3.2. CHARACTERISTICS

3.2.1. Derivation of characteristic ODE.

We return to our basic nonlinear first-order PDE

$$(1) \qquad F(Du, u, x) = 0 \quad \text{in } U,$$

subject now to the boundary condition

$$(2) \qquad u = g \quad \text{on } \Gamma,$$

where $\Gamma \subseteq \partial U$ and $g : \Gamma \to \mathbb{R}$ are given. We hereafter suppose that F, g are smooth functions.

We develop next the method of *characteristics*, which solves (1), (2) by converting the PDE into an appropriate system of ODE. This is the plan. Suppose u solves (1), (2) and fix any point $x \in U$. We would like to calculate $u(x)$ by finding some curve lying within U, connecting x with a point $x^0 \in \Gamma$ and along which we can compute u. Since (2) says $u = g$ on Γ, we know the value of u at the one end x^0. We hope then to be able to calculate u all along the curve, and so in particular at x.

Finding the characteristic ODE. How can we choose the curve so all this will work? Let us suppose it is described parametrically by the function $\mathbf{x}(s) = (x^1(s), \dots, x^n(s))$, the parameter s lying in some subinterval of $\mathbb{R}$. Assuming u is a C^2 solution of (1), we define also

$$(3) \qquad z(s) := u(\mathbf{x}(s)).$$

In addition, set

$$(4) \qquad \mathbf{p}(s) := Du(\mathbf{x}(s));$$

that is, $\mathbf{p}(s) = (p^1(s), \dots, p^n(s))$, where

$$(5) \qquad p^i(s) = u_{x_i}(\mathbf{x}(s)) \quad (i = 1, \dots, n).$$

So $z(\cdot)$ gives the values of u along the curve and $\mathbf{p}(\cdot)$ records the values of the gradient Du. We must choose the function $\mathbf{x}(\cdot)$ in such a way that we can compute $z(\cdot)$ and $\mathbf{p}(\cdot)$.

For this, first differentiate (5):

$$(6) \qquad \dot{p}^i(s) = \sum_{j=1}^n u_{x_i x_j}(\mathbf{x}(s)) \dot{x}^j(s) \qquad \left(\cdot = \frac{d}{ds} \right).$$

This expression is not too promising, since it involves the second derivatives of u. On the other hand, we can also differentiate the PDE (1) with respect to x_i:

$$(7) \qquad \sum_{j=1}^n \frac{\partial F}{\partial p_j}(Du, u, x) u_{x_j x_i} + \frac{\partial F}{\partial z}(Du, u, x) u_{x_i} + \frac{\partial F}{\partial x_i}(Du, u, x) = 0.$$

We are able to employ this identity to get rid of the "dangerous" second derivative terms in (6), provided we first set

$$(8) \qquad \dot{x}^j(s) = \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \quad (j = 1, \dots, n).$$

Assuming now (8) holds, we evaluate (7) at $x = \mathbf{x}(s)$, obtaining thereby from (3), (4) the identity:

$$\sum_{j=1}^n \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) u_{x_i x_j}(\mathbf{x}(s)) + \frac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s)) p^i(s) + \frac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = 0.$$

Substitute this expression and (8) into (6):

$$(9) \qquad \begin{aligned} \dot{p}^i(s) = &-\frac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \\ &- \frac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s)) p^i(s) \quad (i = 1, \dots, n). \end{aligned}$$

Finally we differentiate (3):

$$(10) \qquad \dot{z}(s) = \sum_{j=1}^n \frac{\partial u}{\partial x_j}(\mathbf{x}(s)) \dot{x}^j(s) = \sum_{j=1}^n p^j(s) \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)),$$

the second equality holding by (5) and (8).

We summarize by rewriting equations (8)–(10) in vector notation:

$$(11) \qquad \begin{cases} \text{(a) } \dot{\mathbf{p}}(s) = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \mathbf{p}(s) \\ \text{(b) } \dot{z}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ \text{(c) } \dot{\mathbf{x}}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)). \end{cases}$$

This important system of $2n + 1$ first-order ODE comprises the *characteristic equations* of the nonlinear first-order PDE (1). The functions $\mathbf{p}(\cdot) = (p^1(\cdot), \dots, p^n(\cdot))$, $z(\cdot)$, $\mathbf{x}(\cdot) = (x^1(\cdot), \dots, x^n(\cdot))$ are called the *characteristics*. We will sometimes refer to $\mathbf{x}(\cdot)$ as the *projected characteristic*: it is the projection of the full characteristics $(\mathbf{p}(\cdot), z(\cdot), \mathbf{x}(\cdot)) \subset \mathbb{R}^{2n+1}$ onto the physical region $U \subset \mathbb{R}^n$.

We have proved:

THEOREM 1 (Structure of characteristic ODE). *Let $u \in C^2(U)$ solve the nonlinear, first-order partial differential equation (1) in U. Assume $\mathbf{x}(\cdot)$ solves the* ODE (11)(c), *where $\mathbf{p}(\cdot) = Du(\mathbf{x}(\cdot))$, $z(\cdot) = u(\mathbf{x}(\cdot))$. Then $\mathbf{p}(\cdot)$ solves the* ODE (11)(a) *and $z(\cdot)$ solves the* ODE (11)(b), *for those s such that $\mathbf{x}(s) \in U$.*

We still need to discover appropriate initial conditions for the system of ODE (11), in order that this theorem be useful. We accomplish this in §3.2.3 below.

Remark. The characteristic ODE are truly remarkable in that they form a closed system of equations for $\mathbf{x}(\cdot)$, $z(\cdot) = u(\mathbf{x}(\cdot))$, and $\mathbf{p}(\cdot) = Du(\mathbf{x}(\cdot))$, whenever u is a smooth solution of the general nonlinear PDE (1). The key step in the derivation is our setting $\dot{\mathbf{x}} = D_pF$, so that—as explained above—the terms involving second derivatives drop out. We thereby obtain *closure*, and in particular are not forced to introduce ODE for the second and higher derivatives of u. □

3.2.2. Examples.

Before continuing our investigation of the characteristic equations (11), we pause to consider some special cases for which the structure of these equations is especially simple. We illustrate as well how we can sometimes actually solve the characteristic ODE and thereby explicitly compute solutions of certain first-order PDE, subject to appropriate boundary conditions.

a. F linear.

Consider first the situation that our PDE (1) is linear and homogeneous, and thus has the form

$$F(Du, u, x) = \mathbf{b}(x) \cdot Du(x) + c(x)u(x) = 0 \quad (x \in U). \tag{12}$$

Then $F(p, z, x) = \mathbf{b}(x) \cdot p + c(x)z$, and so

$$D_pF = \mathbf{b}(x). \tag{13}$$

In this circumstance equation (11)(c) becomes

$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)), \tag{14}$$

an ODE involving only the function $\mathbf{x}(\cdot)$. Furthermore equation (11)(b) becomes

$$\dot{z}(s) = \mathbf{b}(\mathbf{x}(s)) \cdot \mathbf{p}(s). \tag{15}$$

Since $\mathbf{p}(\cdot) = Du(\mathbf{x}(\cdot))$, equation (12) simplifies (15), yielding

$$\dot{z}(s) = -c(\mathbf{x}(s))z(s). \tag{16}$$

This ODE is linear in $z(\cdot)$, once we know the function $\mathbf{x}(\cdot)$ by solving (14). In summary,

$$\begin{cases} \text{(a)} & \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)) \\ \text{(b)} & \dot{z}(s) = -c(\mathbf{x}(s))z(s) \end{cases} \tag{17}$$

comprise the characteristic equations for the linear, first-order PDE (12). (We will see later that the equation for $\mathbf{p}(\cdot)$ is not needed.) □

Example 1. We demonstrate the utility of equations (17) by explicitly solving the problem

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in } U \\ u = g & \text{on } \Gamma, \end{cases} \tag{18}$$

where U is the quadrant $\{x_1 > 0, x_2 > 0\}$ and $\Gamma = \{x_1 > 0, x_2 = 0\} \subseteq \partial U$. The PDE in (18) is of the form (12), for $\mathbf{b} = (-x_2, x_1)$ and $c = -1$. Thus the equations (17) read

$$\begin{cases} \dot{x}^1 = -x^2, \ \dot{x}^2 = x^1 \\ \dot{z} = z. \end{cases} \tag{19}$$

Accordingly we have

$$\begin{cases} x^1(s) = x^0 \cos s, \ x^2(s) = x^0 \sin s \\ z(s) = z^0 e^s = g(x^0)\, e^s, \end{cases}$$

where $x^0 \geq 0$, $0 \leq s \leq \frac{\pi}{2}$. Fix a point $(x_1, x_2) \in U$. We select $s > 0$, $x^0 > 0$ so that $(x_1, x_2) = (x^1(s), x^2(s)) = (x^0 \cos s, x^0 \sin s)$. That is, $x^0 = (x_1^2 + x_2^2)^{1/2}$, $s = \arctan\left(\frac{x_2}{x_1}\right)$. Therefore

$$\begin{aligned} u(x_1, x_2) &= u(x^1(s), x^2(s)) = z(s) = g(x^0)\, e^s \\ &= g((x_1^2 + x_2^2)^{1/2})\, e^{\arctan\left(\frac{x_2}{x_1}\right)}. \end{aligned}$$

□

b. F quasilinear.

The partial differential equation (1) is quasilinear should it have the form

(20) $$F(Du,u,x) = \mathbf{b}(x,u(x)) \cdot Du(x) + c(x,u(x)) = 0.$$

In this circumstance $F(p,z,x) = \mathbf{b}(x,z)\cdot p + c(x,z)$; whence

$$D_pF = \mathbf{b}(x,z).$$

Hence equation (11)(c) reads

$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s), z(s)),$$

and (11)(b) becomes

$$\begin{aligned}\dot{z}(s) &= \mathbf{b}(\mathbf{x}(s),z(s))\cdot \mathbf{p}(s)\\ &= -c(\mathbf{x}(s),z(s)), \quad \text{by (20)}.\end{aligned}$$

Consequently

(21) $$\begin{cases}\text{(a)} & \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s),z(s))\\ \text{(b)} & \dot{z}(s) = -c(\mathbf{x}(s),z(s))\end{cases}$$

are the characteristic equations for the quasilinear first-order PDE (20). (Once again the equation for $\mathbf{p}(\cdot)$ is not needed.) □

Example 2. The characteristic ODE (21) are in general difficult to solve, and so we work out in this example the simpler case of a boundary-value problem for a semilinear PDE:

(22) $$\begin{cases} u_{x_1} + u_{x_2} = u^2 & \text{in } U\\ u = g & \text{on } \Gamma.\end{cases}$$

Now U is the half-space $\{x_2 > 0\}$ and $\Gamma = \{x_2 = 0\} = \partial U$. Here $\mathbf{b} = (1,1)$ and $c = -z^2$. Then (21) becomes

$$\begin{cases}\dot{x}^1 = 1, \ \dot{x}^2 = 1\\ \dot{z} = z^2.\end{cases}$$

Consequently

$$\begin{cases} x^1(s) = x^0 + s, \ x^2(s) = s\\ z(s) = \frac{z^0}{1-sz^0} = \frac{g(x^0)}{1-sg(x^0)},\end{cases}$$

where $x^0 \in \mathbb{R}$, $s \geq 0$, provided the denominator is not zero.

Fix a point $(x_1,x_2) \in U$. We select $s > 0$ and $x^0 \in \mathbb{R}$ so that $(x_1,x_2) = (x^1(s),x^2(s)) = (x^0+s,s)$; that is, $x^0 = x_1 - x_2$, $s = x_2$. Then

$$\begin{aligned} u(x_1,x_2) = u(x^1(s),x^2(s)) = z(s) &= \frac{g(x^0)}{1-sg(x^0)}\\ &= \frac{g(x_1-x_2)}{1-x_2g(x_1-x_2)}.\end{aligned}$$

This solution of course makes sense only if $1 - x_2g(x_1-x_2) \neq 0$. □

c. F fully nonlinear.

In the general case, the full characteristic equations (11) must be integrated, if possible.

Example 3. Consider the fully nonlinear problem

(23) $$\begin{cases} u_{x_1}u_{x_2} = u & \text{in } U\\ u = x_2^2 & \text{on } \Gamma,\end{cases}$$

where $U = \{x_1 > 0\}$, $\Gamma = \{x_1 = 0\} = \partial U$. Here $F(p,z,x) = p_1p_2 - z$, and hence the characteristic ODE (11) become

$$\begin{cases}\dot{p}^1 = p^1, \ \dot{p}^2 = p^2\\ \dot{z} = 2p^1p^2\\ \dot{x}^1 = p^2, \ \dot{x}^2 = p^1.\end{cases}$$

We integrate these equations to find

$$\begin{cases} x^1(s) = p_2^0(e^s-1), \ x^2(s) = x^0 + p_1^0(e^s-1)\\ z(s) = z^0 + p_1^0p_2^0(e^{2s}-1)\\ p^1(s) = p_1^0e^s\, p^2(s) = p_2^0e^s,\end{cases}$$

where $x^0 \in \mathbb{R}$, $s \in \mathbb{R}$, and $z^0 = (x^0)^2$.

We must determine $p^0 = (p_1^0,p_2^0)$. Since $u = x_2^2$ on Γ, $p_2^0 = u_{x_2}(0,x^0) = 2x^0$. Furthermore the PDE $u_{x_1}u_{x_2} = u$ itself implies $p_1^0p_2^0 = z^0 = (x^0)^2$, and so $p_1^0 = \frac{x^0}{2}$. Consequently the formulas above become

$$\begin{cases} x^1(s) = 2x^0(e^s-1), x^2(s) = \frac{x^0}{2}(e^s+1)\\ z(s) = (x^0)^2e^{2s}\\ p^1(s) = \frac{x^0}{2}e^s, \ p^2(s) = 2x^0e^s.\end{cases}$$

Fix a point $(x_1,x_2) \in U$. Select s and x^0 so that $(x_1,x_2) = (x^1(s),x^2(s)) = (2x^0(e^s-1), \frac{x^0}{2}(e^s+1))$. This equality implies $x^0 = \frac{4x_2-x_1}{4}$, $e^s = \frac{x_1+4x_2}{4x_2-x_1}$; and so

$$\begin{aligned} u(x_1,x_2) = u(x^1(s),x^2(s)) = z(s) &= (x^0)^2e^{2s}\\ &= \frac{(x_1+4x_2)^2}{16}.\end{aligned}$$

□

3.2.3. Boundary conditions.

We return now to developing the general theory.

a. Straightening the boundary.

We intend in the section following to invoke the characteristic ODE (11) actually to solve the boundary-value problem (1), (2), at least in a small region near an appropriate portion Γ of ∂U. In order to simplify the relevant calculations, it is convenient first to change variables, so as to "flatten out" part of the boundary ∂U. To accomplish this, we first fix any point $x^0 \in \partial U$. Then utilizing the notation from §C.1, we find smooth mappings $\mathbf{\Phi}, \mathbf{\Psi} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{\Psi} = \mathbf{\Phi}^{-1}$ and $\mathbf{\Phi}$ straightens out ∂U near x^0. (See the illustration in §C.1.)

Given any function $u : U \to \mathbb{R}$, let us write $V := \mathbf{\Phi}(U)$ and set

$$v(y) := u(\mathbf{\Psi}(y)) \quad (y \in V). \tag{24}$$

Then

$$u(x) = v(\mathbf{\Phi}(x)) \quad (x \in U). \tag{25}$$

Now suppose that u is a C^1 solution of our boundary-value problem (1), (2) in U. What PDE does v then satisfy in V?

According to (25), we see

$$u_{x_i}(x) = \sum_{k=1}^{n} v_{y_k}(\mathbf{\Phi}(x))\Phi^k_{x_i}(x) \quad (i = 1, \dots, n);$$

that is,

$$Du(x) = Dv(y)D\mathbf{\Phi}(x).$$

Thus (1) implies

$$\begin{aligned} 0 &= F(Du(x), u(x), x) \\ &= F(Dv(y)D\mathbf{\Phi}(\mathbf{\Psi}(y)), v(y), \mathbf{\Psi}(y)). \end{aligned} \tag{26}$$

This is an expression having the form

$$G(Dv(y), v(y), y) = 0 \quad \text{in } V.$$

In addition $v = h$ on Δ, where $\Delta := \mathbf{\Phi}(\Gamma)$ and $h(y) := g(\mathbf{\Psi}(y))$.

In summary, our problem (1), (2) transforms to read

$$\begin{cases} G(Dv, v, y) = 0 & \text{in } V \\ v = h & \text{on } \Delta, \end{cases} \tag{27}$$

for G, h as above. The point is that if we change variables to straighten out the boundary near x^0, the boundary-value problem (1), (2) converts into a problem having the same form.

b. Compatibility conditions on boundary data.

In view of the foregoing computations, if we are given a point $x^0 \in \Gamma$ we may as well assume from the outset that Γ is flat near x^0, lying in the plane $\{x_n = 0\}$.

We intend now to utilize the characteristic ODE to construct a solution (1), (2), at least near x^0, and for this we must discover appropriate initial conditions

$$\mathbf{p}(0) = p^0, \ z(0) = z^0, \ \mathbf{x}(0) = x^0. \tag{28}$$

Now clearly if the curve $\mathbf{x}(\cdot)$ passes through x^0, we should insist that

$$z^0 = g(x^0). \tag{29}$$

What should we require concerning $\mathbf{p}(0) = p^0$? Since (2) implies $u(x_1 \dots x_{n-1}, 0) = g(x_1 \dots x_{n-1})$ near x^0, we may differentiate to find

$$u_{x_i}(x^0) = g_{x_i}(x^0) \quad (i = 1, \dots, n-1).$$

As we also want the PDE (1) to hold, we should therefore insist $p^0 = (p^0_1, \dots, p^0_n)$ satisfies these relations:

$$\begin{cases} p^0_i = g_{x_i}(x^0) & (i = 1, \dots, n-1) \\ F(p^0, z^0, x^0) = 0. \end{cases} \tag{30}$$

These identities provide n equations for the n quantities $p^0 = (p^0_1, \dots, p^0_n)$.

We call (29) and (30) the *compatibility conditions*. A triple $(p^0, z^0, x^0) \in \mathbb{R}^{2n+1}$ verifying (29), (30) is *admissible*. Note z^0 is uniquely determined by the boundary condition and our choice of the point x^0, but a vector p^0 satisfying (30) may not exist or may not be unique.

c. Noncharacteristic boundary data.

So now assume as above that $x^0 \in \Gamma$, that Γ near x^0 lies in the plane $\{x_n = 0\}$, and that the triple (p^0, z^0, x^0) is admissible. We are planning to construct a solution u of (1), (2) in U near x^0 by integrating the characteristic ODE (11). So far we have ascertained $\mathbf{x}(0) = x^0$, $z(0) = z^0$, $\mathbf{p}(0) = p^0$ are appropriate boundary conditions for the characteristic ODE, with $\mathbf{x}(\cdot)$ intersecting Γ at x^0. But we will need in fact to solve these ODE for *nearby* initial points as well, and must consequently now ask if we can somehow appropriately perturb (p^0, z^0, x^0), keeping the compatibility conditions.

In other words, given a point $y = (y_1, \dots, y_{n-1}, 0) \in \Gamma$, with y close to x^0, we intend to solve the characteristic ODE

$$(31)\quad \begin{cases} \text{(a) } \dot{\mathbf{p}}(s) = -D_xF(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_zF(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s) \\ \text{(b) } \dot{z}(s) = D_pF(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ \text{(c) } \dot{\mathbf{x}}(s) = D_pF(\mathbf{p}(s), z(s), \mathbf{x}(s)), \end{cases}$$

with the initial conditions

$$(32)\quad \mathbf{p}(0) = \mathbf{q}(y),\ z(0) = g(y),\ \mathbf{x}(0) = y.$$

Our task then is to find a function $\mathbf{q}(\cdot) = (q^1(\cdot), \dots, q^n(\cdot))$, so that

$$(33)\quad \mathbf{q}(x^0) = p^0$$

and $(\mathbf{q}(y), g(y), y)$ is admissible; that is, the compatibility conditions

$$(34)\quad \begin{cases} q^i(y) = g_{x_i}(y) & (i = 1, \dots, n-1) \\ F(\mathbf{q}(y), g(y), y) = 0 \end{cases}$$

hold for all $y \in \Gamma$ close to x^0.

LEMMA 1 (Noncharacteristic boundary conditions). *There exists a unique solution $\mathbf{q}(\cdot)$ of (33), (34) for all $y \in \Gamma$ sufficiently close to x^0, provided*

$$(35)\quad F_{p_n}(p^0, z^0, x^0) \neq 0.$$

We say the admissible triple (p^0, z^0, x^0) is *noncharacteristic* if (35) holds. We henceforth assume this condition.

Proof. To simplify notation, let us now temporarily write $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. We apply the Implicit Function Theorem (§C.6) to the mapping

$$\mathbf{G} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ \mathbf{G}(p, y) = (G^1(p, y), \dots, G^n(p, y)),$$

where

$$\begin{cases} G^i(p, y) = p_i - g_{x_i}(y) & (i = 1, \dots, n-1) \\ G^n(p, y) = F(p, g(y), y). \end{cases}$$

Now $\mathbf{G}(p^0, x^0) = 0$, according to (29), (30). Also

$$D_p\mathbf{G}(p^0, x^0) = \begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ F_{p_1}(p^0, z^0, x^0) & \dots & & F_{p_n}(p^0, z^0, x^0) \end{pmatrix}_{n \times n},$$

and thus

$$\det D_p\mathbf{G}(p^0, x^0) = F_{p_n}(p^0, z^0, x^0) \neq 0,$$

in view of the noncharacteristic condition (35). The Implicit Function Theorem thus ensures we can uniquely solve the identity $G(p, y) = 0$ for $p = \mathbf{q}(y)$, provided y is close enough to x^0. □

Remark. If Γ is not flat near x^0, the condition that Γ be noncharacteristic reads

$$(36)\quad D_pF(p^0, z^0, x^0) \cdot \nu(x_0) \neq 0,$$

$\nu(x^0)$ denoting the outward unit normal to ∂U at x^0. □

3.2.4. Local solution.

Remember that our aim is to use the characteristic ODE to build a solution u of (1), (2), at least near Γ. So as before we select a point $x^0 \in \Gamma$ and, as shown in §3.2.3, may as well assume that near x^0 the surface Γ is flat, lying in the plane $\{x_n = 0\}$. Suppose further that (p^0, z^0, x^0) is an admissible triple of boundary data, which is noncharacteristic. According to Lemma 1 there is a function $\mathbf{q}(\cdot)$ so that $p^0 = \mathbf{q}(x^0)$ and the triple $(\mathbf{q}(y), g(y), y)$ is admissible, for all y sufficiently close to x^0.

Given any such point $y = (y_1, \dots, y_{n-1}, 0)$, we solve the characteristic ODE (31), subject to initial conditions (32).

Notation. Let us write

$$\begin{cases} \mathbf{p}(s) = \mathbf{p}(y, s) = \mathbf{p}(y_1, \dots, y_{n-1}, s) \\ z(s) = z(y, s) = z(y_1, \dots, y_{n-1}, s) \\ \mathbf{x}(s) = \mathbf{x}(y, s) = \mathbf{x}(y_1, \dots, y_{n-1}, s) \end{cases}$$

to display the dependence of the solution of (31), (32) on s and y. □

LEMMA 2 (Local invertibility). *Assume we have the noncharacteristic condition $F_{p_n}(p^0, z^0, x^0) \neq 0$. Then there exist an open interval $I \subset \mathbb{R}$ containing 0, a neighborhood W of x^0 in $\Gamma \subset \mathbb{R}^{n-1}$, and a neighborhood V of x^0 in $\mathbb{R}^n$, such that for each $x \in V$ there exist unique $s \in I$, $y \in W$ such that*

$$x = \mathbf{x}(y, s).$$

The mappings $x \mapsto s, y$ are C^2.

Proof. We have $\mathbf{x}(x^0, 0) = x^0$. Consequently the Inverse Function Theorem (§C.5) gives the result, provided $\det D\mathbf{x}(x^0, 0) \neq 0$. Now

$$\mathbf{x}(y, 0) = (y, 0) \quad (y \in \Gamma),$$

and so if $i = 1, \dots, n-1$,

$$\frac{\partial x^j}{\partial y_i}(x^0, 0) = \begin{cases} \delta_{ij} & (j = 1, \dots n-1) \\ 0 & (j = n). \end{cases}$$

Furthermore equation (31)(c) implies

$$\frac{\partial x^j}{\partial s}(x^0,0) = F_{p_j}(p^0,z^0,x^0).$$

Thus

$$D\mathbf{x}(x^0,0) = \begin{pmatrix} 1 & & 0 & F_{p_1}(p^0,z^0,x^0) \\ & \ddots & & \vdots \\ 0 & & 1 & \vdots \\ 0 & \cdots & 0 & F_{p_n}(p^0,z^0,x^0) \end{pmatrix}_{n\times n};$$

whence $\det D\mathbf{x}(x^0,0) \neq 0$ follows from the noncharacteristic condition (35). □

In view of Lemma 2 for each $x \in V$, we can locally uniquely solve the equation

$$\text{(37)} \qquad \begin{cases} x = \mathbf{x}(y,s), \\ \text{for } \ y = \mathbf{y}(x), \ s = s(x). \end{cases}$$

Finally, let us define

$$\text{(38)} \qquad \begin{cases} u(x) := z(\mathbf{y}(x), s(x)) \\ \mathbf{p}(x) := \mathbf{p}(\mathbf{y}(x), s(x)) \end{cases}$$

for $x \in V$ and s, y as in (37).

We come finally to our principal assertion, namely, that we can locally weave together the solutions of the characteristic ODE into a solution of the PDE.

THEOREM 2 (Local Existence Theorem). *The function u defined above is C^2 and solves the* PDE

$$F(Du(x),u(x),x) = 0 \quad (x \in V),$$

with the boundary condition

$$u(x) = g(x) \quad (x \in \Gamma \cap V).$$

Proof. 1. First of all, fix $y \in \Gamma$ close to x^0 and, as above, solve the characteristic ODE (31), (32) for $\mathbf{p}(s) = \mathbf{p}(y,s)$, $z(s) = z(y,s)$, and $\mathbf{x}(s) = \mathbf{x}(y,s)$.

2. We assert that if $y \in \Gamma$ is sufficiently close to x^0, then

$$\text{(39)} \qquad f(y,s) := F(\mathbf{p}(y,s), z(y,s), \mathbf{x}(y,s)) = 0 \quad (s \in \mathbb{R}).$$

To see this, note

$$\text{(40)} \qquad \begin{aligned} f(y,0) &= F(\mathbf{p}(y,0), z(y,0), \mathbf{x}(y,0)) \\ &= F(\mathbf{q}(y), g(y), y) = 0, \end{aligned}$$

by the compatibility condition (34). Furthermore

$$\begin{aligned} \frac{\partial f}{\partial s}(y,s) &= \sum_{j=1}^n \frac{\partial F}{\partial p_j}\dot{p}^j + \frac{\partial F}{\partial z}\dot{z} + \sum_{j=1}^n \frac{\partial F}{\partial x_j}\dot{x}^j \\ &= \sum_{j=1}^n \frac{\partial F}{\partial p_j}\left(-\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial z}p^j\right) + \frac{\partial F}{\partial z}\left(\sum_{j=1}^n \frac{\partial F}{\partial p_j}p^j\right) \\ &\quad + \sum_{j=1}^n \frac{\partial F}{\partial x_j}\left(\frac{\partial F}{\partial p_j}\right), \qquad \text{according to (31)} \\ &= 0. \end{aligned}$$

This calculation and (40) prove (39).

3. In view of Lemma 2 and (37)–(39), we have

$$F(\mathbf{p}(x), u(x), x) = 0 \quad (x \in V).$$

To conclude, we must therefore show

$$\text{(41)} \qquad \mathbf{p}(x) = Du(x) \quad (x \in V).$$

4. In order to prove (41), let us first demonstrate for $s \in I$, $y \in W$ that

$$\text{(42)} \qquad \frac{\partial z}{\partial s}(y,s) = \sum_{j=1}^n p^j(y,s)\frac{\partial x^j}{\partial s}(y,s)$$

and

$$\text{(43)} \qquad \frac{\partial z}{\partial y_i}(y,s) = \sum_{j=1}^n p^j(y,s)\frac{\partial x^j}{\partial y_i}(y,s) \quad (i = 1,\dots,n-1).$$

These formulas are obviously consistent with the equality (41) and will later help us prove it. The identity (42) results at once from the characteristic ODE (31)(b),(c). To establish (43), fix $y \in \Gamma$, $i \in \{1,\dots,n-1\}$, and set

$$\text{(44)} \qquad r^i(s) := \frac{\partial z}{\partial y_i}(y,s) - \sum_{j=1}^n p^j(y,s)\frac{\partial x^j}{\partial y_i}(y,s).$$

We first note $r^i(0) = g_{x_i}(y) - q^i(y) = 0$ according to the compatibility condition (34). In addition, we can compute

$$\dot r^i(s) = \frac{\partial^2 z}{\partial y_i \partial s} - \sum_{j=1}^n \left[\frac{\partial p^j}{\partial s}\frac{\partial x^j}{\partial y_i} + p^j \frac{\partial^2 x^j}{\partial y_i \partial s}\right]. \tag{45}$$

In order to simplify this expression, let us first differentiate the identity (42) with respect to y_i:

$$\frac{\partial^2 z}{\partial s \partial y_i} = \sum_{j=1}^n \left[\frac{\partial p^j}{\partial y_i}\frac{\partial x^j}{\partial s} + p^j \frac{\partial^2 x^j}{\partial s \partial y_i}\right]. \tag{46}$$

Substituting (46) into (45), we discover

$$\begin{aligned} \dot r^i(s) &= \sum_{j=1}^n \left[\frac{\partial p^j}{\partial y_i}\frac{\partial x^j}{\partial s} - \frac{\partial p^j}{\partial s}\frac{\partial x^j}{\partial y_i}\right] \\ &= \sum_{j=1}^n \left[\frac{\partial p^j}{\partial y_i}\left(\frac{\partial F}{\partial p_j}\right) - \left(-\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial z}p^j\right)\frac{\partial x^j}{\partial y_i}\right] \quad \text{by (31)(a).} \end{aligned} \tag{47}$$

Now differentiate (39) with respect to y_i:

$$\sum_{j=1}^n \frac{\partial F}{\partial p_j}\frac{\partial p^j}{\partial y_i} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y_i} + \sum_{j=1}^n \frac{\partial F}{\partial x_j}\frac{\partial x^j}{\partial y_i} = 0.$$

We employ this identity in (47), thereby obtaining

$$\dot r^i(s) = \frac{\partial F}{\partial z}\left[\sum_{j=1}^n p^j \frac{\partial x^j}{\partial y_i} - \frac{\partial z}{\partial y_i}\right] = -\frac{\partial F}{\partial z} r^i(s). \tag{48}$$

Hence $r^i(\cdot)$ solves the linear ODE (48), with the initial condition $r^i(0) = 0$. Consequently $r^i(s) = 0$ ($s \in \mathbb{R}$, $i = 1, \dots, n-1$); and so identity (43) is verified.

5. We finally employ (42), (43) in proving (41). Indeed, if $j = 1, \dots, n$,

$$\begin{aligned} \frac{\partial u}{\partial x_j} &= \frac{\partial z}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial z}{\partial y_i}\frac{\partial y^i}{\partial x_j} \quad \text{by (38)} \\ &= \left(\sum_{k=1}^n p^k \frac{\partial x^k}{\partial s}\right)\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\left(\sum_{k=1}^n p^k \frac{\partial x^k}{\partial y_i}\right)\frac{\partial y^i}{\partial x_j} \quad \text{by (42), (43)} \\ &= \sum_{k=1}^n p^k \left(\frac{\partial x^k}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial x^k}{\partial y_i}\frac{\partial y^i}{\partial x_j}\right) \\ &= \sum_{k=1}^n p^k \frac{\partial x^k}{\partial x_j} = \sum_{k=1}^n p^k \delta_{jk} = p^j. \end{aligned}$$

This assertion at last establishes (41), and so finishes up the proof. □

3.2.5. Applications.

We turn now to various special cases, to see how the local existence theory simplifies in these circumstances.

a. F linear.

Recall that a linear, homogeneous, first-order PDE has the form

$$F(Du, u, x) = \mathbf{b}(x) \cdot Du(x) + c(x)u(x) = 0 \quad (x \in U). \tag{49}$$

Our noncharacteristic assumption (36) at a point $x^0 \in \Gamma$ as above becomes

$$\mathbf{b}(x^0) \cdot \nu(x^0) \neq 0, \tag{50}$$

and thus does not involve z^0 or p^0 at all. Furthermore if we specify the boundary condition

$$u = g \quad \text{on } \Gamma, \tag{51}$$

we can uniquely solve equation (34) for $\mathbf{q}(y)$ if $y \in \Gamma$ is near x^0. Thus we can apply the Local Existence Theorem 2 to construct a unique solution of (49), (51) in some neighborhood V containing x^0. Note carefully that although we have utilized the full characteristic equations (31) in the proof of Theorem 2, once we know the solution exists, we can use the reduced equations (17) (which do not involve $\mathbf{p}(\cdot)$) to compute the solution. Observe also the projected characteristics $\mathbf{x}(\cdot)$ emanating from distinct points on Γ cannot cross, owing to uniqueness of solutions of the initial-value problem for the ODE (17)(a).

Example 4. Suppose the trajectories of the ODE

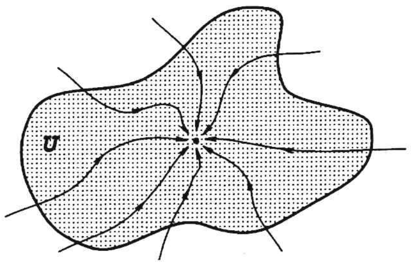
$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)) \tag{52}$$

are as drawn for Case 1.

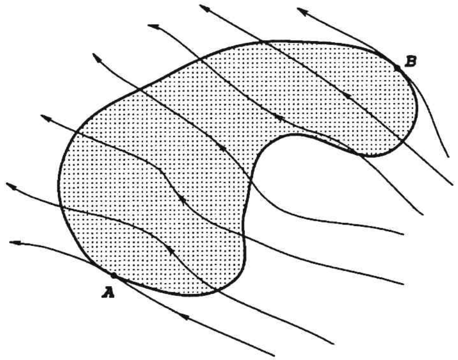
We are thus assuming the vector field $\mathbf{b}$ vanishes within U only at one point, which we will take to be the origin 0, and $\mathbf{b} \cdot \nu < 0$ on $\Gamma := \partial U$. Can we solve the linear boundary-value problem

$$\begin{cases} \mathbf{b} \cdot Du = 0 & \text{in } U \\ u = g & \text{on } \Gamma \text{ ?} \end{cases} \tag{53}$$

Invoking Theorem 2 we see that there exists a unique solution u defined *near* Γ, and indeed that $u(\mathbf{x}(s)) \equiv u(\mathbf{x}(0)) = g(x^0)$ for each solution of the



Case 1: flow to an attracting point



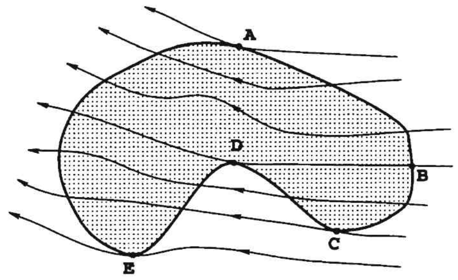
Case 2: flow across a domain

ODE (52), with the initial condition $\mathbf{x}(0) = x^0 \in \Gamma$. However, this solution cannot be smoothly continued to all of U (unless g is constant): any smooth solution of (53) is constant on trajectories of (52), and thus takes on different values near $x = 0$.

On the other hand, now suppose the trajectories of the ODE (52) look like the illustration for Case 2. We are consequently now assuming that each trajectory of the ODE (except those through the characteristic points A, B) enters U precisely once, somewhere through the set

$$\Gamma := \{x \in \partial U \mid \mathbf{b}(x) \cdot \nu(x) < 0\},$$

and exits U precisely once. In this circumstance we can find a smooth solution of (53) by setting u to be constant along each flow line.



Case 3: flow with characteristic points

Assume finally the flow looks like Case 3. We can now define u to be constant along trajectories, but then u will be discontinuous (unless $g(B) = g(D)$).

Note that the point D is characteristic and that the local existence theory fails near D. □

b. F quasilinear.

Should F be quasilinear, the PDE (1) becomes

$$F(Du, u, x) = \mathbf{b}(x, u) \cdot Du + c(x, u) = 0. \tag{54}$$

The noncharacteristic assumption (36) at a point $x^0 \in \Gamma$ reads $\mathbf{b}(x^0, z^0) \cdot \nu(x^0) \neq 0$, where $z^0 = g(x^0)$. As in the preceding example, if we specify the boundary condition

$$u = g \quad \text{on } \Gamma, \tag{55}$$

we can uniquely solve the equations (34) for $\mathbf{q}(y)$ if $y \in \Gamma$ near x^0. Thus Theorem 2 yields the existence of a unique solution of (54), (55) in some neighborhood V of x^0. We can compute this solution in V using the reduced characteristic equations (21), which do not explicitly involve $\mathbf{p}(\cdot)$.

In contrast to the linear case, however, *it is possible that the projected characteristics emanating from distinct points in Γ may intersect outside V*; such an occurrence usually signals the failure of our local solution to exist within all of U.

Example 5 (Characteristics for conservation laws). As an instance of a quasilinear first-order PDE, we turn now to the *scalar conservation law*

$$\begin{aligned} G(Du, u_t, u, x, t) &= u_t + \operatorname{div} \mathbf{F}(u) \\ &= u_t + \mathbf{F}'(u) \cdot Du = 0 \end{aligned} \tag{56}$$