Let $h \to 0^+$ to compute

$$\frac{x-z}{t} \cdot Du(x,t) + u_t(x,t) \geq L\left(\frac{x-z}{t}\right).$$

Consequently

$$\begin{aligned} u_t(x,t) + H(Du(x,t)) &= u_t(x,t) + \max_{q \in \mathbb{R}^n}\{q \cdot Du(x,t) - L(q)\} \\ &\geq u_t(x,t) + \frac{x-z}{t} \cdot Du(x,t) - L\left(\frac{x-z}{t}\right) \\ &\geq 0. \end{aligned}$$

This inequality and (31) complete the proof. □

We summarize:

THEOREM 6 (Hopf–Lax formula as solution). *The function u defined by the Hopf–Lax formula (21) is Lipschitz continuous, is differentiable a.e. in $\mathbb{R}^n \times (0,\infty)$, and solves the initial-value problem*

$$(32) \qquad \begin{cases} u_t + H(Du) = 0 & \text{a.e. in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

3.3.3. Weak solutions, uniqueness.

a. Semiconcavity.

In view of Theorem 6 above it may seem reasonable to define a weak solution of the initial-value problem (18) to be a Lipschitz function which agrees with g on $\mathbb{R}^n \times \{t=0\}$, and solves the PDE a.e. on $\mathbb{R}^n \times (0,\infty)$. However, this turns out to be an inadequate definition, *as such weak solutions would not in general be unique.*

Example. Consider the initial-value problem

$$(33) \qquad \begin{cases} u_t + |u_x|^2 = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R} \times \{t=0\}. \end{cases}$$

One obvious solution is

$$u_1(x,t) \equiv 0.$$

However the function

$$u_2(x,t) := \begin{cases} 0 & \text{if } |x| \geq t \\ x - t & \text{if } 0 \leq x \leq t \\ -x - t & \text{if } -t \leq x \leq 0 \end{cases}$$

is Lipschitz continuous and also solves the PDE a.e. (everywhere, in fact, except on the lines $x = 0, \pm t$). It is easy to see that actually there are infinitely many Lipschitz functions satisfying (33). □

This example shows we must presumably require more of a weak solution than merely that it satisfy the PDE a.e. We will look to the Hopf–Lax formula (21) for a further clue as to what is needed to ensure uniqueness. The following lemma demonstrates that u inherits a kind of "one-sided" second-derivative estimate from the initial function g.

LEMMA 3 (Semiconcavity). *Suppose there exists a constant C such that*

$$(34) \qquad g(x+z) - 2g(x) + g(x-z) \leq C|z|^2$$

for all $x, z \in \mathbb{R}^n$. Define u by the Hopf–Lax formula (21). Then

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C|z|^2$$

for all $x, z \in \mathbb{R}^n$, $t > 0$.

Remark. We say g is *semiconcave* provided (34) holds. It is easy to check (34) is valid if g is C^2 and $\sup_{\mathbb{R}^n} |D^2 g| < \infty$. Note that g is semiconcave if and only if the mapping $x \mapsto g(x) - \frac{C}{2}|x|^2$ is concave for some constant C. □

Proof. Choose $y \in \mathbb{R}^n$ so that $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$. Then, putting $y+z$ and $y-z$ in the Hopf–Lax formulas for $u(x+z,t)$ and $u(x-z,t)$, we find

$$\begin{aligned} u(x+z,t) &- 2u(x,t) + u(x-z,t) \\ &\leq \left[tL\left(\frac{x-y}{t}\right) + g(y+z)\right] - 2\left[tL\left(\frac{x-y}{t}\right) + g(y)\right] \\ &\quad + \left[tL\left(\frac{x-y}{t}\right) + g(y-z)\right] \\ &= g(y+z) - 2g(y) + g(y-z) \\ &\leq C|z|^2, \quad \text{by (34)}. \end{aligned}$$

□

As a semiconcavity condition for u will turn out to be important, we pause to identify some other circumstances under which it is valid. We will no longer assume g to be semiconcave, but will suppose the Hamiltonian H to be uniformly convex.

DEFINITION. *A C^2 convex function $H : \mathbb{R}^n \to \mathbb{R}$ is called* uniformly convex *(with constant $\theta > 0$) if*

$$\sum_{i,j=1}^{n} H_{p_i p_j}(p)\xi_i\xi_j \geq \theta|\xi|^2 \quad \text{for all } p, \xi \in \mathbb{R}^n. \tag{35}$$

We now prove that even if g is not semiconcave, the uniform convexity of H forces u to become semiconcave for times $t > 0$: this is a kind of mild regularizing effect for the Hopf–Lax solution of the initial-value problem (18).

LEMMA 4 (Semiconcavity again). *Suppose that H is uniformly convex (with constant θ) and u is defined by the Hopf-Lax formula* (21). *Then*

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq \frac{1}{\theta t}|z|^2$$

for all $x, z \in \mathbb{R}^n$, $t > 0$.

Proof. 1. We note first using Taylor's formula that (35) implies

$$H\left(\frac{p_1+p_2}{2}\right) \leq \frac{1}{2}H(p_1) + \frac{1}{2}H(p_2) - \frac{\theta}{8}|p_1 - p_2|^2. \tag{36}$$

Next we claim that for the Lagrangian L we have the estimate

$$\frac{1}{2}L(q_1) + \frac{1}{2}L(q_2) \leq L\left(\frac{q_1+q_2}{2}\right) + \frac{1}{8\theta}|q_1 - q_2|^2 \tag{37}$$

for all $q_1, q_2 \in \mathbb{R}^n$. Verification is left as an exercise.

2. Now choose y so that $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$. Then using the same value of y in the Hopf–Lax formulas for $u(x+z,t)$ and $u(x-z,t)$, we calculate

$$\begin{aligned}
&u(x+z,t) - 2u(x,t) + u(x-z,t) \\
&\leq \left[tL\left(\frac{x+z-y}{t}\right) + g(y)\right] - 2\left[tL\left(\frac{x-y}{t}\right) + g(y)\right] \\
&\quad + \left[tL\left(\frac{x-z-y}{t}\right) + g(y)\right] \\
&= 2t\left[\frac{1}{2}L\left(\frac{x+z-y}{t}\right) + \frac{1}{2}L\left(\frac{x-z-y}{t}\right) - L\left(\frac{x-y}{t}\right)\right] \\
&\leq 2t\frac{1}{8\theta}\left|\frac{2z}{t}\right|^2 \leq \frac{1}{\theta t}|z|^2,
\end{aligned}$$

the next-to-last inequality following from (37). □

b. Weak solutions, uniqueness.

In this section we show that semiconcavity conditions of the sorts discovered for the Hopf–Lax solution u in Lemmas 3 and 4 can be utilized as uniqueness criteria.

DEFINITION. *We say that a Lipschitz continuous function $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is a* weak solution *of the initial-value problem:*

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{38}$$

provided

(a) $u(x,0) = g(x)$ $(x \in \mathbb{R}^n)$,

(b) $u_t(x,t) + H(Du(x,t)) = 0$ *for a.e.* $(x,t) \in \mathbb{R}^n \times (0,\infty)$,

and

(c) $u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C\left(1 + \frac{1}{t}\right)|z|^2$

for some constant $C \geq 0$ and all $x, z \in \mathbb{R}^n$, $t > 0$.

Next we prove that a weak solution of (38) is unique, the key point being that this uniqueness assertion follows from the *inequality* condition (c).

THEOREM 7 (Uniqueness of weak solutions). *Assume H is C^2 and satisfies* (19), *and g satisfies* (20). *Then there exists at most one weak solution of the initial-value problem* (38).

Proof*. 1. Suppose that u and $\tilde{u}$ are two weak solutions of (38) and write $w := u - \tilde{u}$.

Observe now at any point (y,s) where both u and $\tilde{u}$ are differentiable and solve our PDE, we have

$$\begin{aligned}
w_t(y,s) &= u_t(y,s) - \tilde{u}_t(y,s) \\
&= -H(Du(y,s)) + H(D\tilde{u}(y,s)) \\
&= -\int_0^1 \frac{d}{dr}H(rDu(y,s) + (1-r)D\tilde{u}(y,s))\,dr \\
&= -\int_0^1 DH(rDu(y,s) + (1-r)D\tilde{u}(y,s))\,dr \cdot (Du(y,s) - D\tilde{u}(y,s)) \\
&=: -\mathbf{b}(y,s) \cdot Dw(y,s).
\end{aligned}$$

Consequently

$$w_t + \mathbf{b} \cdot Dw = 0 \quad \text{a.e.} \tag{39}$$

*Omit on first reading.

2. Write $v := \phi(w) \geq 0$, where $\phi : \mathbb{R} \to [0, \infty)$ is a smooth function to be selected later. We multiply (39) by $\phi'(w)$ to discover

$$v_t + b \cdot Dv = 0 \quad \text{a.e.} \tag{40}$$

3. Now choose $\varepsilon > 0$ and define $u^\varepsilon := \eta_\varepsilon * u$, $\tilde{u}^\varepsilon := \eta_\varepsilon * \tilde{u}$, where η_ε is the standard mollifier in the x and t variables. Then according to §C.4

$$|Du^\varepsilon| \leq \mathrm{Lip}(u), \quad |D\tilde{u}^\varepsilon| \leq \mathrm{Lip}(\tilde{u}), \tag{41}$$

and

$$Du^\varepsilon \to Du, \quad D\tilde{u}^\varepsilon \to D\tilde{u} \quad \text{a.e., as } \epsilon \to 0. \tag{42}$$

Furthermore inequality (c) in the definition of weak solution implies

$$D^2u^\varepsilon, D^2\tilde{u}^\varepsilon \leq C\left(1 + \frac{1}{s}\right) I \tag{43}$$

for an appropriate constant C and all $\varepsilon > 0$, $y \in \mathbb{R}^n$, $s > 2\varepsilon$. Verification is left as an exercise.

4. Write

$$\mathbf{b}_\varepsilon(y, s) := \int_0^1 DH(rDu^\varepsilon(y, s) + (1 - r)D\tilde{u}^\varepsilon(y, s))\, dr. \tag{44}$$

Then (40) becomes

$$v_t + \mathbf{b}_\varepsilon \cdot Dv = (\mathbf{b}_\varepsilon - \mathbf{b}) \cdot Dv \quad \text{a.e.;}$$

hence

$$v_t + \operatorname{div}(v\mathbf{b}_\varepsilon) = (\operatorname{div} \mathbf{b}_\varepsilon)v + (\mathbf{b}_\varepsilon - \mathbf{b}) \cdot Dv \quad \text{a.e.} \tag{45}$$

5. Now

$$\operatorname{div} \mathbf{b}_\varepsilon = \int_0^1 \sum_{k,l=1}^n H_{p_k p_l}(rDu^\varepsilon + (1 - r)D\tilde{u}^\varepsilon)(ru^\varepsilon_{x_l x_k} + (1 - r)\tilde{u}^\varepsilon_{x_l x_k})\, dr$$

$$\leq C\left(1 + \frac{1}{s}\right) \tag{46}$$

for some constant C, in view of (41), (43). Here we note that H convex implies $D^2H \geq 0$.

6. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$, and set

$$R := \max\{|DH(p)| \mid |p| \leq \max(\mathrm{Lip}(u), \mathrm{Lip}(\tilde{u}))\}. \tag{47}$$

Define also the cone

$$C := \{(x, t) \mid 0 \leq t \leq t_0, |x - x_0| \leq R(t_0 - t)\}.$$

Next write

$$e(t) = \int_{B(x_0, R(t_0 - t))} v(x, t)\, dx$$

and compute for a.e. $t > 0$:

$$\begin{aligned}
\dot{e}(t) &= \int_{B(x_0,R(t_0-t))} v_t\, dx - R \int_{\partial B(x_0,R(t_0-t))} v\, dS \\
&= \int_{B(x_0,R(t_0-t))} -\operatorname{div}(v\mathbf{b}_\varepsilon) + (\operatorname{div} \mathbf{b}_\varepsilon)v + (\mathbf{b}_\varepsilon - \mathbf{b}) \cdot Dv\, dx \\
&\quad - R \int_{\partial B(x_0,R(t_0-t))} v\, dS \quad \text{by (45)} \\
&= -\int_{\partial B(x_0,R(t_0-t))} v(\mathbf{b}_\varepsilon \cdot \nu + R)\, dS \\
&\quad + \int_{B(x_0,R(t_0-t))} (\operatorname{div} \mathbf{b}_\varepsilon)v + (\mathbf{b}_\varepsilon - \mathbf{b}) \cdot Dv\, dx \\
&\leq \int_{B(x_0,R(t_0-t))} (\operatorname{div} \mathbf{b}_\varepsilon)v + (\mathbf{b}_\varepsilon - \mathbf{b}) \cdot Dv\, dx \quad \text{by (41), (44)} \\
&\leq C\left(1 + \frac{1}{t}\right) e(t) + \int_{B(x_0,R(t_0-t))} (\mathbf{b}_\varepsilon - \mathbf{b}) \cdot Dv\, dx
\end{aligned}$$

by (46). The last term on the right hand side goes to zero as $\varepsilon \to 0$, for a.e. $t_0 > 0$, according to (41), (42) and the Dominated Convergence Theorem. Thus

$$\dot{e}(t) \leq C\left(1 + \frac{1}{t}\right) e(t) \quad \text{for a.e. } 0 < t < t_0. \tag{48}$$

7. Fix $0 < \varepsilon < r < t$ and choose the function $\phi(z)$ to equal zero if

$$|z| \leq \varepsilon[\mathrm{Lip}(u) + \mathrm{Lip}(\tilde{u})]$$

and to be positive otherwise. Since $u = \tilde{u}$ on $\mathbb{R}^n \times \{t = 0\}$,

$$v = \phi(w) = \phi(u - \tilde{u}) = 0 \quad \text{at } \{t = \varepsilon\}.$$

Thus $e(\varepsilon) = 0$. Consequently Gronwall's inequality (see §B.2) and (48) imply

$$e(r) \leq e(\varepsilon)e^{\int_\varepsilon^r C(1+\frac{1}{s})ds} = 0.$$

Hence

$$|u - \tilde{u}| \leq \varepsilon[\mathrm{Lip}(u) + \mathrm{Lip}(\tilde{u})] \quad \text{on } B(x_0, R(t_0 - r)).$$

This inequality is valid for all $\varepsilon > 0$, and so $u \equiv \tilde{u}$ in $B(x_0, R(t_0 - r))$. Therefore, in particular, $u(x_0, t_0) = \tilde{u}(x_0, t_0)$. □

In light of Lemmas 3, 4 and Theorem 7, we have

THEOREM 8 (Hopf–Lax formula as weak solution). *Suppose H is C^2 and satisfies* (19), *and g satisfies* (20). *If either g is semiconcave or H is uniformly convex, then*

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}$$

is the unique weak solution of the initial-value problem (38) *for the Hamilton–Jacobi equation.*

Examples. (i) Consider the initial-value problem:

$$\begin{cases} u_t + \frac{1}{2}|Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = |x| & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{49}$$

Here $H(p) = \frac{1}{2}|p|^2$ and so $L(q) = \frac{1}{2}|q|^2$. The Hopf–Lax formula for the unique, weak solution of (49) is

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{\frac{|x-y|^2}{2t} + |y|\right\}. \tag{50}$$

Assume $|x| > t$. Then

$$D_y\left(\frac{|x-y|^2}{2t} + |y|\right) = \frac{y-x}{t} + \frac{y}{|y|} \quad (y \neq 0);$$

and this expression equals zero if $x = y + \frac{y}{|y|}t$, $y = (|x| - t)\frac{x}{|x|} \neq 0$. Thus $u(x,t) = |x| - \frac{t}{2}$ if $|x| > t$. If $|x| \leq t$, the minimum in (50) is attained at $y = 0$. Consequently

$$u(x,t) = \begin{cases} |x| - t/2 & \text{if } |x| \geq t \\ \frac{|x|^2}{2t} & \text{if } |x| \leq t. \end{cases}$$

Observe that the solution becomes semiconcave at times $t > 0$, even though the initial function $g(x) = |x|$ is not semiconcave. This accords with Lemma 4.

(ii) We next examine the problem with reversed initial conditions:

$$\begin{cases} u_t + \frac{1}{2}|Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = -|x| & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{51}$$

Then

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{\frac{|x-y|^2}{2t} - |y|\right\}.$$

Now

$$D_y\left(\frac{|x-y|^2}{2t} - |y|\right) = \frac{y-x}{t} - \frac{y}{|y|} \quad (y \neq 0),$$

and this equals zero if $x = y - \frac{y}{|y|}t$, $y = (|x| + t)\frac{x}{|x|}$. Thus

$$u(x,t) = -|x| - \frac{t}{2} \quad (x \in \mathbb{R}^n,\ t \geq 0).$$

The initial function $g(x) = -|x|$ is semiconcave, and the solution remains so for times $t > 0$. □

In Chapter 10 we will again study Hamilton–Jacobi PDE and discover another notion of weak solution, which is applicable even if H is not convex.

3.4. INTRODUCTION TO CONSERVATION LAWS

In this section we investigate the initial-value problem for scalar conservation laws in one space dimension:

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1}$$

Here $F : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given and $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is the unknown, $u = u(x,t)$. As noted in §3.2, the method of characteristics demonstrates that there does not in general exist a smooth solution of (1), existing for all times $t > 0$. By analogy with the developments in §3.3.5, we therefore look for some sort of weak or generalized solution.