LECTURE 9, March 28, 2023

STABILITY of VISCO. SOLS. W.R.T. UNIFORM CONVERGENCE of DATA (i.e. CONT. DEPENDENCE).

Prop. $u_n \in C(\Omega)$ v. SUBSOLS. (resp. SUPERSOLS) of $F_n(Du, u, x) = 0$ in Ω, $F_n \in C(\mathbb{R}^N \times \mathbb{R} \times \Omega)$ &

$$\begin{cases} u_n \to u & \text{locally uniformly} \\ F_n \to F & \text{locally uniformly} \end{cases} \Rightarrow u \text{ is a v. SUBSOL. (resp. SUPERSOL.) of } F(Du, u, x) = 0 \text{ in } \Omega.$$

☺ Rmk: No information on Du_n, Du !

Pf. Only "$\le$" ("$\ge$" HW). Fix $\varphi \in C^1(\Omega)$, $\bar{x} \in \operatorname{argmax}(u - \varphi)$ STRICT MAX. GOAL: $F(D\varphi, u, \cdot)\big|_{x=\bar{x}} \le 0$.

By lemma of last time $\exists\, x_n \to \bar{x}$: x_n is a loc. max pt. of $u_n - \varphi$. Use u_n subsol. of $F_n = 0$:

$$\begin{array}{ccccccc} F_n\big(D\varphi(x_n), & u_n(x_n), & x_n\big) & \le 0 \\ \downarrow & \text{HW} \downarrow & \downarrow & \\ F\big(D\varphi(\bar{x}), & u(\bar{x}), & \bar{x}\big) & \le 0 \end{array}$$

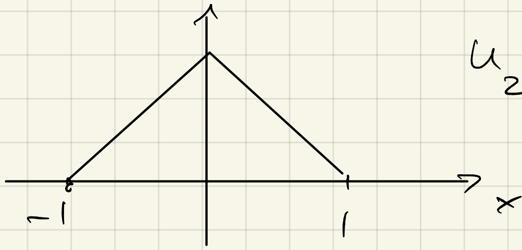
$\blacksquare$ "$\le$"

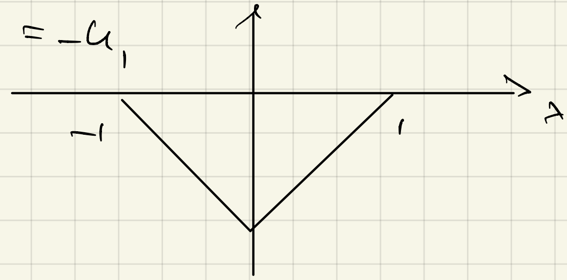
Important example $N=1$ ← eikonal eq.

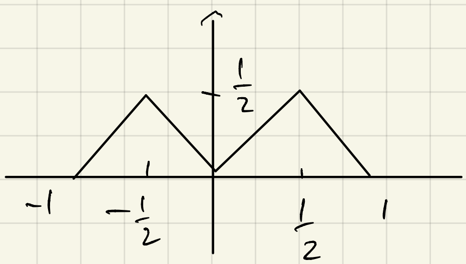
$$(P) \begin{cases} |u'| - 1 = 0 & \text{in }]-1, 1[\\ u(-1) = 0 = u(1) \end{cases}$$

$$u_1(x) = 1 - |x|$$

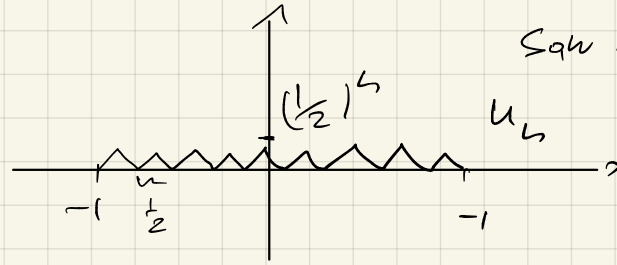
$u_2(x) = |x| - 1 = -u_1$







Saw-tooth ?



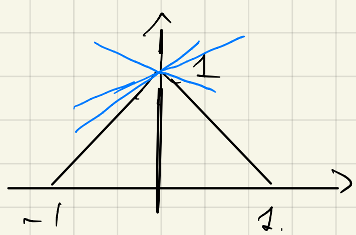
u_n are Lip. weak sols. of (P), $\sup |u_n| = \frac{1}{2^n} \to 0$ as $n \to \infty$

$u_n \to 0 = u(x)$ UNIFORMLY in $[-1, 1]$

BUT $u(x) \equiv 0$ DOES NOT solve the eq. at ANY POINT!

<u>Q</u> Is some u_n a VISCO. SOLN. ?

1. $u_1 = 1 - |x|$



$p \in [-1, 1]$

$D^+u(0) = \{ p \in \mathbb{R} :$
$u(x) \leq u(0) + px + o(x) \}$
$\quad\quad\quad\; \overset{\shortparallel}{1} \quad\quad\quad\quad x \to 0$

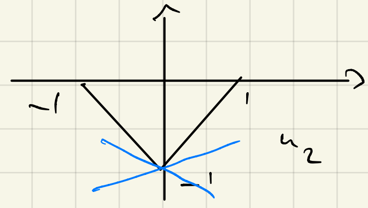
$D^+u(0) = [-1, 1]$

u_1 is V.SUBSOL. $\iff \forall p : |p| \leq 1 \quad |p| - 1 \leq 0$ OK.

" " V.SUPERSOL. $D^-u(0) = \emptyset \implies u_1$ is SUPERSOL.

$\implies u_1$ is a VISCO SOL. of (P).

2. $u_2 = |x| - 1 = -u_1$



$D^-u_2(0) = [-1, 1]$, $D^+u_2(0) = \emptyset$

$\implies u_2$ is SUB-SOL.

<u>Q</u> Supersol. ? $p \in D^-u_2(0)$ $|p| - 1 \overset{?}{\geq} 0$? <u>NO</u> $0 \in D^-u_2(0)$!

$\implies u_2$ is NOT a V.Sol. of (P)

All other u_n have corner $\vee$ where $|u'|-1 \geq 0$ is NOT satisfied.

Conclusion. Among all u_n only u_1 is a V. SOLUTION.

Rmk. 1. $u_1(x) = 1-|x| = \mathrm{dist}(x, \partial\Omega)$ $\quad \Omega =]-1,1[$

Rmk. 2. I'll show that (P) has a UNIQUE VISCO. SOLN.

Rmk. 3. $u_2(x) = |x|-1$ $\quad D^+u_2(0) = \emptyset$, $D^-u_2(0) = [-1,1]$.

$\Rightarrow$ $|p|-1 \leq 0$ $\forall p \in D^-u_2(0)$

 $\quad \parallel$

 $1-|p| \geq 0$ $\quad \Rightarrow$ u_2 is a SUPERSOL. of

$$1-|u'| = 0 \quad \text{in fact a SOLN.}$$

$$\parallel$$

$$-(|u'|-1)$$

N.B. $-(|u|'-1) = 0$ IS NOT EQUIVALENT TO $|u'|-1 = 0$

Rmk 4 (Rule for changing sign). HW

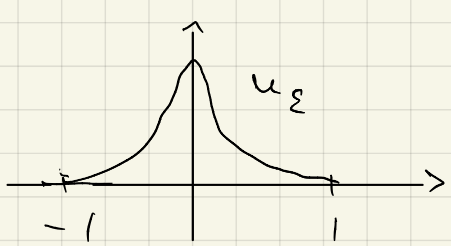
u subsol. of $F(Du, u, x) = 0$ $\iff$ $v = -u$ is SUPERSOL.

of $-F(-Dv, -v, x) = 0$.

Rmk 5. Further motivation for NOT EQUIVAL. of $F=0$ & $-F=0$

VANISH. VISCOSITY.
$$\begin{cases} |u_\varepsilon'|-1 = \varepsilon u_\varepsilon'' & \text{in }]-1,1[\\ u_\varepsilon(-1) = 0 = u_\varepsilon(1) \end{cases}$$

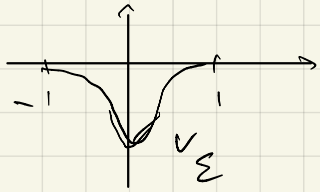
HW SOLVE IT EXPLICITLY



HW $\lim_{\varepsilon \to 0^+} u_\varepsilon = 1-|x| = u_1(x)$

u_2 comes from
$$\begin{cases} 1-|v_\varepsilon'| = \varepsilon v_\varepsilon'' &]-1,1[\\ v_\varepsilon(-1) = 0 = v_\varepsilon(1) \end{cases}$$

$v_\varepsilon = -u_\varepsilon$



$\lim_{\varepsilon\to 0^+} v_\varepsilon(x) = u_2(x)$

COMPARISON PRINCIPLE for DIRICHLET PROBLEM.
for a model pb.

(SE) $\qquad u + H(Du, x) = 0 \quad$ in $\Omega \subseteq \mathbb{R}^n$ open BOUNDED.

Rmk. $\quad u, v \in C^1(\Omega) \cap C(\bar\Omega)$

$$u + H(Du, x) \overset{(I)}{\leq} v + H(Dv, x) \quad \text{in } \Omega.$$

$$u \overset{(BC)}{\leq} v \text{ on } \partial\Omega \quad \Longrightarrow \quad u \leq v \text{ in } \bar\Omega.$$

Pf $\quad \Phi(x) = u(x) - v(x) \in C(\bar\Omega)$ by Weierst. Thm $\exists\, \bar x \in \bar\Omega$:

$\Phi(\bar x) = \max_{\bar\Omega} \Phi \quad : \quad \Phi(x) \leq \Phi(\bar x) \ \forall\, x \in \bar\Omega.$

Goal: $\quad \Phi(\bar x) \leq 0.$

Case 1 $\quad \bar x \in \partial\Omega. \quad \overset{(BC)}{\Longrightarrow} \quad \Phi(\bar x) \leq 0$

Case 2 $\quad \bar x \in \Omega \quad Du(\bar x) = Dv(\bar x) \quad$ by (I)

$$u + \cancel{H(\underset{\substack{\| \\ Dv}}{Du}, \cdot)}\Big|_{\bar x} \leq v + \cancel{H(\underset{\substack{\| \\ Du}}{Dv}, \cdot)}\Big|_{\bar x} \quad \Longrightarrow u(\bar x) \leq v(\bar x).$$

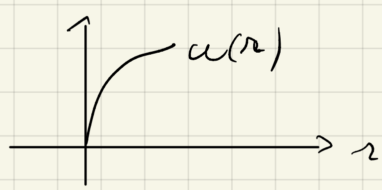
∎

Note. $\forall H$! No regularity of H is needed for classical solutions.

For visco. sol. I'll need a regularity cond. on H.

Def. $\omega : [0, +\infty) \to [0, +\infty)$ is a MODULUS if NONDECREASING &

$\lim_{r \to 0^+} \omega(r) = 0$



Rmk. $f: \overset{X \subseteq}{\mathbb{R}^d} \to \mathbb{R}$ UNIFORMLY CONT. $\iff$

$\exists\, \omega_f$ modulus (MOD. of CONT. of f) :

$$|f(x) - f(y)| \le \omega_f(|x-y|) \quad \forall x,y \in \bar{X}$$

Pf. "$\Leftarrow$" trivial. "$\Rightarrow$" HW ... build ω_f .... $\square$

Assumption on H : $\exists\ \omega$ modulus s.t.

$$(RH) \quad |H(p,x) - H(p,y)| \le \omega\big(|x-y|(1+|p|)\big) \quad \forall x,y \in \bar{\Omega},\ \forall p \in \mathbb{R}^n$$

N.B. $\Rightarrow$ H UNIF. CONT. in x, unif for $|p|$ bdd.

Thm. (Comparison Principle for (SE). [BCD] p.51-53).
$H \in C(\mathbb{R}^n \times \bar{\Omega})$ satisf. (RH), Ω open BOUNDED,
$u, v \in C(\bar{\Omega})$, u vis. subsolut. of (SE), v v.S. SUPER sol of (SE)
in Ω. & (BC) $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω.

Proof. Idea (S.N. Kružkov) "doubling of variables"

$$\Phi(x,y) = u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon}, \quad \varepsilon > 0$$

Φ has max in $\bar{\Omega} \times \bar{\Omega}$ in $(x_\varepsilon, y_\varepsilon)$.

$$\max_{\bar{\Omega}}(u-v) = \max_{\bar{\Omega}} \Phi(x,x) \le \max_{\bar{\Omega}\times\bar{\Omega}} \Phi = \Phi(x_\varepsilon, y_\varepsilon)$$
$$\le u(x_\varepsilon) - v(y_\varepsilon)$$

$$\boxed{\text{GOAL}: \quad \liminf_{\varepsilon \to 0+} \big(u(x_\varepsilon) - v(y_\varepsilon)\big) \le 0}$$

Estimates on $|x_\varepsilon - y_\varepsilon|$:

$$\Phi(x_\varepsilon, x_\varepsilon) \le \Phi(x_\varepsilon, y_\varepsilon) \implies$$

$$\cancel{u(x_\varepsilon)} - v(x_\varepsilon) \le \cancel{u(x_\varepsilon)} - v(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \implies$$

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \le v(x_\varepsilon) - v(y_\varepsilon) \le 2 \max_{\bar\Omega} |v| \qquad (1)$$

$$\implies |x_\varepsilon - y_\varepsilon|^2 \le 4\varepsilon \max|v| \xrightarrow[\varepsilon \to 0]{} 0$$

$(1) + v \in UC(\bar\Omega) \implies$

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \le \omega_v(|x_\varepsilon - y_\varepsilon|) \to 0 \quad \varepsilon \to 0 \qquad (2).$$

Case 1 $\exists\, \varepsilon_n \downarrow 0 \,!\, (x_{\varepsilon_n}, y_{\varepsilon_n}) \in \partial(\bar\Omega \times \bar\Omega) =$

$$= (\partial\Omega \times \bar\Omega) \cup (\bar\Omega \times \partial\Omega)$$

Case 2 $\forall\, \varepsilon \in (0, \bar\varepsilon]$ $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$.

Case 1A $x_{\varepsilon_n} \in \partial\Omega$ $\forall \varepsilon_n$ $\varepsilon_n \to 0$.

$$u(x_{\varepsilon_n}) - v(y_{\varepsilon_n}) \overset{BC}{\le} v(x_{\varepsilon_n}) - v(y_{\varepsilon_n}) \le \omega_v(|x_{\varepsilon_n} - y_{\varepsilon_n}|) \to 0 \quad \varepsilon_n \to 0.$$

Case 1B $y_{\varepsilon_n} \in \partial\Omega$ $\qquad u \in UC(\bar\Omega)$.

$$u(x_{\varepsilon_n}) - v(y_{\varepsilon_n}) \overset{BC}{\le} u(x_{\varepsilon_n}) - u(y_{\varepsilon_n}) \to 0 \quad \varepsilon_n \to 0$$

Case 2. $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ $\forall \varepsilon < \bar\varepsilon$.

$$\varphi(x) = v(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon} \in C^\infty \qquad u - \varphi \text{ has a max at } x_\varepsilon$$

φ can be used as TEST FUNCTION. at $x = x_\varepsilon$

$$D\varphi(x_\varepsilon) = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} =: p_\varepsilon$$

$$\psi(y) = u(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon} \in C^\infty \qquad \psi - v \text{ has a MAX at } y_\varepsilon$$

$\Rightarrow$ $v - \psi$ has a $\underline{\text{min}}$ at y_ε, ψ is TEST FN at y_ε

$$D\psi(y_\varepsilon) = -\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}(-1) = p_\varepsilon .$$

u vis. SUB SOL., v vis. SUPERS. $\Rightarrow$

$$u(x_\varepsilon) + H(p_\varepsilon, x_\varepsilon) \le 0 \le v(y_\varepsilon) + H(p_\varepsilon, y_\varepsilon)$$

$$\Rightarrow u(x_\varepsilon) - v(y_\varepsilon) \le H(p_\varepsilon, y_\varepsilon) - H(p_\varepsilon, x_\varepsilon) \overset{(RH)}{\le}$$

$$\le \omega\big(|x_\varepsilon - y_\varepsilon|(1 + |p_\varepsilon|)\big) = \omega\Big(\underbrace{|x_\varepsilon - y_\varepsilon|}_{\substack{\text{by (1)} \\ \searrow 0}} + \underbrace{\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}}_{\substack{\text{by (2)} \\ \searrow 0}}\Big)$$

$\varepsilon \to 0$

$$\Rightarrow \limsup_{\varepsilon \to 0+} \big(u(x_\varepsilon) - v(y_\varepsilon)\big) \le 0. \qquad \Rightarrow \text{GOAL.} \qquad \blacksquare$$

COMMENTS on the assumptions of COMP. PRINC.

(RH) $|H(p,x) - H(p,y)| \le \omega(|x-y|(1+|p|))$
is satisfied if:

Ex. 1 $\quad H(p,x) = H_1(p) + f(x)$, $\quad H_1 \in C(\mathbb{R}^n)$

$f \in C(\bar\Omega) \Rightarrow f \in UC(\bar\Omega) \Rightarrow \exists\, \omega_f$ mod. of cont. of f.

$$|H_1(p) + f(x) - H_1(p) - f(y)| \le \omega_f(|x-y|) \le$$
$$\le \omega_f(|x-y|(1+|p|)) \Rightarrow (RH) \text{ holds with } \omega = \omega_f.$$

<u>OK</u> H from mechanics $H = |p|^2 + V(x)$ is OK if $V \in C(\bar\Omega)$.

<u>Ex 2</u> $H(p,x) = g(x)|p| + f(x)$, $f \in C(\bar\Omega)$, $g \in Lip(\bar\Omega)$

$$\Rightarrow |H(p,x) - H(p,y)| \le |g(x)|p| - g(y)|p|| + |f(x) - f(y)|$$
$$\le L_g |x-y||p| + \omega_f(|x-y|) \Rightarrow (RH) \text{ OK}$$

with $\omega(r) = 2\max\{L_g r, \omega_f(r)\}$.

<u>Ex</u> of (RH) NOT satisfied: $H(p,x) = g(x)|p|^2$

(RH) does not hold unless $g \equiv$ const.

<u>Corollary</u> (UNIQUENESS of sol. to Dirichlet problem)

$H \in C(\mathbb{R}^n \times \bar\Omega)$, (RH), Ω open bounded $\Rightarrow \exists$ at most one visc. sol. in $C(\bar\Omega)$ of

$$(DP) \quad \begin{cases} u + H(Du, x) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

<u>Pf.</u> $u, v \in C(\bar\Omega)$ vis. sol. of (DP) $\Rightarrow u = g = v$ on $\partial\Omega$.

$u \le v$ by Com Princ. & also $v \le u$ by Comp. Princ.

$\Rightarrow \quad u = v \quad \text{in } \bar{\Omega}. \quad \blacksquare$

Rmk. Comp. Princ. & uniqueness are OK for

$$\lambda(x) u + H(Du, x) = 0 \qquad \text{if} \quad \lambda(x) \geq \lambda_0 > 0 \quad \forall x \in \Omega$$

$$\lambda \in C(\bar{\Omega}).$$

Pf. HW.

A nontrivial application: EIKONAL EQUATION in homogeneous media.

$$(DP) \quad \begin{cases} |Du| - 1 = 0 & \text{in } \Omega \subseteq \mathbb{R}^n \text{ open bdd} \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that eq. is not of the form (SE)!