LECTURE 12, April 6, 2023

OPTIMAL CONTROL: (S) $\begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s > t \\ y(t) = x \end{cases}$

$\alpha \in \mathcal{A}$ measurable $[0,T] \to A$. OPEN LOOP $\quad y(s) = y_x(s; \alpha, t)$.

$$J(x,t,\alpha(\cdot)) = \int_t^T \ell(y(s), \alpha(s))\,ds + g(y(T))$$

GOAL: $\min J$ (for $t=0$)

Ref. W. Fleming - R. Rishel: Deterministic & Stochastic optimal control, Springer 1975.

Methods (1) • Existence of an optimal control

• necessary conditions of optimality
(Pontryagin Maximum Principle ~ 60-70)

(2) Dynamic Programming Method: sufficient conditions of optimality & a "constructive strategy" to build optimal

DYNAMIC PROGRAMMING METHOD.

Consider the VALUE FUNCTION

$$v(x,t) := \inf_{\alpha \in \mathcal{A}} J(x,t,\alpha) = \text{"minimal cost as a function of initial position \& time."}$$

Will see v solves a H-J PDE.

• if you solve such equation, can build optimal controls from the solution.

HYPOTHES (simplified!) [E]

$$\begin{cases} g: \mathbb{R}^n \to \mathbb{R}, \quad \ell: \mathbb{R}^n \times A \to \mathbb{R} \quad \text{CONT.} \\ A \text{ compact } (\subseteq \mathbb{R}^m) \\ |\ell(x,a)|, |g(x)| \le C_1 \qquad \forall x, a, z \\ |g(x) - g(z)| \le C_1 |x-z| \\ |\ell(x,a) - \ell(z,a)| \le C_1 |x-z| \end{cases}$$

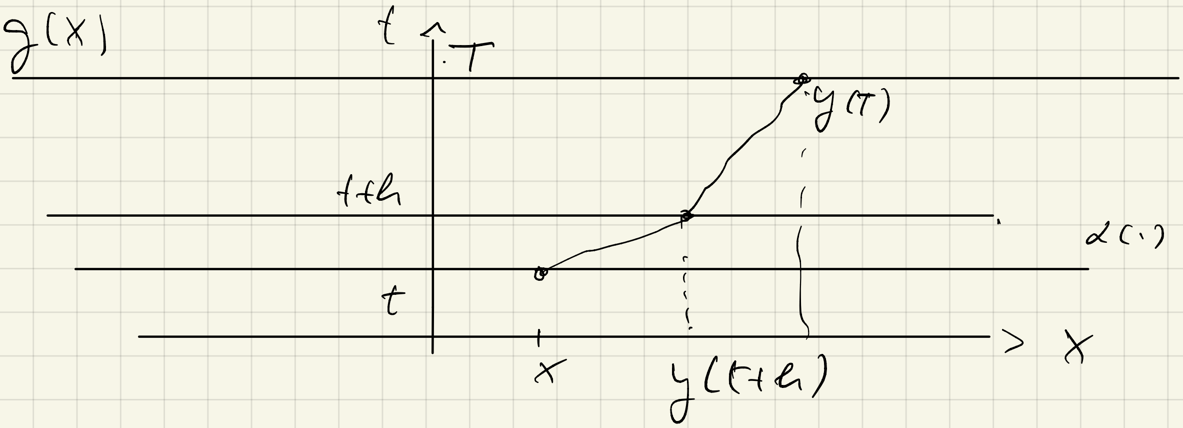
<u>Thm.</u>: (Dynamic Programming Principle: ... Caratheodory ..., R. Bellman, ... R. Isaacs ...) $\forall x \in \mathbb{R}^n$, $0 \le t$, $h > 0$, $t+h < T$

$$V(x,t) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_t^{t+h} \ell(y(s), \alpha(s))\,ds + V(y(t+h), t+h) \right\} \qquad \text{(DPP)}$$

where $y(s) = y_x(s; \alpha, t)$.

<u>Rmk.</u> if $t+h = T$ DPP = def. of value

$V(x,T) = g(x)$



<u>Def</u> $\varepsilon > 0$, $\bar{\alpha} \in \mathcal{A}$ is ε-optimal for J at (x,t)

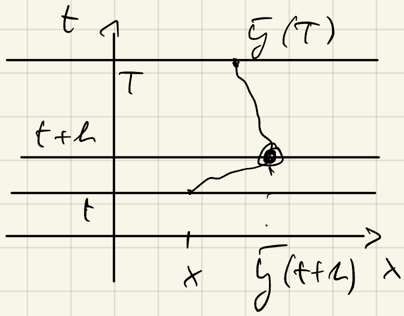
if $J(x,t,\bar{\alpha}) \le V(x,t) + \varepsilon$.

<u>Proof of DPP</u> Only Mayer pb. $\ell \equiv 0$

GOAL: $V(x,t) = \inf_{\alpha \in \mathcal{A}} V(y(t+h), t+h))$

" $\geq$ " $\varepsilon > 0$, $\bar{\alpha}$ ε-opt. for (x,t). $\bar{y}(s) = y_x(s;\bar{\alpha},t)$

$$v(x,t) \geq g(\bar{y}(T)) - \varepsilon$$



$$v(\bar{y}(t+h), t+h) \leq g(\bar{y}(T))$$

$$\Rightarrow \quad v(x,t) \geq v(\bar{y}(t+h), t+h)) - \varepsilon$$

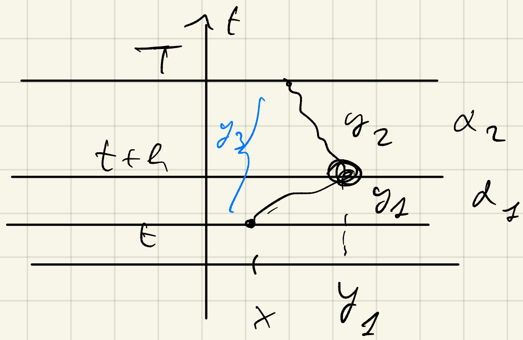
$$\Rightarrow \quad v(x,t) \geq \inf_{\alpha} v(y(t+h), t+h) - \varepsilon$$

$\varepsilon \to 0$

$\Rightarrow$ " $\geq$ " in DPP $\frac{1}{2}$ GOAL.

Step 2 " $\leq$ " Fix arbitrary $\alpha_1 \in \mathcal{A}$

$y_1(s) := y_x(s; \alpha_1, t)$ Fix $\varepsilon > 0$, α_2 ε-opt. for $(y_1(t+h), t+h)$ y_2 traj.



$$g(y_2(T)) \leq v(y_1(t+h), t+h) + \varepsilon$$

$$\alpha_3(s) = \begin{cases} \alpha_1(s) & t \leq s \leq t+h \\ \alpha_2(s) & t+h < s \leq T \end{cases}$$

$\alpha_3 \in \mathcal{A}$ $\leadsto$ y_3 the traj. $\begin{cases} \dot{y}_3 = f(y_3, \alpha_3) \\ y_3(t) = x \end{cases}$

$$y_3(s) = \begin{cases} y_1(s) & t \leq s \leq t+h \\ y_2(s) & t+h < s < T \end{cases}$$

by uniqueness of sol. of (S)

$$\Rightarrow \quad y_3(T) = y_2(T)$$

$$v(x,t) \leq J(x,t,\alpha_3) = g(y_3(T)) = g(y_2(T)) \leq v(y_1(t+h), t+h) + \varepsilon$$

$\forall \alpha_1 \in \mathcal{A}$

$$\Rightarrow \quad v(x,t) \le \inf_{\alpha} v\big(y_\cdot(t+h), t+h\big) + \varepsilon \to 0$$

$\Rightarrow$ GOAL "$\le$" in DPA $\boxtimes$

Prop Under the standing ass. on f, ℓ, g, A, $v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is bounded & Lipschitz.

Pf. Step 1 $\quad |v(x,t)| \le (T-t)\sup|\ell| + \sup|g| \le (T+1)C \quad \boxtimes$

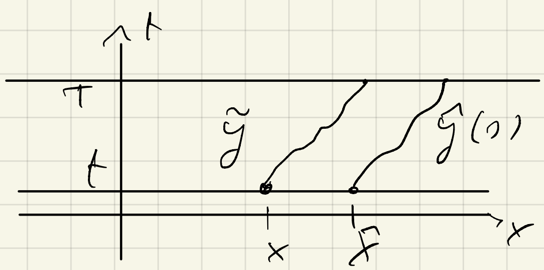
From now on $\ell \equiv 0$. general case [Ev].

Step 2 Lip in x Goal $\exists\, C'$:

$$|v(x,t) - v(\hat{x},t)| \le C' |x - \hat{x}|.$$

$v(x,t) - v(\hat{x},t) \le \ldots$ fix $\varepsilon > 0$, $\hat{\alpha}, \hat{y}$ ε-optimal for $(\hat{x},T)$

$v(\hat{x},t) \ge g(\hat{y}(T)) - \varepsilon$. Use $\hat{\alpha}$ for (x,t)



$\tilde{y}(s) = y_x(s; \hat{\alpha}, t)$ Use (E2)

$$|\tilde{y}(T) - \hat{y}(T)| \le e^{L(T-t)} |x - \hat{x}|$$

$$v(x,t) - v(\hat{x},t) \le g(\tilde{y}(T)) - g(\hat{y}(T)) - \varepsilon \le$$

$$\le C|\tilde{y}(T) - \hat{y}(T)| - \varepsilon \le \underbrace{C e^{L(T-t)}}_{\le C e^{LT} =: C'} |x - \hat{x}| - \varepsilon$$

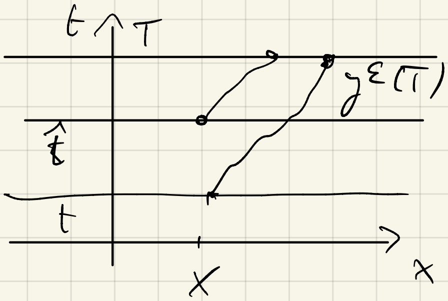
$\varepsilon \to 0$

$\le C' |x - \hat{x}|$. Repeat with $x, \hat{x}$ exchanged.

& get. $\quad |v(x,t) - v(\hat{x},t)| \le C' |x - \hat{x}| \quad \boxtimes$ Lip in x

Step 3 Lip in t : $0 \le t < \hat{t} \le T$

$\frac{1}{2}$. Goal : $\exists\, C''$: $v(x,\hat{t}) - v(x,t) \le C'' |\hat{t} - t| = C''(\hat{t} - t)$



Fix $\varepsilon > 0$: α^ε ε-opt. for (x,t)
$\rightarrow y^\varepsilon$ the trajectory.

$$\hat{\alpha}(s) = \alpha^\varepsilon(s - (\hat{t} - t)) \, , \quad \hat{y}(s) = y_x(s; \hat{\alpha}, \hat{t})$$

$\hat{y}$ solves $(\hat{S}) \begin{cases} \dot{y} = f(y, \hat{\alpha}) \\ y(\hat{t}) = x \end{cases}$

CLAIM : $\hat{y}(s) = y^\varepsilon(s - (\hat{t} - t))$ (Eq)

Pf of claim. $\hat{y}(\hat{t}) = x \overset{?}{=} y^\varepsilon(\hat{t} - (\hat{t} - t)) = y^\varepsilon(t) = x$ OK

$$\frac{dy^\varepsilon}{ds}(s - (\hat{t} - t)) = f\big(y^\varepsilon(s - (\hat{t} - t)), \underbrace{\alpha^\varepsilon(s - (\hat{t} - t))}_{\hat{\alpha}(s)}\big)$$

$\Rightarrow y^\varepsilon(\cdot - (\hat{t} - t))$ solves $(\hat{S})$, by uniqueness

$\Rightarrow$ (Eq).

$$v(x,\hat{t}) - v(x,t) \overset{y^\varepsilon_t\ \varepsilon\text{-opt.}}{\le} g(\hat{y}(T)) - g(y^\varepsilon(T)) + \varepsilon =$$

$$\overset{(Eq)}{=} g(y^\varepsilon(T - (\hat{t} - t))) - g(y^\varepsilon(T)) + \varepsilon$$

$g \in \mathrm{Lip}$

$$\le C_1 |y^\varepsilon(T - (\hat{t} - t)) - y^\varepsilon(T)| + \varepsilon$$

$y(\cdot) \in \mathrm{Lip}$

$$\le \underbrace{C_1 M}_{C''}(\hat{t} - t) + \varepsilon \rightarrow 0$$

$$\Rightarrow v(x,\hat{t}) - v(x,t) \le C''(\hat{t} - t).$$

Step. 5. $v(x,t) - v(x,\hat{t}) \leq C_3(\hat{t}-t)$

Pf. : SIMILAR, skip it. or see [EV.]. ☒

Def. $$H(p,x) := \max_{a\in A} \{-f(x,a)\cdot p - \ell(x,a)\}.$$
$$= -\min_{a\in A} \{f(x,a)\cdot p + \ell(x,a)\}.$$

Thm. Under the standing ass. on f, ℓ, g, A, the value function v is the UNIQUE VISCOS. SOLUT. in $BUC(\mathbb{R}^n\times[0,T])$ of the TERMINAL VALUE PROBLEM

$$(CT) \quad \begin{cases} -u_t + H(D_x u, x) = 0 & \text{in } \Omega = \mathbb{R}^n\times]0,T[\\ u(x,T) = g(x). & \text{in } \mathbb{R}^n. \end{cases}$$

OK $\longrightarrow$

Proof. Part 1 : v solves (CT)

Part 2 : UNIQUENESS.

Part 1. Step. 0 :

$$v(x,t) = \inf_{\alpha\in\mathcal{A}} \left\{ \int_t^T \ell(y(s),\alpha(s))\,ds + g(y(T)) \right\}$$

$y(T) = x$ if $t = T$

$\Rightarrow$ $v(x,T) = g(x)$.

For simplicity $\ell \equiv 0$.

Step. 1 v visc. subsol. GOAL: $\forall \phi\in C^1(\Omega)$ s.t.

$v-\phi$ has a max (x,t), $t<T$

$$\left[-\phi_t + \max_{a\in A}\left(-f(\cdot,a)\cdot D_x\phi\right)\right]_{(x,t)} \leq 0 .$$

Fix $\bar a \in A$, $\bar\alpha(s) = \bar a \ \forall s$, $\bar y(s) = y_x(s;\bar\alpha, t)$

$$\dot{\bar y}(s) = f(\bar y(s), \bar a) \quad \forall\, s\in[t,T] .$$

$$0 = (v-\phi)(x,t) \geq (v-\phi)(\bar y(s), s) \qquad s>t,\ s\approx t$$

$$v(x,t) - v(\bar y(s), s) \geq \phi(x,t) - \phi(\bar y(s), s)$$

(DPP) $\quad v(x,t) = \inf_{\alpha\in\mathcal{A}} v(y(s), s)$

$$\Rightarrow \qquad 0 \geq v(x,t) - v(\bar y(s), s) \qquad s = t+h,\ h>0$$

$$\Rightarrow \qquad \frac{\phi(x,t) - \phi(\bar y(t+h), t+h)}{h} \leq 0$$

let $h \searrow 0+$

$$\Rightarrow \qquad -\phi_t(x,t) - D\phi(x,t)\cdot \underbrace{\dot{\bar y}(t)}_{= f(\bar y(t), \bar a),\ \bar y(t) = x} \leq 0$$

$\forall \bar a \in A$

$$\Rightarrow \qquad -\phi_t(x,t) + \underbrace{\max_{a\in A}\left\{-D\phi(x,t)\cdot f(x,a)\right\}}_{H(D\phi(x,t), x)} \leq 0$$

$\blacksquare$ "$\leq$"

Step 2 $\quad v$ is a supersol. (DPP) $v(x,t) = \inf_{\alpha\in\mathcal{A}} v(y_x(s), s)$ $\quad s>t$

$s = t+h \qquad \varepsilon > 0 \qquad$ "error εh"

$\exists\, \bar\alpha \in \mathcal{A}$ (dep. on εh), $\bar y(s) = y_x(s;\bar\alpha, t)$:

$$v(x,t) \geq v(\bar{y}(t+h), t+h) - \varepsilon h$$

Take $\phi \in C^1(\Omega)$! $v-\phi$ has a min at (x,t)

$$0 = (v-\phi)(x,t) \leq (v-\phi)(\bar{y}(t+h), t+h)$$

$$-\varepsilon h \leq v(x,t) - v(\bar{y}(t+h)) \leq \phi(x,t) - \phi(\bar{y}(t+h), t+h)$$

Problem: $\dot{\bar{y}}(t)$ may NOT exist!

Use Fund. thm. of calc. on R.H.S.

$$\boxed{-\varepsilon h \leq} \int_t^{t+h} \Big(-\underbrace{\frac{d}{ds} \phi(\bar{y}(s), s)}_{\exists \text{ a.e. } s} \Big) ds =$$

$$= -\int_t^{t+h} \Big[\phi_t(\bar{y}(s), s) + D_x \phi(\bar{y}(s), s) \cdot \underbrace{\dot{\bar{y}}(s)}_{= f(\bar{y}(s), \bar{\alpha}(s))} \Big] ds \overset{\text{a.e.}}{=}$$

Use $\bar{y}(s) = x + O(h)$
$t < s < t+h$

$$= O(h^2) + \int_t^{t+h} \Big[-\phi_t(x,t) - \underbrace{D_x \phi(x,t) \cdot f(x, \bar{\alpha}(s))}_{} \Big] ds$$

Divide by h: $\boxed{\bar{\alpha}(s) \in A \ \forall s}$ $\Rightarrow \leq \max_{a \in A} \left[-D_x \phi(x,t) \cdot f(x, a) \right] = H(D_x \phi(x,t), x)$

$$-\varepsilon \leq O(h) + \frac{1}{\cancel{h}} \cancel{h} \left[-\phi_t(x,t) + H(D_x \phi(x,t), x) \right]$$

$\varepsilon \to 0+$, $h \to 0+$ $\Rightarrow$ $0 \leq -\phi_t(x,t) + H(D_x \phi(x,t), x)$.

$\Rightarrow$ v is super sol. ∎

End of Part 1: $v \in BUC(\bar{\Omega})$ & solves visco sense (CT). ∎