LECTURE 8, March 23, 2023.

<u>Proposition</u>: $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$, $u^j \in C^2_x \cap C^1_t(\Omega)$ solutions of

$$\frac{\partial u^j}{\partial t} + H(D_x u^j, x) = \varepsilon^j \Delta_x u^j \quad \text{in } \Omega \subseteq \mathbb{R}^n \times (0,\infty), \quad \varepsilon^j \to 0+$$

$u^j \to u$ loc. uniformly $\Rightarrow$ u is a viscosity solution of

$$(HJ) \quad \frac{\partial u}{\partial t} + H(D_x u, x) = 0 \quad \text{in } \Omega.$$

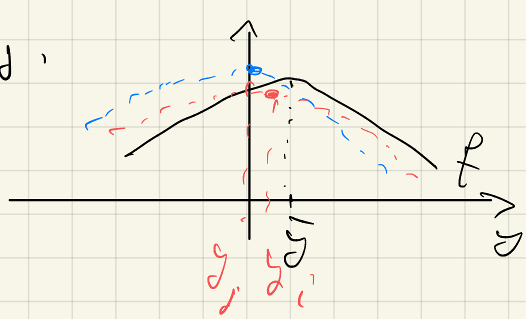
<u>NOTE</u>. NO INFO on $Du^j \to Du$??

nor $|\Delta_x u^j| \leq C$? ... ?

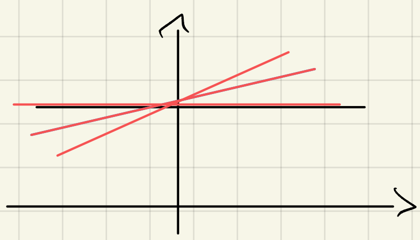
<u>Pf.</u>: Need a Lemma.

<u>Lemma</u>: $f_j : \Omega \to \mathbb{R}$ cont., $\Omega \subseteq \mathbb{R}^N$ open.
$f_j \to f$ loc. uniformly, $\bar{y} \in \Omega$ loc. STRICT MAX point of f (i.e. $\exists r > 0 : f(\bar{y}) > f(y) \ \forall y : |y - \bar{y}| < r$)
$\Rightarrow \exists y_j \to \bar{y}$, y_j loc. maxpts of f_j.

<u>Meaning</u>.



FALSE if MAX is NOT STRICT.



<u>Pf.</u> $\exists r > 0$, $\exists \delta > 0$:

$\max_{\partial B(\bar{y},r)} f < f(\bar{y}) - \delta$. $f_j \to f$ unif in $\overline{B}(\bar{y},r)$

$\max_{\partial B(\bar{y},r)} f_j < f_j(\bar{y}) - \frac{\delta}{2} < f_j(\bar{y})$ for $j \geq \bar{J}$

Weierstrass Thm. $\Rightarrow$ $\exists\, y_j \in \overline{B}(\bar{y},r)$ max point of f_j in $\overline{B}(\bar{y},r)$

$y_j \in B(\bar{y},r)$. Now take $r_j \to 0+$, repeat the argument in $\overline{B}(\bar{y},r_j)$ $\rightsquigarrow$ loc. max points of f_j :

$y_j \to \bar{y}$ as $j \to \infty$. $\square$

<u>Proof of Prop.</u> Only "$\leq$" . u V.subsol. of (H-J)

("$\geq$" very similar HW). $\varphi \in C^2(\Omega)$, $(\bar{x},\bar{t}) \in \arg\max(u-\varphi)$

<u>GOAL</u>: $\varphi_t + H(D_x\varphi, \cdot)\big|_{(\bar{x},\bar{t})} \leq 0$.

<u>Case 1</u> $(\bar{x},\bar{t})$ STRICT MAX POINT of $u-\varphi$

Lemma: $\Rightarrow$ $\exists (x_j, t_j) \xrightarrow[j\to\infty]{} (\bar{x},\bar{t})$ max pts. of $u^j - \varphi$.

In (x_j, t_j) $u^j_t = \varphi_t$ & $D_x u^j = D_x\varphi$,

$D^2_x(u^j - \varphi) \leq 0$ SEMIDEF $\Rightarrow$ $\Delta_x(u^j - \varphi) \leq 0$ at (x_j, t_j)

$0 = u^j_t + H(D_x u^j, x) - \varepsilon \Delta_x u^j$ at (x_j, t_j)

$\geq \varphi_t + H(D_x\varphi, x) - \varepsilon \Delta_x \varphi \Big|_{(x_j,t_j)}$

let $j \to \infty$ $\quad 0$

$$\longrightarrow \varphi_t(\bar{x},\bar{t}) + H(D_x\varphi(\bar{x},\bar{t}), \bar{x}) \leq 0 \qquad \boxed{}$$

the GOAL.

<u>Case 2</u> General case of $(\bar{x},\bar{t})$ merely max pt. NOT strict. I need

<u>Lemma</u> In the def. of visco. sub (or resp. SUPERSOL.) it is not restrictive to assume $u-\varphi$ has a STRICT loc MAX at $(\bar{x},\bar{t})$ (resp. STRICT min at $(\bar{x},\bar{t})$).

<u>Pf</u>. Take $u-\varphi$ $\varphi\in C^2$ with loc max at $(\bar{x},\bar{t})$.

$$\bar{\varphi}(x,t) = \varphi(x,t) + |x-\bar{x}|^2 + |t-\bar{t}|^2 \quad \begin{cases} > \varphi(x,t) & \forall (x,t) \neq (\bar{x},\bar{t}) \\ = \varphi(\bar{x},\bar{t}) \end{cases}$$

$$\bar{\varphi} \in C^2$$

$\Rightarrow$ $u-\bar{\varphi}$ has a STRICT MAX at $(\bar{x},\bar{t})$.

$$\Rightarrow \quad \bar{\varphi}_t + H(D_x\bar{\varphi}, \cdot)\Big|_{(\bar{x},\bar{t})} \leq 0 \qquad (*)$$

$$\bar{\varphi}_t = \varphi_t + 2(t-\bar{t}) \Rightarrow \bar{\varphi}_t(\bar{x},\bar{t}) = \varphi_t(\bar{x},\bar{t})$$

$$D_x\bar{\varphi} = D_x\varphi + 2(x-\bar{x}) \Rightarrow D_x\bar{\varphi} = D_x\varphi \text{ at } (\bar{x},\bar{t})$$

$$(*) \Rightarrow \quad \varphi_t + H(D_x\varphi, \cdot)\Big|_{(\bar{x},\bar{t})} \leq 0. \qquad \boxed{}$$

This completes the proof of Prop. $\boxed{}$

Next theorem gives a further motivation to the choice of "$\leq$" or "$\geq$" in the def. of visco soln.:

Hopf - Lax formula.

Thm. $H : \mathbb{R}^n \to \mathbb{R}$ convex & superlinear, $L = H^*$,
$g \in Lip(\mathbb{R}^n)$ $\quad u(x,t) = \min_{y \in \mathbb{R}^n} \{ tL(\frac{x-y}{t}) + g(y) \}$.

$\Rightarrow$ u is a visc. sol. of (HJ) $u_t + H(D_x u) = 0$ in $\mathbb{R}^n \times (0,\infty)$.

Pf. "$\leq$" (DPP) $u(x,t) = \min_y \{ (t-s) L(\frac{x-y}{t-s}) + u(y,s) \}$
$0 \leq s < t$. Fix $q \in \mathbb{R}^n$, $h = t-s$ $\quad y = x - hq$ ... as in the previous proof ...

$$u(x,t) - u(y,s) \overset{(+)}{\leq} h L(q) \qquad \forall h > 0$$

Take $\varphi \in C^2$: $u - \varphi$ has a max pt. at (x,t)

$$(u-\varphi)(x,t) \geq (u-\varphi)(y,s) \qquad \forall (y,s) \approx (x,t)$$
$$y = x - hq, \quad s = t - h, \quad h \approx 0$$

$$u(x,t) - u(x-hq, t-h) \geq \varphi(x,t) - \varphi(x-hq, t-h)$$

by (+) $\qquad h > 0$

$$\varphi(x,t) - \varphi(x-hq, t-h) \leq h L(q) \qquad \text{let } h \to 0+$$

$$\Rightarrow \quad \varphi_t + \underbrace{q \cdot \nabla_x \varphi - L(q)}\Big|_{(x,t)} \leq 0 \qquad \forall q$$

$$\Rightarrow \quad \varphi_t + \max_q \{ \quad \} \leq 0$$

$$\Rightarrow \quad \varphi_t + H(\nabla_x \varphi)\Big|_{(x,t)} \leq 0. \qquad \frac{1}{2}$$

Part 2 "$\geq$" now $u - \varphi$ has a MIN at (x,t).
As in previous proof $z \in \mathbb{R}^n$:

$$u(x,t) = t\,L\left(\frac{x-z}{t}\right) + g(z) \quad , \quad \text{Choose } y\,!$$

$$\frac{x-z}{t} = \frac{y-z}{s} \quad \ldots\ldots \quad y = x - h\,\frac{x-z}{t} \qquad s<t$$

(DPP) $$u(x,t) - u(y,s) \overset{(++)}{\geq} (t-s)\,L\left(\frac{x-z}{t}\right)$$

$$(u-\varphi)(x,t) \leq (u-\varphi)(y,s) \qquad (y,s) \approx (x,t)$$

$$u(x,t) - u(y,s) \leq \varphi(x,t) - \varphi(y,s)$$

$$\Rightarrow \quad \varphi(x,t) - \varphi(y,s) \geq (t-s)\,L\left(\frac{x-z}{t}\right) \qquad s \to t^-$$

$$\Rightarrow \quad \varphi_t + \frac{x-z}{t}\cdot D_x\varphi - L\left(\frac{x-z}{t}\right)\Big|_{(x,t)} \geq 0 \qquad q = \frac{x-z}{t}$$

$$\Rightarrow \quad \varphi_t + \underbrace{\max_q \{ q\cdot D_x\varphi - L(q) \}}_{= H(D_x\varphi)}\Big|_{(x,t)} \geq 0$$

goal "$\geq$" ∎.

To come : H-Lax is the UNIQUE visco. soln. of the Cauchy pb. for (HJ).

MORE GENERAL CONTEXT : general fully nonlinear PDE

(E) $$F(Du, u, x) = 0 \quad \text{in } \Omega \subseteq \mathbb{R}^N \text{ open.}$$

$F : \mathbb{R}^N \times \mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ continuous.

Ref ch II [BCD].

Recall $u \in USC(\Omega)$ upper semicont. if $\forall x_0 \in \Omega$

$$u(x_0) \geq \limsup_{x \to x_0} u(x).$$

$u \in LSC(\Omega)$ lower s.cont. if $u(x_0) \le \liminf_{x \to x_0} u(x)$.

Rmk. $u \in C(\Omega) \iff u \in USC(\Omega) \cap LSC(\Omega)$.

Weierstrass thm. $\begin{cases} u \in USC(\bar\Omega) \quad \bar\Omega \text{ compact} \Rightarrow u \text{ has max} \\ u \in LSC(\bar\Omega) \quad \text{"} \quad \text{"} \Rightarrow u \text{ has min.} \end{cases}$

Def. (i) $u \in USC(\Omega)$ is a visco sub-sol. of (E) if

$\forall \varphi \in C^1(\Omega)$: $u - \varphi$ has a loc. max pt. at $\bar x$ we have

$$F(D\varphi, u, \cdot)\Big|_{x=\bar x} \le 0$$

(ii) $u \in LSC(\Omega)$ is v.-supersol. if $\forall \varphi \in C^1(\Omega)$, $\tilde x$ loc min of $u - \varphi$ $F(D\varphi, u, \cdot)\Big|_{\tilde x} \ge 0$.

(iii) $u \in C(\Omega)$ is visc. solution of (E) if it is sub & supersol.

Rmk 1 In the def. it is equivalent to consider $\varphi \in C^1$ or C^2 or C^∞ (Pf. HW: use C^∞ is dense in C^1 for uniform convergence)

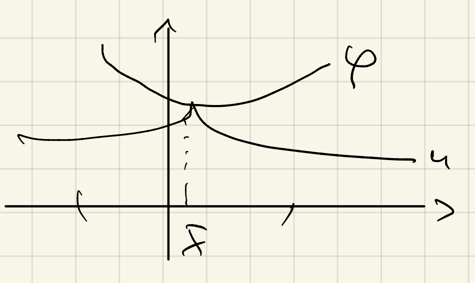
Rmk 2 In the def. it is equivalent to take $\bar x$ STRICT MAX pt. & $\tilde x$ STRICT min pt. & $u(\bar x) = \varphi(\bar x)$

Then. $\bar x \in \operatorname{argmax}(u - \varphi)$ & $u = \varphi$ at $\bar x$:

$0 = u(\bar x) - \varphi(\bar x) \ge u(x) - \varphi(x) \quad \forall x \simeq \bar x$

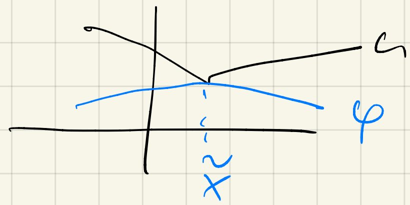
$\iff u(x) \le \varphi(x)$

"graph of φ touches graph u from above".



For supersol. $0=(u-\varphi)(\tilde{x}) \le (u-\varphi)(x)$

$\Rightarrow \quad u(x) \ge \varphi(x)$



graph φ touches graph u from BELOW at $\tilde{x}$. ∎

CONSISTENCY with CLASSICAL SOLS.

Prop. (Cons. 1). (i) $u \in C(\Omega)$ with 1st partial ders. & solves (E) pointwise $\Rightarrow$ u is visco sol.

(ii) u visco sol. of (E) & $u \in C^1(\Omega)$ $\Rightarrow$ u solves (E) pointwise.

Pf. (i) "$\le$" $\varphi \in C^1$, $\bar{x} \in \operatorname{argmax}(u-\varphi)$. $Du(\bar{x}) = D\varphi(\bar{x})$

at $\bar{x}$ $\Rightarrow$ $F(\underbrace{Du(\bar{x})}_{= D\varphi(\bar{x})}, u(\bar{x}), \bar{x}) = 0$ $\quad \ge 0$ ∎

(ii) $u \in C^1(\Omega)$ viscosol.! take $\varphi = u$. $u-\varphi$ has a max and min at EACH $x \in \Omega$. $F(\underbrace{D\varphi}_{= Du}, u, \cdot)\big|_x \; \begin{matrix} \le 0 \\ \ge 0 \end{matrix} \quad \forall x$. ∎

———o———

Recall u differentiable at $\bar{x} \in \Omega$ if $\exists p \in \mathbb{R}^n$:

$$u(x) = u(\bar{x}) + p\cdot(x-\bar{x}) + o(|x-\bar{x}|) \qquad x \to \bar{x}.$$

Def $u \in USC(\bar{\Omega})$, $p \in \mathbb{R}^n$ belongs to the SUPER DIFFERENTIAL of u at $\bar{x}$, $p \in D^+u(\bar{x})$, if

$$u(x) \leq u(\bar{x}) + p\cdot(x-\bar{x}) + o(|x-\bar{x}|) \qquad x \to \bar{x}.$$

$u \in LSC(\bar{\Omega})$, $p \in \mathbb{R}^N$ is in the SUBDIFFERENTIAL of u at $\bar{x}$, $p \in D^- u(\bar{x})$, if

$$u(x) \geq u(\bar{x}) + p\cdot(x-\bar{x}) + o(|x-\bar{x}|) \qquad x \to \bar{x}.$$

Rmk. u diff. at $\bar{x}$ $\iff$ $D^+u(\bar{x}) \cap D^-u(\bar{x}) \neq \emptyset$

& then $D^+u(\bar{x}) \cap D^-u(\bar{x}) = \{Du(\bar{x})\}$.

Rmk. u convex $\Rightarrow$ $\partial u(x) \subseteq D^-u(x)$.

Lemma. (i) $p \in D^+u(\bar{x}) \iff \exists \varphi \in C^1(\Omega)$: $u-\varphi$ has a loc. max at $\bar{x}$ & $D\varphi(\bar{x}) = p$ (& $u(\bar{x}) = \varphi(\bar{x})$)

(ii) $p \in D^-u(\bar{x}) \iff \exists \varphi \in C^1(\Omega)$: $u-\varphi$ has a loc. min at $\bar{x}$ & $D\varphi(\bar{x}) = p$ (& $u(\bar{x}) = \varphi(\bar{x})$).

Pf. $\cdot$ (i) "$\Leftarrow$" $0 = u(\bar{x}) - \varphi(\bar{x}) \geq u(x) - \varphi(x) \qquad x \approx \bar{x}$

$$u(x) - u(\bar{x}) \leq \varphi(x) - \varphi(\bar{x}) = \underbrace{D\varphi(\bar{x})}_{p}\cdot(x-\bar{x}) + o(|x-\bar{x}|), \quad x \to \bar{x}$$

$\Rightarrow u(x) \leq u(\bar{x}) + p\cdot(x-\bar{x}) + o(|x-\bar{x}|) \quad \Rightarrow p \in D^+u(\bar{x})$

$\boxtimes \Leftarrow D^+$

For D^-: $p \in D^-u(\bar{x}) \iff -p \in D^+(-u)(\bar{x})$

HW

Then get "$\Leftarrow$" for $p \in D^-u(\bar{x})$.

"$\Rightarrow$" more tricky ... but elementary see [E] or [BCD].

$\boxtimes$

Corollary (i) $u \in USC(\Omega)$ is a vis. SUB SOL. of (E) $\iff$

$\forall x \in \Omega \ \forall p \in D^+u(x) \quad F(p, u(x), x) \leq 0.$

(ii) $u \in LSC(\Omega)$ is a vis SUPER SOL. of (E) $\iff$ $\forall x \in \Omega$

$\forall p \in D^-u(x) \quad F(p, u(x), x) \geq 0.$

Pf: Lemma. ∎

Cor (Consistency 2). If u is a visco. sol. of (E) in Ω &

u is diff.ble at $\bar{x} \in \Omega \Rightarrow F(Du(\bar{x}), u(\bar{x}), \bar{x}) = 0.$

Pf. u diff.ble at $\bar{x} \Rightarrow Du(\bar{x}) \in D^+u(\bar{x}) \cap D^-u(\bar{x})$.

$$\left.\begin{array}{l}\text{(i) } u \text{ v. subsol.} \Rightarrow F(Du(\bar{x}), u(\bar{x}), \bar{x}) \leq 0 \\ \text{(ii) } u \text{ v. SUPERSOL.} \Rightarrow F(Du(\bar{x}), u(\bar{x}), \bar{x}) \geq 0\end{array}\right\} \Rightarrow F(\ldots) = 0$$

∎

Rmk. u v. sol. of (E) & $u \in Lip(\Omega)$

$\Rightarrow \quad F(Du, u, x) = 0 \quad$ a. e. in Ω.