LECTURE 5, March 14, 2023

<u>Thm.</u> $K \subseteq \mathbb{R}^n$ convex, $f : K \to \mathbb{R}$ convex $\Rightarrow$ $\forall\, x_0 \in \mathring{K}$ $\exists\, r \in \mathbb{R}^n$:

$$f(x) \geq f(x_0) + r \cdot (x - x_0) \qquad \forall\, x \in K.$$

<u>Proof.</u> $A = (\mathring{\mathrm{epi}\, f})$, $B = \{(x_0, f(x_0))\}$ are convex

$A \cap B = \emptyset$, A open : Thm on separation of convex sets in $\mathbb{R}^n \times \mathbb{R}$ $\Rightarrow$ $\exists\ v = (p, \gamma) \in \mathbb{R}^n \times \mathbb{R} \neq (0,0)$ & $\alpha \in \mathbb{R}$:

$$p \cdot x + t\gamma \overset{(*)}{\leq} \alpha \overset{(**)}{\leq} p \cdot x_0 + \gamma f(x_0).$$

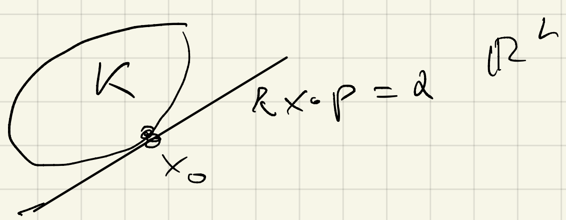
$\forall\, t > f(x)$ $\forall\, x \in \mathring{K}$.

<u>Claim 1</u> $\gamma \leq 0$: if not $\gamma > 0$, let $t \to +\infty$

l.h.s. $\to +\infty$ $\leq \alpha$ $\otimes$ contr.!

<u>Claim 2</u> $\gamma \neq 0$. If $\gamma = 0$ $\Rightarrow$ $p \neq 0$ &

$p \cdot x \leq \alpha \leq p \cdot x_0$ $\quad \forall\, x \in K$ $\quad \otimes$ with $x_0 \in \mathring{K}$



□ Divide $(*)$ & $(**)$ by $|\gamma|$

$$\frac{p}{|\gamma|} \cdot x - t \leq \alpha \leq \frac{p}{|\gamma|} \cdot x_0 - f(x_0) \qquad \forall\, t > f(x)$$

let $t \downarrow f(x)^+$, $r := \dfrac{p}{|\gamma|}$ $\Rightarrow$ $r \cdot x - f(x) \leq \alpha \leq r \cdot x_0 - f(x_0)$

$\Rightarrow$ $f(x) \geq f(x_0) + r \cdot (x - x_0)$. ∎

CONVEX CONJUGATION (FENCHEL TRANSFORM).

"Lagrangian" $L(q,x)$ convex in q. $\forall x$ fixed: I'll omit x

Hypothesis:

(C) $L: \mathbb{R}^n \to \mathbb{R}$ convex

(S) $\lim_{|q|\to\infty} \frac{L(q)}{|q|} = +\infty$ SUPERLINEAR.

Def. The convex conjugate of L is

$$L^*(p) := \sup_{q\in\mathbb{R}^n} \{ q \cdot p - L(q) \}$$

Lemma. $L^*(p) < +\infty$ $\forall p \in \mathbb{R}^n$ & sup is a max.

Pf. $\frac{q\cdot p - L(q)}{|q|} \to -\infty$ as $|q| \to \infty$ by (S)

$\Rightarrow$ $q\cdot p - L(q) \to -\infty$ " $\Rightarrow$ by (a cor. of)

Weierstrass Thm. $\Rightarrow$ $q\cdot p - L(q)$ has a max in $\mathbb{R}^n$. ∎

CONNECTION with LEGENDRE TRANSFORM in Calc. Var.:

$L\in C^1$, L_q is bijective, $p = L_q(q)$ has a UNIQUE SOL.

$q = Q(p)$ (& $Q \in C^1$). Compute L^*:

$$D_q\{\dots\} = p - L_q(q) = 0 \iff q = Q(p)$$

$$L^*(p) = p\cdot Q(p) - L(Q(p)) = H(p)$$ def. last week.

Def $\forall L$ satisf. (C) (S) $H(p) = L^*(p)$.

Thm. (CONVEX DUALITY, FENCHEL)

Ass. L sats. (C) (S) $\Rightarrow$ $H = L^*$ sats:

- $p \mapsto H(p)$ is convex in $\mathbb{R}^n$
- $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty$ $\quad$ H superlinear.
- $H^*(q) = L(q) \quad \forall q \in \mathbb{R}^n$

N.B. CONJUGACY IS INVOLUTIVE! $(L^*)^* = L$.

Pf.: 1. $H = L^*$ convex: $p, \hat{p} \in \mathbb{R}^n$, $0 \le \tau \le 1$.

$$H(\tau p + (1-\tau)\hat{p}) = \max_{q\in\mathbb{R}^n} \{ \tau p \cdot q + (1-\tau)\hat{p}\cdot q - \underbrace{\ }_{\tau + (1-\tau)} L(q) \}$$

$$\le \max_q \{ \tau p \cdot q - \tau L(q) \} + \max_q \{ (1-\tau)\hat{p}\cdot q - (1-\tau) L(q) \}.$$

$(\tau > 0 \ (1-\tau) > 0)$

$$= \tau H(p) + (1-\tau) H(\hat{p}). \qquad \boxed{} \ 1.$$

2. Goal $\frac{H(p)}{|p|} \to +\infty$ as $|p| \to \infty$. Fix $\lambda > 0$

$p \neq 0 \quad q = \frac{\lambda p}{|p|} \qquad H(p) \ge p \cdot \frac{\lambda p}{|p|} - L\left(\frac{\lambda p}{|p|}\right) \ge$

has norm λ

$$\ge \lambda |p| - \underbrace{\max_{|\xi| = \lambda} L(\xi)}_{C_\lambda}.$$

$$\frac{H(p)}{|p|} \ge \lambda - \frac{C_\lambda}{|p|} \Rightarrow \liminf_{|p|\to\infty} \frac{H(p)}{|p|} \ge \lambda \qquad \forall \lambda > 0$$

$$\Rightarrow \lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty. \qquad \boxed{} \ 2.$$

3. Goal 3.1 $H^* \leq L$

$$\xi = q$$

$$H(p) + L(q) = \sup_{\xi \in \mathbb{R}^n} \{ \xi \cdot p - L(\xi) \} + L(q) \geq q \cdot p - \cancel{L(q)} + \cancel{L(q)}$$

$$\geq q \cdot p \quad \forall\, q, p \in \mathbb{R}^n .$$

$$\Rightarrow \quad L(q) \geq q \cdot p - H(p) \quad \forall\, p \in \mathbb{R}^n$$

$$\geq \sup_{p \in \mathbb{R}^n} \{ q \cdot p - H(p) \} = H^*(q) . \quad \boxed{3.1}$$

Goal 3.2. : $H^* \geq L$. $\underbrace{}_{}$ $L(p)$ — $-\sup = \inf$ —

$$H^*(q) = \sup_p \{ p \cdot q - \sup_\xi \{ \xi \cdot p - L(\xi) \} =$$

$$= \sup_p \inf_\xi \{ p \cdot q - p \cdot \xi + L(\xi) \} \qquad (+)$$

Then support hyperplane: $\exists\, s \in \partial L(q)$,

$$L(\xi) \underset{(++)}{\geq} L(q) + s \cdot (\xi - q) \qquad \forall\, \xi \in \mathbb{R}^n$$

In $(+)$ $p = s$:

$$H^*(q) \geq \inf_\xi \{ s \cdot \cancel{(q - \xi)} \overset{(++)}{+} L(q) + s \cdot \cancel{(\xi - q)} \} = L(q)$$

$\boxed{}$.

Ref. T. Rockafellar, Convex Analysis ~70.

Examples. • $L(q) = \frac{|q|^2}{2} \Rightarrow H(p) = \frac{|p|^2}{2}$

HW • $L(q) = \frac{|q|^\alpha}{\alpha}$, $\alpha > 1$ | Q : $L^*(p) = \dots ?$

JENSEN INEQUALITY.

Notat: $g : \mathbb{R}^k \supseteq U \to \mathbb{R}^n$: $\fint_U g(x)\,dx := \frac{1}{|U|}\int_U g(x)\,dx$

$|U|$ = Lebesgue meas of U.

Thm: $f : \mathbb{R}^n \to \mathbb{R}$ convex, $u : \mathbb{R}^k \supseteq U \to \mathbb{R}^n$ integrable.

$$\Rightarrow \quad f\left(\fint_U u\,dx\right) \leq \fint_U f(u(x))\,dx .$$

Rmk. generalizes $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)}{2} + \frac{f(y)}{2}$

$$f\left(\frac{1}{N}\sum_{i=1}^N x_i\right) \leq \frac{1}{N}\sum_{i=1}^N f(x_i). \qquad (HW) \qquad \blacksquare$$

Pf. $p = \fint_U u\,dx \in \mathbb{R}^n$. Take $r \in \partial f(p)$:

$$f(q) \geq f(p) + r\cdot(q-p) \quad \forall q \in \mathbb{R}^n. \text{ Take } q = u(x)$$

$$\Rightarrow f(u(x)) \geq f(p) + r\cdot(u(x)-p) \quad \forall x \in U$$

$$\fint_U \downarrow \quad \geq \fint_U$$

$$\Rightarrow \fint_U f(u(x)) \geq f(p) + r\cdot\Big(\underbrace{\fint_U u(x) - p}_{=0}\Big) = f\left(\fint_U u\,dx\right). \qquad \blacksquare$$

Connection between Calc. Vars & H-J eqs. continued.

$$(CP)\quad \begin{cases} u_t + H(D_x u, x) = 0 & \text{in } \mathbb{R}^n \times]0,T[\\ u(x,0) = g(x) \end{cases}$$

$H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ cont. & $\begin{cases} p \mapsto H(p,x) \text{ convex } \forall x \text{ fixed} \\ \lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty \end{cases}$,,

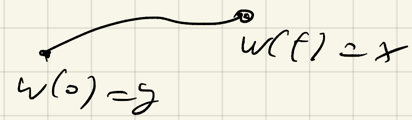
Def. $L(\cdot, x) := H^*(\cdot, x) \quad \forall x$ fixed.

Duality thm. $\Rightarrow \quad H(p,x) = \max_{q \in \mathbb{R}^n} \{ q \cdot p - L(q,x) \}$.

last week I proposed the candidate sol. of (CP)!

$$v(x,t) := \inf \{ I[w] + g(w(0)) : w \in C^1([0,t], \mathbb{R}^n),\ w(t) = x \}$$

$$I[w] = \int_0^t L(\dot{w}(s), w(s))\, ds.$$



Rmk. $v(0,x) = g(x)$

Notat.: $F : \Omega \to \mathbb{R} \qquad \operatorname{argmax}_x F = \{ \bar{x} \in \Omega : F(\bar{x}) = \max F \}$
$F(\bar{x}) \geq F(x)\ \forall x \in \Omega$.

Prop. (Verification theorem). $u \in C^1(\mathbb{R}^n \times (0,\bar{t})) \cap C(\mathbb{R}^n \times [0,\bar{t}])$ sol. of (CP). Then

(i) $u(x,t) \leq v(x,t)$

(ii) $u(x,t) = v(x,t) \iff \exists\, \underline{X} \in C^1([0,\bar{t}], \mathbb{R}^n)$:

$$\text{(E)} \quad \dot{\underline{X}}(s) \cdot Du(\underline{X}(s), s) - L(\dot{\underline{X}}(s), \underline{X}(s)) = H(Du(\underline{X}(s)), \underline{X}(s)) \quad \forall s \in (0,\bar{t}).$$

and in such a case

$$v(x,t) \underset{(+)}{=} I[\underline{X}(\cdot)] + g(\underline{X}(0)) = \min_w \{ I[w] + g(w(0)) : w \in C^1,\ w(t) = x \}.$$

N.B. 1 $\quad H(p,x) = \max_q \{ q \cdot p - L(q,x) \}$, (E) $\iff$

$$(\star) \begin{cases} \dot{\underline{X}}(s) \in \operatorname{argmax}_q \{ q \cdot Du(\underline{X}(s), s) - L(q, \underline{X}(s)) \} \\ \underline{X}(t) = x \end{cases}$$

"DIFFERENTIAL INCLUSION", it becomes an ODE if argmax is a singleton.

N.B. 2 (†) means $\underline{x}$ solving (E) (or ✡) is OPTIMAL for the minimization of $I[w] + g(w(0))$, $w(t) = x$.

Pf. x, $0 < t \leq \bar{t}$ $\quad w \in \mathcal{Q}_t$ admissible traj.

$$\varphi(s) := u(w(s), s) + \int_s^t L(\dot{w}(s), w(s))\, ds.$$

$$\dot{\varphi}(s) = u_t(w(s), s) + \underbrace{D_x u(w(s), s) \cdot \dot{w}(s) - L(\dot{w}(s), w(s))}_{}$$

$$\leq \max_q \{ D_x u(w(s), s) \cdot q - L(q, w(s)) \}$$

$$= H(D_x u(w(s), s), w(s)) = 0$$

by HJ eq. in (CP).

& $= 0 \iff w(s) = \underline{x}(s)$ satisf. (E) or ✡

$$\Rightarrow \begin{cases} \varphi(t) \leq \varphi(0) & \forall t \leq \bar{t} \quad \forall w(\cdot) \in \mathcal{Q}_t \\ \varphi(t) = \varphi(0) & \text{if } w = \underline{x} \text{ solv. (E)} \end{cases}$$

$$\varphi(t) = u(w(t), t) = u(x, t) \leq \varphi(0) = u(w(0), 0) + \int_0^t L(\dot{w}, w)\, ds$$

$$\underset{u|_{t=0} = g}{=} g(w(0)) + I[w]$$

$$\Rightarrow u(x, t) \leq \inf_w \left(g(w(0)) + I[w] \right) = v(x, t)$$

& "=" holds $\iff w = \underline{x}$ satg. (E) (i.e. ✡). ∎

Corollary Under the ass. of Verif. Thm. +

(I_pCV) $L \in C^1$, $H \in C^1$, $\exists\, Q(p,x) \in C^1$: $p = L_q(Q(p,x), x)$ (*).

Then the sol. of
$$(\star)\quad \begin{cases} \dot{X}(s) = H_p(Du(X(s),s), X(s)) \\ X(t) = x \end{cases}$$
if $\exists$ $\forall\, t \in [0,T]$, it is optimal for $\min\,[I[w] + g(w(0))]$ among $w \in C^1$: $w(t) = x$.

Pf. We saw (I_pCV) $\operatorname{argmax}_q \{q \cdot p - L(q,p)\} = \{Q(p,x)\}$

Then ($\star$) becomes the ODE
$$(\star\star)\quad \begin{cases} \dot{X}(s) = Q(Du(X(s),s), X(s)). \\ X(t) = x \end{cases}$$
Remains to prove $Q = H_p$:
$$H(p,x) = Q(p,x)\cdot p - L(Q(p,x),x)$$
$$H_p = Q + \cancel{Q_p \cdot p} - \cancel{L_q\, Q_p} = Q$$
∎

Rmk. If $H_p = Q$ is C^1 and Du is loc. Lip, then $Q(Du,\cdot)$ is loc. Lip. $\Rightarrow$ the trajectory of ($\star\star$) exists at least in some interval $[0,\tilde{t}]$, $\tilde{t} \le t$ small enough. ∎