) \leq 0. \tag{12}$$

We further observe that since the mapping $(y,s) \mapsto -\Phi(x_0, y, t_0, s)$ has a minimum at the point (y_0, s_0),

$$\tilde{u} - \tilde{v} \text{ has a minimum at } (y_0, s_0)$$

for

$$\tilde{v}(y,s) := u(x_0,t_0) - \lambda(t_0+s) - \frac{1}{\epsilon^2}(|x_0-y|^2 + (t_0-s)^2) - \epsilon(|x_0|^2+|y|^2).$$

As $\tilde{u}$ is a viscosity solution of (1), we know then that

$$\tilde{v}_s(y_0,s_0) + H(D_y\tilde{v}(y_0,s_0), y_0) \geq 0.$$

Consequently

$$-\lambda + \frac{2(t_0-s_0)}{\epsilon^2} + H\left(\frac{2}{\epsilon^2}(x_0-y_0) - 2\epsilon y_0, y_0\right) \geq 0. \tag{13}$$

6. Next, subtract (13) from (12):

$$2\lambda \leq H\left(\frac{2}{\epsilon^2}(x_0-y_0) - 2\epsilon y_0, y_0\right) - H\left(\frac{2}{\epsilon^2}(x_0-y_0) + 2\epsilon x_0, x_0\right). \tag{14}$$

In view of hypothesis (3) therefore,

$$\lambda \leq C\epsilon(|x_0|+|y_0|) + C|x_0-y_0|\left(1 + \frac{|x_0-y_0|}{\epsilon^2} + \epsilon(|x_0|+|y_0|)\right). \tag{15}$$

We employ estimates (10), (11) in (15), and then let $\epsilon \to 0$, to discover $0 < \lambda \leq 0$. This contradiction completes the proof. □

10.3. CONTROL THEORY, DYNAMIC PROGRAMMING

It remains for us to establish the existence of a viscosity solution to our initial-value problem for the Hamilton–Jacobi partial differential equation. One method would be now to prove the existence of a smooth solution u^ϵ of the regularized equation (2) in §10.1 and then to make good enough uniform