LECTURE 6, March 14, 2023

PLAN : Recap: • If $\exists$ classical sol. in $[0, \bar{t}]$
we expect it is unique
• but don't expect $\exists$ globally in time
in fact the value fn. v is not C^1 ....

• want. v "generalized solution" of HJ
• want also UNIQUENESS of GEN. SOLN.

Special case $H = H(p)$ $\rightsquigarrow$ Hopf-Lax formula (1960-70)

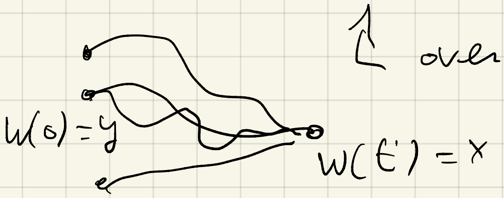
for (CP)
$$\begin{cases} u_t + H(D_x u) = 0 \\ u(x,0) = g(x) \end{cases}$$

Ass.
• $H : \mathbb{R}^n \to \mathbb{R}$ indep. of x & CONVEX & SUPERLINEAR.
• $g : \mathbb{R}^n \to \mathbb{R}$ Lipschitz.

Def. : $L = H^*$. Recall : CANDIDATE SOLUT.

$$v(x,t) = \inf \left\{ \int_0^t L(\dot{w}(s))\,ds + g(y) : w(0) = y,\ w(t) = x,\ w \in C^1 \right\}$$

$\uparrow$ over y & w



Thm. $v(x,t) = u(x,t) := \min_{y \in \mathbb{R}^n} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\} \quad \forall t > 0.$ (H-L)

$\rightarrow$ H-L formula.

Note : the R.H.S. is a FINITE-DIMENSIONAL MINIMIZATION PROBLEM!

$\to$ much easier to compute v !

Proof. 1. $\exists$ min in (H−L). Fix $x \in \mathbb{R}^n$, $t>0$

$$|g(y)-g(0)| \le L_g|y| \implies |g(y)| \le |g(0)| + L_g|y|$$

$$\left|\frac{x-y}{t}\right| = \frac{|y|}{t} + o(|y|) \quad \text{as } |y|\to\infty . \quad L \text{ superlinear} \implies$$

$$\frac{L\left(\frac{x-y}{t}\right)}{\frac{|y|}{t}} \to +\infty \quad \text{as } |y|\to\infty \implies$$

bdd

$$\frac{tL\left(\frac{x-y}{t}\right) + g(y)}{|y|} \to +\infty \quad \text{as } |y|\to\infty$$

$$\implies \underbrace{tL\left(\frac{x-y}{t}\right) + g(y)}_{} \to +\infty$$

$$\exists \min_y \{ \quad t \quad \} \qquad \boxtimes \ 1.$$

2. "$v \le u$". Choose $w(s) = y + \frac{s}{t}(x-y)$ $\quad w(0)=y$, $w(t)=x$

$w \in C^1$, $\dot w(s) = \frac{x-y}{t}$

$$v(x,t) \le \int_0^t L\left(\frac{x-y}{t}\right) ds + g(y) = \underbrace{tL\left(\frac{x-y}{t}\right) + g(y)}_{} \quad \forall y \in \mathbb{R}^n$$

$$\implies v(x,t) \le \min_y \{ \quad \} = u(x,t) . \qquad \boxtimes$$

3. "$v \ge u$". Fix any $w \in C^1$, $w(0)=y$, $w(t)=x$

$$L\Big(\underbrace{\frac{1}{t}\int_0^t \dot w(s)\,ds}_{\frac{x-y}{t}}\Big) \underset{\text{Jensen}}{\le} \frac{1}{t}\int_0^t L(\dot w(s))\,ds$$

$\forall y\ \forall w \in \mathcal{A}_t$

$$tL\left(\frac{x-y}{t}\right) + g(y) \le \int_0^t L(\dot w(s))\,ds + g(y)$$

$\Rightarrow \inf_y \underbrace{\quad}_{} \le \inf_{y,w} \underbrace{\quad}_{} = v(x,t)$

" $u(x,t)$.

$\boxtimes$

Def. (HL) $u(x,t) := \begin{cases} \min_y \{ t L(\frac{x-y}{t}) + g(y) \}, & t > 0 \\ g(x), & t = 0. \end{cases}$

this is a candidate sol. of (CP).

Lemma (a simplified version of Dynamic Programming Principle)

$\forall x \in \mathbb{R}^n \quad \forall\ 0 \le s < t$

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s) L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}$$

Rmk. $s = 0 \quad u(y,s) = g(y) \longrightarrow$ def. of u.

Proof. 0. min $\exists$. We'll see that $y \mapsto u(y,s)$ is Lip, so I can repeat previous argument. $\Rightarrow \{ \ \} \to +\infty$ as $|y| \to \infty \Rightarrow \exists \min_y \{ \ \}$. $\boxtimes$

1. "$\le$" Fix y, s, choose $z \in \mathbb{R}^n$:

$$u(y,s) = s L\left(\frac{y-z}{s}\right) + g(z)$$

$u(x,t) \le t L\left(\frac{x-z}{t}\right) + g(z) \le \oplus$ use convexity of L.

$$\frac{x-z}{t} = \frac{x-y}{t} + \frac{y-z}{t} = \left(1 - \frac{s}{t}\right) \frac{x-y}{t-s} + \frac{s}{t} \frac{y-z}{s} =$$

$\left(1 - \frac{s}{t}\right) = \frac{t-s}{t}$ $\qquad \lambda = \frac{s}{t} \in]0,1[$

$\textcircled{+} \leq \cancel{t} \frac{t-s}{\cancel{t}} L\left(\frac{x-y}{t-s}\right) + \cancel{t} \frac{s}{\cancel{t}} L\left(\frac{y-z}{s}\right) + g(z) =$

$$= (t-s) L\left(\frac{x-y}{t-s}\right) + \underbrace{s L\left(\frac{y-z}{s}\right) + g(z)}_{u(y,s)} \qquad \forall y$$

$$\Rightarrow \quad u(x,t) \leq \min_y \left\{ (t-s) L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}. \qquad \square \text{ "}\leq\text{"}$$

2. " $\geq$ " similar, see [Evans]. $\square$

<u>Lemma</u>: $u(x,t) = (HL) = \begin{cases} \min\{ \quad \} & t>0 \\ g(x), & t=0 \end{cases}$

is Lipschitz in $\mathbb{R}^n \times [0, +\infty)$, in particular

$$\lim_{t \to 0+} u(x,t) = g(x).$$

<u>Proof</u>. 1. Lip in x, $t>0$ fixed. $x, \hat{x} \in \mathbb{R}^n$

$u(\hat{x},t) - u(x,t) \leq \ldots$ choose y: $u(x,t) = tL\left(\frac{x-y}{t}\right) + g(y)$

$$\leq \cancel{tL\left(\frac{\hat{x}-(\hat{x}-x+y)}{t}\right)} + g(\hat{x}-x+y) - \cancel{tL\left(\frac{x-y}{t}\right)} - g(y)$$

$$= g(\hat{x}-x+y) - g(y) \leq L_g |\hat{x}-x|.$$

Exchange the roles of $\hat{x}$ & x $\Rightarrow$

$$|u(\hat{x},t) - u(x,t)| \leq L_g |\hat{x}-x|. \qquad \square \text{ Lip } x$$

2. Lip in t at $t=0$! Look for C_1 :

$$|u(x,t)-g(x)| \le C_1 t \qquad \forall t>0 \qquad \underline{\text{GOAL}}$$

$$u(x,t)-g(x) \underset{y=x}{\le} tL(0)+\cancel{g(x)}-\cancel{g(x)} = tL(0)$$

☒ $\frac{1}{2}$

Remarks : C_1? ! $g(x)-u(x,t)\le C_1 t$ i.e.

$u(x,t) \ge g(x) - C_1 t$ ← Goal. Use $g(y)-g(x) \ge -L_g|y-x|$

$$u(x,t) = \min_y \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} \pm g(x)$$

$$\ge g(x) + \min_y \left\{ tL\Big(\underbrace{\frac{x-y}{t}}_{z}\Big) - L_g \underbrace{|y-x|}_{|z|t} \right\} =$$

$$= g(x) + \min_{z\in\mathbb{R}^n} \left\{ tL(z) - tL_g|z| \right\} =$$

$$C_1|z| = \max_{|p|=C_1} z\cdot p \qquad -C_1|z| = \min_{|p|=C_1}(-z\cdot p)$$

$$= g(x) + \min_{z\in\mathbb{R}^n} \min_{|p|\le L_g} t\left\{ L(z) - z\cdot p \right\}$$

$$= g(x) - t\max_z \max_{|p|\le L_g} \left\{ z\cdot p - L(z) \right\}$$

$$= g(x) - t\max_{|p|\le L_g} L^*(p) \underset{\text{Duality}}{=} g(x) - t\underbrace{\max_{|p|\le L_g} H(p)}_{C_{H,g}}$$

$$u(x,t) - g(x) \ge -tC_{H,g}$$

$\Rightarrow$ GOAL with $\bar{C} = L(0) \vee C_{H,g}$ ☒ 2

3. Lip in $t>0$: $0<\hat{t}<t$ Goal :

$$|u(x,t) - u(x,\hat{t})| \le C|t-\hat{t}|.$$

(DPP) $$u(x,t) = \min_y \left\{ (t-\hat{t}) L\left(\frac{x-y}{t-\hat{t}}\right) + \underline{u(y,\hat{t})} \right\}$$

instead of $g(y)$

Repeat the proof done for $\hat{t}=0$ with g replaced by $u(\cdot,\hat{t})$: they have the same Lip const $\boxtimes$

4. $$|u(x,t) - u(\hat{x},\hat{t})| \le L_g |x-\hat{x}| + \bar{C}|t-\hat{t}| . \quad \boxtimes$$

$\exists\, u(x,\hat{t})$

Thm. Let u be given by (H-L) be differentiable at (x,t), $t>0$. Then

$$\left[u_t + H(D_x u)\right]\Big|_{(x,t)} = 0.$$

Proof. 1. "$\le$" $\quad 0 \ne q \in \mathbb{R}^n$, $\quad t-s = h > 0$

(DPP) $\Rightarrow$ (DPP)

$$u(x+hq, t+h) = \min_y \left\{ h L\left(\frac{x+hq-y}{h}\right) + u(y,t)\right\} \overset{y=x}{\le}$$

$$\le h L\left(\frac{\cancel{h}q}{\cancel{h}}\right) + u(x,t)$$

$$\Rightarrow \frac{u(x+hq, t+h) - u(x,t)}{h} \le L(q) \qquad h \searrow 0_+$$

$$\left(u_t + D_x u \cdot q\right)\Big|_{(x,t)} = \frac{\partial u}{\partial(q,1)}(x,t) \le L(q) \qquad \forall q$$

$$\Rightarrow \quad \underbrace{u_t + \sup_{q\in\mathbb{R}^n} \{ D_x u \cdot q - L(q) \}} \le 0$$

by Duality thm. $L^*(D_x u) = H(D_x u)$ ▯ "$\le$"

$\underline{2}$. "$\ge$" Given (x,t), $t>0$: GOAL $\exists q \in \mathbb{R}^n$:

$$u_t + D_x u \cdot q - L(q) \ge 0 \qquad (G)$$

$$\left(\Leftrightarrow \quad u_t + \sup_q \{ \underbrace{}_{} \} \ge 0 \right).$$
$$\ge H(D_x u)$$

$$u(x,t) - u(y,s) \ge tL\left(\frac{x-z}{t}\right) + \cancel{g(z)} - sL\left(\frac{y-z}{s}\right) - \cancel{g(z)} = (\times)$$

choose z : $u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$

Look for (y,s) : $\frac{x-z}{t} \overset{(\star)}{=} \frac{y-z}{s}$ $\qquad y = z + s\frac{x-z}{t} =$

$h = t - s > 0 \quad s = t - h \quad \frac{s}{t} = 1 - \frac{h}{t} \qquad = \cancel{z} + x - \cancel{z} - \frac{h}{t}(x-z)$

$(\star) \Leftrightarrow \quad y = x - \frac{h}{t}(x-z) \qquad$ Back to $(\times)$

$$u(x,t) - u\left(x - \frac{h}{t}(x-z)\right) \ge \underset{h}{(t-s)}\, L\left(\frac{x-z}{t}\right)$$

$$\frac{u(x,t) - u\left(x - \frac{h}{t}(x-z)\right)}{h} \ge L\left(\frac{x-z}{t}\right) \qquad \text{Let } h \to 0+$$

$$\left(u_t + D_x u \cdot \frac{x-z}{t} \right)\Big|_{(x,t)} \ge L\left(\frac{x-z}{t}\right) \qquad q := \frac{x-z}{t}$$

$$u_t + D_x u \cdot q \Big|_{(x,t)} \ge L(q) \qquad \text{the GOAL } (G). \ ▯$$

Thm (Rademacher [Evans - Gariepy. Measure theory ...])

$f: \mathbb{R}^n \supseteq A \to \mathbb{R}$ Lipschitz $\Rightarrow$ f is differentiable almost everywhere.

Cor. u def. by (H-L) is Lip in $\mathbb{R}^n \times [0,+\infty)$ & satisfies

("CP") $\begin{cases} u_t + H(D_x u) = 0 & \text{a.e. in } \mathbb{R}^n \times]0,+\infty) \\ u(x,0) = g(x) & \forall x. \end{cases}$ ∎

Can call "weak sol." of (CP) a sol. of "CP" i.e. a Lip. fn solving PDE a.e.

Q: Is it UNIQUE? Answ.: NO

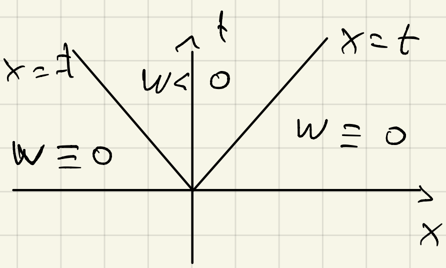
Example $n=1$ $\begin{cases} u_t + u_x^2 = 0 & \mathbb{R} \times (0,\infty) \\ u(x,0) = 0 \end{cases}$

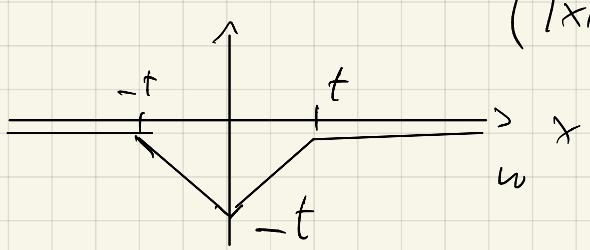
$H(p) = p^2$, $L(q) = \frac{q^2}{4}$

(HL): $u(x,t) = \min_{y \in \mathbb{R}} \left\{ t \frac{(x-y)^2}{4t^2} + 0 \right\} = 0$

$u(x,t) \equiv 0$ is a CLASSICAL SOL.N !

Another weak sol. is $w(x,t) = \begin{cases} 0, & |x| \geq t \\ |x| - t, & |x| < t \end{cases}$

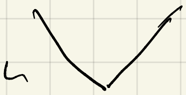




w is Lip.

for $|x| < t$ $w_t = -1$ $w_x = \text{sign } x$

$$(w_x)^2 = 1 \quad \Rightarrow \quad w_t + (w_x)^2 = 0 \qquad \forall\ |x| < t$$

$\Rightarrow$ HJ satisfied everywhere, except on 

<u>Conclusion</u>: Weak sols: "Lip + PDE a.e." is OK for Existence, NOT for uniqueness.

<u>Rmk</u>. Classical sols are too strong for GLOBAL existence, OK for uniqueness: see Verif. thm, & next time! "COMPARISON PRINCIPLE" $\Rightarrow$ UNIQUENESS of CLASSICAL sols.