LECTURE 10, March 30, 2023

$$(DP)\quad \begin{cases} |Du|-1=0 & \text{in } \Omega\subseteq\mathbb{R}^n \text{ open \& bounded} \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

EIKONAL EQUATION in HOMOGENEOUS MEDIA

Candidate solution is

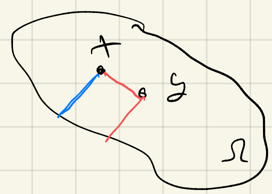
$$u(x) := \mathrm{dist}(x,\partial\Omega) := \inf\{|x-y| : y\in\partial\Omega\}$$

$\partial\Omega$ bdd. $\qquad = \min\{|x-y| : y\in\partial\Omega\}.$

<u>Theorem</u> $u(x)=\mathrm{dist}(x,\partial\Omega)$ is the UNIQUE VISCOSITY sol. of DP.

<u>Proof</u> <u>PART I</u> (Existence): u is a solution.

<u>Step 1</u> $u\in \mathrm{Lip}(\bar\Omega)$



$u(x)\le u(y)+|x-y|$

$\Rightarrow \quad |u(x)-u(y)|\le |x-y| \qquad \forall x,y\in\bar\Omega$

$\Rightarrow \quad \mathrm{Lip}(u)\le 1.$

<u>Step 2</u> u is SUBSOL. of $|Du|-1=0$.

$\varphi\in C^1(\Omega)$, $x\in\Omega$: $(u-\varphi)(x)\ge(u-\varphi)(y) \quad \forall y\approx x$

GOAL: $|D\varphi(x)|\le 1$

$$= \varphi(x)-\varphi(y)\le u(x)-u(y)\le |x-y|$$

$D\varphi(x)\cdot(x-y)+o(|x-y|)$ as $y\to x$ $\qquad q=\dfrac{x-y}{|x-y|}$

Divide by $|x-y|$:

$D\varphi(x)\cdot q + o(1) \leq 1 \qquad y \to x \quad \Rightarrow$

$D\varphi(x)\cdot q \leq 1 \qquad \forall q : |q| = 1 \quad \Rightarrow |D\varphi(x)| \leq 1.$

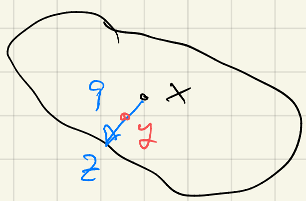
$\left(|p| = \max_{|q|=1} p\cdot q \right)$ $\boxtimes$ u SUBSOL.

Step 3. u SUPERSOL. : $\varphi \in C^1(\Omega)$, $x \in \Omega$ min pt. of $u-\varphi$

$(u-\varphi)(x) \leq (u-\varphi)(y) \quad \forall y \approx x$. GOAL : $|D\varphi(x)| \geq 1$,

or, find $q \in \mathbb{R}^n$, $|q|=1$: $D\varphi(x)\cdot q \geq 1$

(1) $\quad u(x) - u(y) \leq \varphi(x) - \varphi(y) \qquad \forall y \approx x$



$\exists z \in \partial\Omega : |x-z| = u(x)$

$y \in [x,z]$: $u(x) = |x-y| + |y-z| \geq$

$\geq |x-y| + u(y)$ (2)

(1)+(2) $\Rightarrow$ $\varphi(x) - \varphi(y) \geq |x-y|$ (3) $\qquad q := \dfrac{x-z}{|x-z|}$

$y = x + tq \qquad t \in (0, |x-z|)$

$\varphi(x) - \varphi(y) = D\varphi(x)\cdot(x-y) + o(|x-y|) \qquad y \to x$

Use it in (3) & divide by $|x-y|$:

$D\varphi(x)\cdot q + \underbrace{o(1)}_{\to 0} \geq 1 \qquad y \to x$

$\Rightarrow \quad D\varphi(x)\cdot q \geq 1 \qquad$ the GOAL. $\boxtimes$ SUPERSOL.

PART II. UNIQUENESS. Need

Prop. (change of UNKNOWN). $u \in C(\Omega)$ visc. sol. of $F(Du, u, x) = 0$ in Ω. $\Phi \in C^1(\mathbb{R})$, $\Phi'(t) > 0 \quad \forall t$

$\Psi = \Phi^{-1}$. Then $v = \Phi(u)$ is a visco sol. of

$$F(\Psi'(v)Dv, \Psi(v), x) = 0 \quad \text{in } \Omega.$$

Rmk: $u = \Psi(v)$ $Du = \Psi'(v)Dv$ if $u \in C^1$.

Proof: Only "$\le$" $x \in \Omega$, $p \in D^+v(x)$!

GOAL: $\boxed{F(\Psi'(v(x))\,p, \Psi(v(x)), x) \le 0}$

$$p \in D^+v(x) \iff v(y) \le v(x) + p\cdot(y-x) + o(|x-y|) \quad y \to x$$

Ψ increasing. $u(y) = \Psi(v(y)) \le \Psi(v(x) + p\cdot(y-x) + o(|x-y|))$

$$= \Psi(v(x)) + \Psi'(v(x))\big(p\cdot(y-x) + o(|x-y|)\big) + o(\underbrace{p\cdot(y-x) + o(|x-y|)}_{})$$

$$= u(x) + \underbrace{\Psi'(v(x))p}_{q}\cdot(y-x) + o(|x-y|) \quad y \to x$$

$\Rightarrow q \in D^+u(x)$. u subsol. $\Rightarrow F(q, u(x), x) \le 0$

$= F(\Psi'(v(x))p, \Psi(v(x)), x) \le 0$ ∎

HW: do the proof for SUPERSOLS. ∎

Back to UNIQUENESS for (DP).

Kružkov transform $\Phi(r) = 1 - e^{-r}$ $\in C^\infty(\mathbb{R})$

$\Phi'(r) = e^{-r} > 0$ $1 - e^{-r} = t$ $e^{-r} = 1 - t$

$-r = \log(1-t)$ $\Psi(t) = -\log(1-t)$ $\Psi :]-\infty, 1[\to \mathbb{R}$

$\Psi'(t) = \dfrac{1}{1-t} > 0$.

Want COMP. PRINC. between u subsol. of $|Du| - 1 = 0$, & v supersol., $u \le v$ on $\partial\Omega$. GOAL: $u \le v$ in $\overline{\Omega}$.

Call $V = \Phi(v)$, $U = \Phi(u)$. $U \le V$ on $\partial\Omega$.

Equivalent Cond.: $U \le V$ in $\bar\Omega$.

Prop. $\Rightarrow$ U, V are sub- & supersol. of

$$\left| \frac{DW}{1-W} \right| - 1 = 0 \qquad \text{For } W < 1$$

$1 - W(x) > 0$

$$\frac{|DW|}{1-W} - 1 = 0 \quad \Rightarrow \quad |DW| - 1 + W = 0 \quad \text{in } \Omega.$$

HW

Can use the Comparison Principle for (SE) with

$H(P) = |P| - 1$ it satisfies (RH)

$\Rightarrow$ $U \le V$ in $\bar\Omega$ $\Rightarrow$ Comparison holds for $u \le v$

$\Rightarrow$ UNIQUENESS for (DP). $\square$

Rmk. Does comp. princ. hold for $|Du| - n(x) = 0$ in Ω?

$n(x) \ge 0$. Answer: YES if $n(x) > 0$ $\forall x \in \Omega$

pf. HW.

NO if $\exists\, x_0 \in \Omega$: $n(x_0) = 0$ even if $n(x) > 0$ $\forall x \ne x_0$

Example $n=1$

$$(DP)\begin{cases} |u'| - 2|x| = 0 & \text{in }]-1,1[\\ u(-1) = 0 = u(1) & n(x) = 2|x| \end{cases}$$

$u_1(x) = x^2 - 1$ & $u^2(x) = -u_1(x)$ are C^1 & solve (DP)

HW find ∞-many visco sols. [BCD ch II].

Comparison Principles for EVOLUTIVE EQUATIONS, H-J eqs. & Cauchy problems.

(E) $$u_t + H(D_x u) + f(x,t) = 0 \quad \text{in } \Omega = \mathbb{R}^n \times]0,T[.$$

<u>Thm</u> (Comp Princ. #1). $u, v : \bar\Omega \to \mathbb{R}$ Lipschitz & bounded, resp. sub- & supersol. of (E), $H \in C(\mathbb{R}^n)$, $f \in UC(\bar\Omega)$, $u(x,0) \le v(x,0)$ $\forall x \in \mathbb{R}^n$ $\Longrightarrow$ $u \le v$ in $\bar\Omega$.

<u>Cor</u> $\exists$ at most ONE visc. sol. in $L^\infty(\bar\Omega) \cap Lip(\bar\Omega)$ of (E) s.t. $u(x,0) = g(x)$.

<u>Rmk</u>.
- Ω is UNBOUNDED.
- u_t in (E) instead of u in (SE)
- No CONDITIONS at $t = T$
- $u, v \in Lip$ not just continuous.

<u>Lemma</u> (behavior of u & v at $t=T$). $H \in C(\mathbb{R}^n \times \bar\Omega)$, $u \in C(\bar\Omega)$ subsol. (resp. supersol.) of

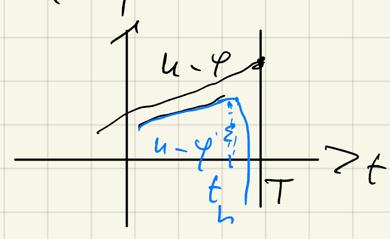
$$u_t + H(D_x u, x, t) = 0 \quad \text{in } \Omega, \quad \varphi \in C^1(\bar\Omega):$$

$u - \varphi$ has a MAX pt. (resp. a MIN) at (x_0, T) $\Longrightarrow$

$$\varphi_t + H(D_x\varphi, \cdot, \cdot)\Big|_{(x,t)=(x_0,T)} \le 0 \quad (\text{resp} \ge 0).$$

<u>Pf</u>. Can assume (x_0, T) is STRICT max of $u - \varphi$

For $\varepsilon > 0$ $$\varphi^\varepsilon(x,t) := \varphi(x,t) + \frac{\varepsilon}{T-t} \ge \varphi(x,t)$$



$$\lim_{t\to T-} \varphi^{\varepsilon}(x,t) = +\infty \quad \Rightarrow \quad \lim_{t\to T-} \underbrace{u-\varphi^{\varepsilon}}_{=\psi^{\varepsilon}} = -\infty$$

CLAIM on a sequence $\varepsilon_n \searrow 0$, $u-\varphi^{\varepsilon_n}$ has a max at (x_n, t_n), $t_n < T$, $x_n \to x_0$, $t_n \to T$ as $n \to \infty$.

Pf.: postponed.

Then, u subsol. in Ω $\Rightarrow$

$$\varphi_t^{\varepsilon_n} + H(D_x \varphi^{\varepsilon_n}, \cdot, \cdot)\Big|_{(x_n, t_n)} \le 0 \qquad (*)$$

$$D_x \varphi^{\varepsilon} = D_x \varphi \qquad \varphi_t^{\varepsilon_n} = \varphi_t + \frac{\varepsilon}{(T-t)^2} \ge \varphi_t$$

$$(*) \Rightarrow \quad \cancel{\frac{\varepsilon_n}{(T-t_n)^2}} + \varphi_t(x_n, t_n) + H(D_x\varphi, \cdot, \cdot)\Big|_{(x_n, t_n)} \le 0$$

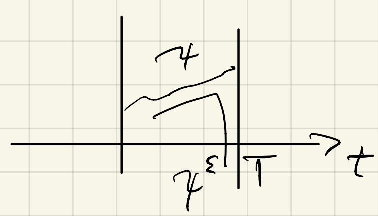
$n \to \infty$

$$\varphi_t(x_0, T) + H(D_x \varphi, \cdot, \cdot)\Big|_{(x_0, T)} \le 0$$

which is the GOAL.

Pf. of the CLAIM $\psi = u - \varphi$ $\quad \psi^{\varepsilon} = \psi - \frac{\varepsilon}{T-t} \le \psi$

Ass.: ψ has STRICT MAX at (x_0, T)



$\psi^{\varepsilon} \to -\infty$ as $t \to T-$. Fix $r > 0$

use variant of Weierstrass in $\bar{B}(x_0, r) \times [T-r, T[$ $\Rightarrow$

ψ^{ε} has max in , $(x_{\varepsilon}, t_{\varepsilon})$.

Extract $x_n \to \tilde{x}$, $t_n \to \tilde{t} \in [T-r, T]$ $\quad \tilde{x} \in \bar{B}(x_0, r)$

$$\psi^{\varepsilon_n}(x_n, t_n) \to \ell \quad (\ge -\infty), \quad \varepsilon_n \searrow 0+$$

Goal $\tilde{x} = x_0$ $\quad \tilde{t} = T$.

$$\psi^{\varepsilon_n}(x_n, t_n) \le \psi(x_n, t_n) \xrightarrow[n\to\infty]{} \psi(\tilde{x}, \tilde{t})$$

$$\Rightarrow \ell \le \psi(\tilde{x}, \tilde{t})$$

$$\psi^{\varepsilon_n}(x_n, t_n) \ge \psi^{\varepsilon_n}(x_0, T-\sqrt{\varepsilon_n}) = \underbrace{\psi(x_0, T-\sqrt{\varepsilon_n})}_{\to \psi(x_0,T)} - \underbrace{\frac{\varepsilon_n}{\sqrt{\varepsilon_n}}}_{\to 0}$$

$$\Rightarrow \ell \ge \psi(x_0, T).$$

$$\Rightarrow \psi(x_0, T) \le \psi(\tilde{x}, \tilde{t}) \qquad (x_0, T) \text{ strict max pt. of } \psi$$

$$\Rightarrow (x_0, T) = (\tilde{x}, \tilde{t}). \qquad \text{the GOAL.} \qquad \square \text{ Lemma.}$$

Rmk. on Comp Princ. #1 It works for

$$(CP)\begin{cases} u_t + |Du|^2 + V(x) = 0 & \text{MECHANICAL H-J eq.} \\ u(x,0) = g(x) & \text{if } V \in UC(\mathbb{R}^n),\ g \in BC(\mathbb{R}^n) \end{cases}$$

$\Rightarrow$ (CP) has at most ONE vis. sol. in $L^\infty(\bar\Omega) \cap Lip(\bar\Omega)$.

Proof of CP #1. Doubling the variables: $\varepsilon > 0$, $\beta > 0$, $\eta > 0$

$$\Phi(x,y,t,s) := u(x,t) - v(y,s) - \frac{|x-y|^2}{2\varepsilon} - \frac{|t-s|^2}{2\varepsilon} - \beta(\ell(x) + \ell(y)) - \eta(t+s)$$

$$\lim_{|x|\to\infty} \ell(x) = +\infty, \quad \ell \in C^\infty(\mathbb{R}^n) \Rightarrow \Phi \to -\infty \text{ as } |x| \text{ or } |y| \to \infty$$

$\Rightarrow \Phi$ has a MAX in $(\bar\Omega)^2 = \mathbb{R}^{2n} \times [0,T]^2$

Choose $\ell(x) = \log(1+|x|^2)$, $\ell_{x_i} = \dfrac{2x_i}{1+|x|^2} \Rightarrow |D\ell| \le 2.$

CONTINUES NEXT TIME!