LECTURE 7, March 21, 2023

COMPARISON PRINCIPLE FOR CLASSICAL SOLUTIONS of H-J EQUATIONS.

<u>Prop.</u> Supp. $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$, $u, v \in C(\mathbb{R}^n \times [0,T])$ with 1st order partial derivatives $\forall x \in \mathbb{R}^n$, $0 < t \leq T$, & $u - v$ has compact support in x $\forall t$ fixed,

$$\begin{cases} u_t + H(D_x u, x) \leq v_t + H(D_x v, x) & \text{in } \mathbb{R}^n \times]0,T] \\ u \leq v & \text{at } t = 0 \end{cases}$$

$\Rightarrow$ $u(x,t) \leq v(x,t)$ $\forall x \in \mathbb{R}^n, \forall t \in [0,T]$.

<u>Cor.</u> (UNIQUENESS). Supp. u, v as above & satisfy

$$(CP) \begin{cases} w_t + H(D_x w, x) = 0 & \text{in } \mathbb{R}^n \times]0,T] \\ w(x,0) = g(x) \end{cases}$$

$\Rightarrow$ $u = v$ in $\mathbb{R}^n \times [0,T]$.

<u>Pf of Cor.</u> by Comp. princ. $\Rightarrow u \leq v$ $\forall x, t$

& also $v \leq u$. ∎

<u>Proof of Prop.</u>: (Idea: similar to Maximum Principle for $\Delta u = f(x)$)

<u>St. 1</u> Ass. in addition $u_t + H(D_x u, x) < v_t + H(D_x v, x)$ $\forall x, t$.

$u - v$ has a max in $\mathbb{R}^n \times [0,T]$ in (x_0, t_0).

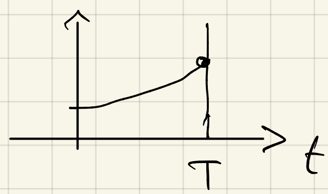
<u>goal</u>: $(u-v)(x_0,t_0) \le 0$

If. $t_0 = 0$ done by initial condition.

If $t_0 > 0$ $D_x u = D_x v$ at (x_0, t_0) $\Rightarrow$ $u_t < v_t$ at (x_0, t_0)

If $t_0 < T$ $u_t = v_t$ at (x_0, t_0) $\otimes$

If $t_0 = T$



$(u-v)_t (x_0, T) \ge 0$

$\otimes$ with $u_t < v_t$

∎ if "<".

<u>Step 2</u>. General case $u_t + H(D_x u, x) \le v_t + H(D_x v, x)$

$\eta > 0$ $u_\eta(x,t) = u(x,t) - \eta t$ at $t = 0$ $u_\eta = u$

$$\frac{\partial u_\eta}{\partial t} = \frac{\partial u}{\partial t} - \eta \qquad D_x u = D_x u_\eta \Rightarrow \frac{\partial u_\eta}{\partial t} + H(D_x u_\eta, x) = u_t - \eta + H(D_x u, x)$$

$$\le v_t + H(D_x v, x) - \eta < v_t + H(D_x v, x)$$

By Step 1 $u_\eta \le v$ $\forall x, t$

$\quad u(x,t) - \eta t$ $\quad \eta \to 0 \Rightarrow u(x,t) \le v(x,t)$ $\forall x, t$.

∎

Look for a notion of solution weaker than classical & stronger than a.e. + Lip, having ∃ & uniq. of soln. of Cauchy problem.

→ "SEMICONCAVE sols" for $p \mapsto H(p,x)$ CONVEX

- S. N. Kružkov ~1960-67
- A. Douglis ~1961-1965

→ "VISCOSITY SOLS." • ok for nonconvex H

• ok for a much larger class of eqs., also 2^{nd} order.

• M. CRANDALL, P.L. LIONS, L.C. EVANS ~1983.

SEMICONCAVE FUNCTIONS

[Cannarsa - Sinestrari book 20...].

Def. $f \in C(\mathbb{R}^n)$ is SEMICONCAVE with constant $C_1 \in \mathbb{R}$ if

$$(SC) \quad f(x+z) - 2f(x) + f(x-z) \leq C_1 |z|^2 \qquad \forall x, z \in \mathbb{R}^n$$

Ex. 1 f CONCAVE $\Rightarrow$ f is SeConc. with $C_1 = 0$

Pf $x = \frac{x+z}{2} + \frac{x-z}{2}$ $\quad f(x) \overset{\text{conc.}}{\geq} \frac{1}{2} f(x+z) + \frac{1}{2} f(x-z)$

$\Rightarrow 0 \geq -2f(x) + f(x+z) + f(x-z)$ $\Rightarrow$ (SC) w. $C_1 = 0$ ∎

Vicev. HW : f Se.Conc. with $C_1 = 0$ $\Rightarrow$ f concave.

Rmk (SC) $\Leftrightarrow$ $\frac{1}{|z|}\left(\frac{f(x+z) - f(x)}{|z|} - \frac{f(x) - f(x-z)}{|z|} \right) \leq C_1$

1^{st} ord. diff. quot.

(SC) means : boundedness from above of 2^{nd} ord. diff quot.

Ex 2. $f\in C^2(\mathbb{R}^n)$. Then f is s.conc. w. C $\iff$

$D^2 f - C\, Id_n \le 0$ neg. semidef.

Pf. HW! use Taylor expansion ☐

Prop. $f\in C(\mathbb{R}^n)$ is s.conc. w. C $\iff$

$x\mapsto f(x) - C\frac{|x|^2}{2}$ is CONCAVE.

Pf. "$\Longleftarrow$" only. $x = \frac{x+z}{2} + \frac{x-z}{2}$

$$f(x) - C\frac{|x|^2}{2} \overset{\text{conc}}{\ge} \frac{1}{2}\left(f(x+z) - C\frac{|x+z|^2}{2} + f(x-z) - C\frac{|x-z|^2}{2}\right) =$$

$$\left[\ |x+z|^2 + |x-z|^2 = |x|^2+|z|^2+\cancel{2x\cdot z} + |x|^2+|z|^2 - \cancel{2x\cdot z}\ \right]$$

$$= \frac{1}{2}\left[f(x+z)+f(x-z) - \frac{C}{2}\left(2|x|^2+2|z|^2\right)\right] \implies$$

$$2f(x) - f(x+z) - f(x-z) \ge -C|z|^2 \implies (SC)$$

☐ $\frac{1}{2}$

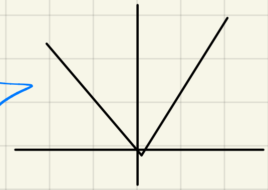
Ex.3 $f(x) = |x|^2 - \alpha|x|$, $\alpha>0$

$n=1$ f not diff.le at $x=0$

it is se.con. w. $C=2$

$$f(x) - \frac{C}{2}|x|^2 = -\alpha|x| \quad \text{concave.}$$ ☐

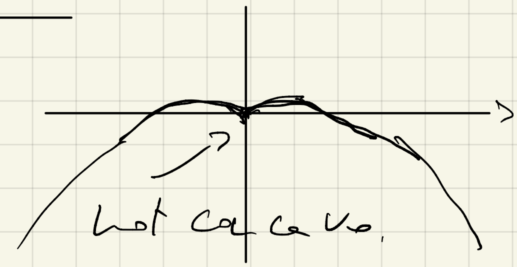
Ex. 5 $f(x) = |x|$



Test. is $g(x) = |x| - c\frac{|x|^2}{2}$ concave?

No $\forall c$ $\qquad g \sim |x|$ as $x \to 0$

$|x|$ NOT concave



Cor. f se. conc. $\Rightarrow$ f loc. Lip.

$\not\Leftarrow$

Pf. $f(x) = \text{conc.} + c\frac{|x|^2}{2} \Rightarrow f$ Lip. loc. $\boxtimes$

Lip loc

So semiconc. is intermediate prop. between Loc Lip & C^2 with $D^2 f \le c$.

Thm. (Douglis - Kružkov) (CP) $\begin{cases} u_t + H(D_x u) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$

H convex & superlinear.

$g \in \mathrm{Lip}(\mathbb{R}^n)$, $H \in C^2(\mathbb{R}^n)$ $\Rightarrow$ $\exists$ at most one generalized sol of (CP) in the foll.g sense:

$u : \mathbb{R}^n \times [0, +\infty[\to \mathbb{R}$ Lip., $u(x,0) = g(x)$ $\forall x$

$u_t + H(D_x u, x) = 0$ a.e. in $\mathbb{R}^n \times (0, +\infty)$ & $\exists K$:

(1) $u(x+z,t) - 2u(x,t) + u(x-z,t) \le K\left(1 + \frac{1}{t}\right)|z|^2$ $\quad \forall t > 0$ $\forall x, z \in \mathbb{R}^n$.

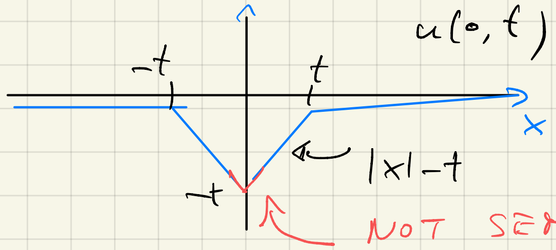
Pf NO see [Evans]. $\boxtimes$

N.B. (1) is semiconcavity of $x \mapsto u(x,t)$ $t > 0$ w. $c = K\left(1 + \frac{1}{t}\right)$. $\boxtimes$

Ex.

$n=1$ $\quad \begin{cases} u_t + u_x^2 = 0 \\ u(x,0) = 0 \end{cases}$ $\qquad u_{HL}(x,t) \equiv 0$ is se.con.

The other weak sol.n.



NOT SEMICOCAVE.

$\to$ NOT GEN. SOL.

$\blacksquare$

Cor. $H \in C^2(\mathbb{R}^n)$, conv., superl., $g \in Lip(\mathbb{R}^n)$ & SEMICONC.

$\Rightarrow$ $u(x,t)$ = Hopf-Lax formula is the UNIQUE GEN.ED SOLN. of (CP) in the sense of the Thm.

Pf. Remains to check only (1) $\forall t>0$ for

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t H^*\left(\frac{x-y}{t}\right) + g(y) \right\}$$

Goal: $\exists K$:

$$(\star) = u(x+z,t) - 2u(x,t) + u(x-z,t) \le K\left(1+\frac{1}{t}\right)|z|^2$$

Choose y : $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$

$$(\star) \le tL\left(\frac{x+z-y_1}{t}\right) + g(y_1) - 2tL\left(\frac{x-y}{t}\right) - 2g(y) + tL\left(\frac{x-z-y_2}{t}\right) + g(y_2)$$

$\forall y_1, y_2 \in \mathbb{R}^n$

Choose y_1: $x+z-y_1 = x-y$ $\qquad y_1 = z+y$

y_2: $x-z-y_2 = x-y$ $\qquad y_2 = y-z$

$$(\star) \le g(z+y) - 2g(y) + g(y-z) \le C_g |z|^2$$

$\Rightarrow$ Goal with $K = C_g$. $\blacksquare$

Other cases [Evans]. $g \in Lip$ (NOT se. conc.)

$D^2 H - c_0 Id \geq 0$, $c_0 > 0$ $\Rightarrow$ (1) holds for Hopf-Lax $\Rightarrow$ Hopf-Lax is the unique sol. in gen. sense of (CP). ∎

INTRODUCTION TO VISCOSITY SOLUTIONS.

Ref Chp. 10 of [Evans], Chapt. 2 [BCD].

Idea 0 "download the missing derivatives on smooth test functions"

Recall the CLASSICAL "weak or distributional solutions" of div-form eqs.

(DE) $\quad \mathrm{div}\, \underline{F}(x,u) = 0 \quad$ in $\Omega \subseteq \mathbb{R}^N$ open

$\underline{F}$ is C^1 vector field. $\quad \varphi \in C^1_c(\Omega)$ ← compact support.

multiply (DE) by φ & $\int_\Omega$

$$0 = \int_\Omega \varphi(x)\, \mathrm{div}\, \underline{F}(x, u(x))\, dx = \int_\Omega \left(\mathrm{div}(\varphi \underline{F})\, dx - \underline{F} \cdot \nabla \varphi \right) dx$$

$$\mathrm{div}(\varphi \underline{F}) = \varphi\, \mathrm{div} F + \underline{F} \cdot \nabla \varphi$$

div Thm

$$= \int_{\partial\Omega} \varphi \underline{F} \cdot \underbrace{\underline{n}_e}_{\text{exter. normal}} d\sigma - \int_\Omega \underline{F} \cdot \nabla \varphi\, dx$$

$= 0$

$\Rightarrow$ Any classical sol. of (DE) sats. also

$$(*) \quad \int_\Omega \underline{F}(x,u(x)) \cdot D\varphi(x)\,dx = 0 \qquad \forall \varphi \in C_c^1(\Omega).$$

this makes sense even for $u \in L^1_{loc}(\Omega)$

Def. weak sol of (DE) is $u \in L^1_{loc}(\Omega)$: $(*)$ holds.

OK for CONSERVATION LAWS !

BUT $\quad u_t + H(D_x u, x) = 0 \quad$ (HJ) $\quad$ not in div form.

weak don't work !

Idea 1 "move the missing derivatives using "maximum principle" & approximate by "vanishing viscosity"."

Def. • $u \in C(\mathbb{R}^n \times (0,\infty))$ is a VISCOSITY SUB-SOLUTION of (HJ) if $\forall\ \varphi \in C^2(\mathbb{R}^n \times (0,\infty))$ & $\forall\ (\bar{x},\bar{t})$ points of MAXIMUM of $u - \varphi$

$$\varphi_t + H(D_x\varphi, \cdot)\Big|_{(\bar{x},\bar{t})} \leq 0$$

• u is visc. SUPERSOLUTION if $\forall \varphi \in C^2$ & $\forall\ (\tilde{x},\tilde{t})$ point of MINIMUM of $u - \varphi$

$$\varphi_t + H(D_x\varphi, \cdot)\Big|_{(\tilde{x},\tilde{t})} \geq 0.$$

• u is a v-Sol. of (HJ) if it is SUB & SUPER Sol.

MOTIVATION: vanishing viscosity approximation for (HJ). $\varepsilon > 0$

$$(HJ_\varepsilon) \qquad u^\varepsilon_t + H(D_x u^\varepsilon, x) = \varepsilon \underbrace{\Delta_x u^\varepsilon}_{\text{artificial viscosity}}$$

$$\Delta_x = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

FACTS (HJ_ε) + init. condit. has sols in C^2 in x & C^1 in t, one expects $u^\varepsilon \to u$ as $\varepsilon \to 0+$ (in some sense ...) and "u solves HJ".

FACTS KNOWN ESTIMATES on u^ε sol of (HJ_ε) (with $|u(\cdot,0)| \le C$)

- $|u^\varepsilon(x,t)| \le C \quad \forall \varepsilon \ \forall x, t$
  $\{u^\varepsilon\}$ is EQUIBOUNDED
- $|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \le \omega(|x-y| + |t-s|)$

$\lim_{r \to 0} \omega(r) = 0$ EQUICONTINUITY.

- NO UNIFORM est.s on Du^ε or u^ε_t

Ascoli-Arzelà Thm. $\exists\ \varepsilon_j \searrow 0+$! $u^{\varepsilon_j} \to u$ locally uniformly as $j \to \infty$ (i.e. $\forall K \subset\subset \Omega = \mathbb{R}^n \times [0, \infty)$

$$\sup_K |u^{\varepsilon_j} - u| \to 0 \quad \text{as } j \to \infty \ ).$$

Prop. $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$, $u^j \in C^2_x \cap C^1_t(\Omega)$ sols of

$$u^j_t + H(D_x u^j, x) = \varepsilon^j \Delta_x u^j \quad , \quad \varepsilon^j \to 0+ \quad ,$$

$u^\delta \to u$ loc unif. $\Rightarrow$ u is a VISCOSITY SOLUTION

of (HJ).