LECTURE 4 , March 8, 2023

On the interval of existence of the solution to

$$(CP) \begin{cases} u_t + H(D_x u) = 0 \\ u(x,0) = g(x) \end{cases}$$

Corollary $H, g \in C^2(\mathbb{R}^n)$, D^2H, D^2g bounded,

$$\bar{T} := \sup \{ t \geq 0 : \det(I + t D^2H(Dg(y)) D^2g(y)) > 0 \quad \forall y \in \mathbb{R}^n \}$$

$\Rightarrow$ $\bar{T} > 0$ & u def. by (D) solves (CP) in $\mathbb{R}^n \times]0, \bar{T}[$.

Pf. We must check $\underline{Y}(\cdot, t) = \underline{X}(\cdot, t)^{-1}$ $\exists$ $\forall$ $t \in]0, \bar{T}[$.

- $\bar{T} > 0$ because $\det(I + t D^2H \ldots) \simeq 1$ $\forall t \in [0, \varepsilon[$, $\forall y$
- $t < \bar{T}$ use the inverse fn. thm to invert

$$\underline{X}(y,t) = y + t DH(Dg(y)), \quad D_y \underline{X} = I + t D^2H(Dg(y)) D^2g(y)$$

(see last lecture)

$\det D_y \underline{X} > 0 \Rightarrow \exists$ $\underline{Y}(x,t)$ inverse of $y \mapsto \underline{X}(y,t)$

$\Rightarrow$ u $\exists$ $\forall t < \bar{T}$ & Thm of Lect. 3 gives the conclusion.

Corollary 2 Ass. $H, g \in C^2(\mathbb{R}^n)$ both convex or both concave. Then $\bar{T} = +\infty$ & $\exists$ global classical sol.

to (CP). (in $\mathbb{R}^n \times (0,+\infty)$).

Proof $n=1$, $H'', g'' \geq 0$

$$\frac{\partial X}{\partial y} = 1 + t\, H''(g'(y))\, g''(y) \geq 1 \quad \forall t \geq 0 \quad \forall y \quad \Rightarrow \bar{T} = +\infty.$$

$n > 1$ Need facts from LINEAR ALGEBRA:

A, B symm matrices, POSIT. SEMIDEF. $(A \geq 0,\ B \geq 0)$

$\overset{HW}{\Longrightarrow}$ eigenvalues of AB are all ≥ 0

(Warning! $AB \geq 0$ FALSE).

$A = D^2 H(Dg(y)) \geq 0$, $B = D^2 g(y) \geq 0$ $\quad \forall y \in \mathbb{R}^n$

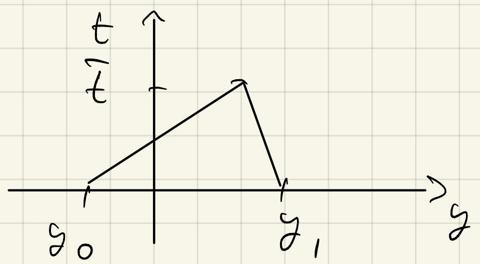
$\Rightarrow$ Eigenvalues of $(I + tAB) = 1 + t\, \underbrace{\text{eigenv}(AB)}_{\geq 0} \geq 1 \quad \forall y$

$$\det(I + tAB) = \prod_{i=1}^{n} \text{eigenvalues}(I + tAB) \geq 1 \quad \forall y$$

$\Rightarrow \bar{T} = +\infty$. $\boxtimes$

Examples where the characteristics cross.

Ex: $n = 1$ $\quad X(y,t) = y + t\, H'(g'(y))$ straight lines



Look for $\bar{t} > 0$: occurs

$$y_0 + tH'(g'(y_0)) \overset{?}{=} y_1 + tH'(g'(y_1))$$

$$y_1 - y_0 = t\left(H'(g'(y_0)) - H'(g'(y_1))\right)$$

$$\bar{t} = \frac{y_1 - y_0}{H'(g'(y_0)) - H'(g'(y_1))} \quad \overset{>0}{} \quad > 0 \quad \text{if DEN.} > 0$$

$\Leftrightarrow \quad H'(g'(y_0)) > H'(g'(y_1))$ e.g. $\forall y_0 < y_1$ if

$H' \circ g'$ is strictly decreasing. Ex $H'' > 0$, $g'' < 0$.

<u>Note</u>: $\bar{t}$ is similar to the crossing time for cons. laws. Here we expect a jump not in u but in $D_x u$ at $t = \bar{t}$.

CONNECTION between H-J & cons. law for $\underline{\underline{n=1}}$.

$$\begin{cases} u_t + H(u_x) = 0 \\ u(x,0) = g(x) \end{cases} \qquad \text{assume } u \in C^2 \qquad v := u_x$$

$\partial/\partial x \quad \downarrow$

$$\begin{cases} v_t + H(v)_x = 0 \\ v(x,0) = g'(x) \end{cases} \qquad \text{is a cons. law for } v \text{ with flux } f = H$$

Special ex.: $\quad u_t + \frac{u_x^2}{2} = 0 \quad \longleftrightarrow \quad v_t + (v^2)_x = 0$

H-J for class. mech. $\Leftrightarrow$ Burger, $v_t + v\, v_x = 0$

"$\Rightarrow$" $\bar{t}$ is the same for the 2 equations. (CHECK IT!)

& discont. of v at $t = \bar{t}$ $\Leftrightarrow$ discont. of u_x ∎

<u>Q</u>! IMPORTANT! what happens after $t = \bar{t}$.

"ANSWER": $\exists$ global generalised solutions .....

... TO COME ...

CONNECTIONS AMONG H-J PDEs & Calc. of Vars. & Analytic Mechanics.

Start from $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ Lagrangian.

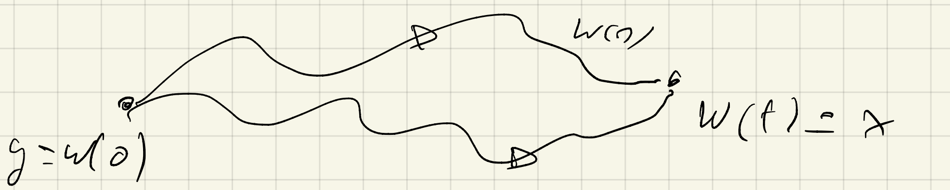
$$(q, x) \mapsto L(q, x)$$

Def. Action functional.

$$I[w(\cdot)] = \int_0^t L(\dot{w}(s), w(s))\, ds \qquad w(\cdot) \in \mathcal{A}$$

$$\mathcal{A} := \{ w \in C^1([0,t], \mathbb{R}^n) : w(0) = y,\ w(t) = x \}.$$

Probl. $\min_{\mathcal{A}} I$



Thm (Necessary conds. of optimality) Supp. $L \in C^2$, $\underline{X}(\cdot) \in \mathcal{A}$ minimizing I, $\underline{X}(\cdot) \in C^2$ $\Rightarrow$ $\underline{X}(\cdot)$ solves the Euler-Lagrange system of ODEs.

$$\text{(EL)} \qquad -\frac{d}{ds} L_q(\dot{\underline{X}}, \underline{X}) + L_x(\dot{\underline{X}}, \underline{X}) = 0 \qquad 0 < s < t$$

Pf. e.g. [Evans]. $\boxtimes$

Ex. "Mechanical" case $L(q, x) = \frac{m|q|^2}{2} - \phi(x)$

potential

$L_q = mq$, $L_x = -\nabla\phi$ $\Rightarrow$

$$\text{(EL)} \qquad -(m\dot{\underline{X}})' - D\phi = 0, \qquad m\ddot{\underline{X}} = -D\phi$$

Newton's law with force $F = -\nabla\phi$.

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CONNECTION (EL) $\Leftrightarrow$ Hamiltonian system.

Def. for $\bar{x}(\cdot) \in \mathcal{Q}$, generalized moment is

$$p(s) := L_q(\dot{\bar{x}}(s), \bar{x}(s)).$$

Hypothesis on L: $\forall x, p \in \mathbb{R}^n$ $\exists$ unique sol. q to

$p = L_q(q, x)$, & call it $q =: Q(p, x)$ & ass.

$Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1, i.e. $p = L_q(Q(p,x), x)$.

Rmk. $\dot{\bar{x}}(s) = Q(p(s), \bar{x}(s))$. $(\ast)$

Def. Hamiltonian ass. to $L \in C^1$ is

$$H(p, x) := p \cdot Q(p, x) - L(Q(p, x), x) \quad \forall x, p \in \mathbb{R}^n.$$

Ex.: Mechanical: $L = \frac{m|q|^2}{2} - \phi(x)$, $L_q = mq \overset{?}{=} p$

$q = \frac{p}{m} =: Q(p, x)$ $\quad H(p, x) = p \cdot \frac{p}{m} + \phi(x) - \cancel{m} \frac{|p|^2}{2m^{\cancel{2}}} =$

$$= \frac{|p|^2}{2m} + \phi(x) \quad \blacksquare$$

Thm. $\bar{x}(\cdot)$ solves (EL), $p(s) = L_q(\dot{\bar{x}}(s), \bar{x}(s))$ $\Rightarrow$

$$\Rightarrow \begin{cases} \dot{\bar{x}}(s) = H_p(p(s), \bar{x}(s)) \\ \dot{p}(s) = -H_x(p(s), \bar{x}(s)) \end{cases}$$

Hamiltonian Syst. of H (ass. to L.)

Pf: See [Evans.] $\blacksquare$

N.B. Cal. Vars. $\longleftrightarrow$ (EL) eqs. $\xleftrightarrow{\text{gen. moment.}}$ Ham. Syst.

$\updownarrow$ method of charact.

Can we use ? $\longleftarrow - - - \longrightarrow$ H-J PDE

Consider charact. eq (6) for $z(\cdot)$ with $\underline{x}(\cdot), p(\cdot)$ solving the Ham syst. :

$$\dot{z}(s) = p(s)\cdot H_p(p(s), \underline{x}(s)) - H(p(s), \underline{x}(s)) =$$

use $H \leftrightarrow L$)

$$= \cancel{p(s)\cdot \dot{\underline{x}}(s)} - \cancel{p(s)\cdot \underbrace{Q(p(s), \underline{x}(s))}_{=\dot{\underline{x}}}} + L(Q(p(s), \underline{x}(s)), \underline{x}(s))$$

by (*)

$$\Rightarrow \quad \dot{z}(s) = L(\dot{\underline{x}}(s), \underline{x}(s)) \quad \Rightarrow$$

$$z(t) = g(y) + \int_0^t L(\dot{\underline{x}}(s), \underline{x}(s))\, ds$$

This suggest to consider

$$v(x,t) := \text{"min"} \Big\{ \underbrace{\int_0^t L(\dot{w}(s), w(s))\, ds}_{I[w]} + g(y) : w \in \mathcal{Q} \Big\}$$

$\uparrow$ if it $\exists$ $\qquad$ $(w(0)=y,\ w(t)=x)$

value function ass. to minimiz. of $I[w] + g(y)$

I expect that v "solves" the

$$\text{(CP)} \quad \begin{cases} u_t + H(D_x u, x) = 0 \\ u(x,0) = g(x). \end{cases}$$

We will prove that.

- If $p \mapsto H(p,x)$ is convex in $\mathbb{R}^n$ $\forall x$ (not nec. C^1)

can define a L s.t. $H \Longleftrightarrow L$

- if $\exists$ sol. nec' of (CP) in $]0, \bar{T}[$ $\Rightarrow$

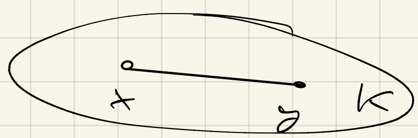
$$u(x,t) = v(x,t) = \text{value fn.}$$

- v is a "generalized solution" of (CP) in $\mathbb{R}^n \times (0, +\infty)$, ... the unique ONE.

INTRODUCTION TO CONVEX ANALYSIS.

Recalls. • $K \subseteq \mathbb{R}^n$ is convex if $\forall$ $x, y \in K$ $\forall t \in [0,1]$

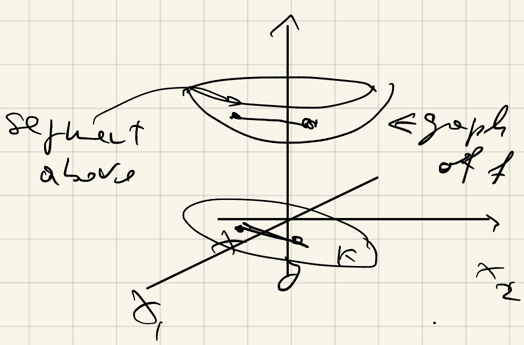
$tx + (1-t)y \in K$.



- if K is convex, $f: K \to \mathbb{R}$ is convex if

$\forall x, y \in K$ $\forall t \in [0,1]$

$$f(tx + (1-t)y) \leq f(x) + (1-t)f(y)$$



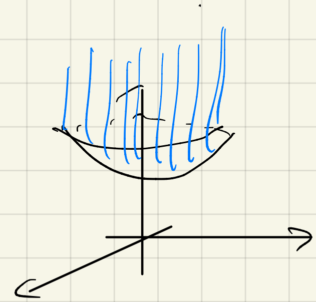
Basic properties

1. f convex $\Rightarrow$ $\forall c \in \mathbb{R}$ $K_c := \{x \in K : f(x) \leq c\}$ sub-level set

K_c is CONVEX

2. $\text{epi} f := \{(x,t) : x \in K, t \geq f(x)\}$

is convex in $\mathbb{R}^n \times \mathbb{R}$

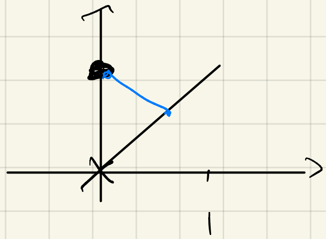


& also $(\overset{\circ}{\text{epi } f}) =$ interior of epi f is convex

3. f convex $\Rightarrow$ f is locally Lip in $\overset{\circ}{K}$

Pf: see W. Fleming: Fns. of several vars, 1977.

Rmk.



$K = [0,1]$

$$f(x) = \begin{cases} 1 & x = 0 \\ x & 0 < x \le 1 \end{cases}$$

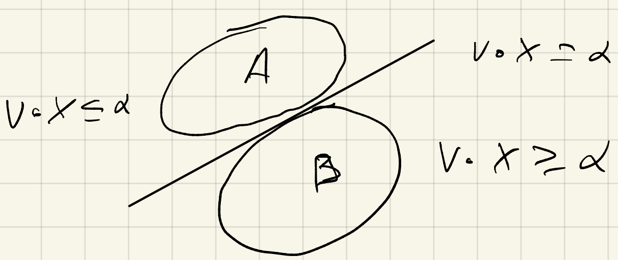
disc. at 0, but convex $\boxed{}$

4. Thm. of separation of convex sets.
(geometric form of Hahn-Banach thm. in dim $= n < +\infty$)
(Ref.: H. Brezis: Analisi Funzionale).

Thm. $A, B \subseteq \mathbb{R}^n$ convex, $A, B \neq \emptyset$, $A \cap B = \emptyset$, A open.
$\Rightarrow \exists v \in \mathbb{R}^n : v \neq 0$ & $\alpha \in \mathbb{R}$:

$\forall x \in A \quad v \cdot x \le \alpha \le v \cdot y \quad \forall y \in B$ i.e.

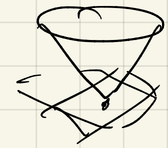
the hyperplane $\{v \cdot x = \alpha\}$ separates A & B.



Pf No.

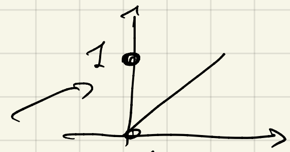
Thm. $K \subseteq \mathbb{R}^n$ convex, $f: K \to \mathbb{R}$ convex $\Rightarrow$
$\forall x_0 \in \overset{\circ}{K} \ \exists z \in \mathbb{R}^n : f(x) \ge f(x_0) + z \cdot (x - x_0) \ \forall x \in K$,

i.e. $\forall x_0 \in \overset{\circ}{K}$ graph of f has a "supporting hyperplane" passing through $(x_0, f(x_0))$.



<u>Rmk.</u> if f diff.le at x_0 $\quad r = Df(x_0)$

<u>Rmk.</u> $x_0 \in \partial K$ Thm NOT TRUE



no supporting line.

<u>Def.</u> $\partial f(x_0)$ = the SUBGRADIENT of f at x_0 is the set of all $r \in \mathbb{R}^n$ s.t. $(*)$ holds $\forall x \in K$.