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Abstract. An important generalization of a Nash equilibrium is the case when the players must choose strategies which depend on the other players. The case in zero-sum differential games with players y and z when there is a constraint of the form $g(y,z) \leq 0$ is introduced. The Isaacs' equations for the upper value and the lower value of a zero sum differential game are derived and a condition guaranteeing existence of value is derived. It is also proved that the value functions are the limits of penalized games.

1. Introduction

A generalized zero sum differential game refers to the dynamical system optimization problem with dynamics given by

$$\frac{d\xi}{d\tau} = f(\tau, \xi(\tau), \eta(\tau), \zeta(\tau)), \quad 0 \leq t < \tau \leq T \tag{1.1}$$

$$\xi(t) = x \in \mathbb{R}^n \tag{1.2}$$

$$P(\eta, \zeta) = h(\xi(T)) \tag{1.3}$$

$$g(\eta(\tau), \zeta(\tau)) \leq 0, \quad t \leq \tau \leq T. \tag{1.4}$$

The players are the maximizer η and the minimizer ζ of P. This is the simplest case used to model a system in which the controls must satisfy the constraint $g(y,z) \leq 0$. In this case, we will require the use of the hamiltonians associated to the upper value and the lower value defined by

$$H^+(t,x,r,p) = \min_{z \in Z_g(r)} \max_{y \in Y_z} p \cdot f(t,x,y,z), \qquad H^-(t,x,r,p) = \max_{y \in Y_g(r)} \min_{z \in Z_y} p \cdot f(t,x,y,z), \tag{1.5}$$

with

$$\begin{cases} Y_z = \{y \in Y : g(y,z) \leq 0\}, \qquad Y_g(r) = \left\{y \in Y \mid \max_{z \in Z} g(y,z) \leq r\right\}, \\ Z_y = \{z \in Z : g(y,z) \leq 0\}, \qquad Z_g(r) = \left\{Z \in Z \mid \max_{y \in Y} g(y,z) \leq r\right\}, \end{cases} \tag{1.6}$$

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This paper is an attempt to introduce the important topic of generalized games extended to dynamic games. A generalized game is one in which the player's choice of control may depend on the other players. More precisely, if we consider the N-person game with players $i = 1, 2, \dots, N$ and payoffs $P_i\left(x_i, x^{-i}\right)$ for player i, the generalized game is to

$$\min_{x_i \in X_i(x^{-i})} P_i\left(x_i, x^{-i}\right), \quad i = 1, 2, \dots, N,$$

where $x^{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, x_{i+2}, \dots, x_N)$ and $X_i\left(x^{-i}\right)$, a subset of some Euclidean space, is the strategy set for player i which may depend on the other players. This problem has very important applications in situations in which players cannot choose independently of the other players, for instance when there are shared resource constraints. Refer to Facchinei and Kanzow [7] and the references there for the theory of generalized Nash equilibria.

Suppose $N = 2$ and the two players are denoted y and z with strategy sets Y and Z, respectively. In the generalized game we would have the strategy sets $Y_z = \{y \in Y \mid g(y, z) \leq 0\}$ for player y and $Z_y = \{z \in Z \mid g(y, z) \leq 0\}$ for player z $\big($these correspond to $X_i(x^{-i})$ in the notation above$\big)$. If each player has their own payoff one could consider the standard two-person non-zero sum penalized game for each player:

$$\begin{aligned}
&\min_{y \in Y} P_1(y, z) + \frac{1}{\varepsilon} g^+(y, z) \\
&\min_{z \in Z} P_2(y, z) + \frac{1}{\varepsilon} g^+(y, z)
\end{aligned}$$

where $g^+(y, z) = \max(g(y, z), 0)$. A Nash equilibrium would be a point $({y_\varepsilon}^*, {z_\varepsilon}^*) \in Y \times Z$ which solves this game. The Nash equilibrium for the generalized game would be any limit point of $(y_\varepsilon, z_\varepsilon)$. More directly, a Nash equilibrium of the generalized game is a point $(y^*, z^*) \in \{(y, z) \in Y \times Z \mid g(y^*, z^*) \leq 0\}$ such that

$$\min_{\{y \in Y \mid g(y, z^*) \leq 0\}} P_1(y, z^*) = P_1(y^*, z^*) \qquad \text{and} \qquad \min_{\{z \in Z \mid g(y^*, z) \leq 0\}} P_2(y^*, z) = P_2(y^*, z^*).$$

In a zero sum game in which player 1 is a minimizer and player 2 is a maximizer $P_2 = -P_1$ and so it is logical that the penalty term for the maximizer with payoff P_2 should be $-\dfrac{1}{\varepsilon} g^+(y, z)$. When each player's goal is to *minimize* their own payoff it is clear that the penalty term should be $+\frac{1}{\varepsilon}$, but in a two-person zero-sum game this is no longer true. This has important ramifications for exactly how to penalize to get rid of the constraint as we see in the last section of this paper.

For differential N-person games the situation becomes much more complicated. A Nash equilibrium, even for a standard N-person game is generally assumed to be open loop or closed loop and may not exist at all. For the system of Hamilton-Jacobi equations

$$V_{i,t} + H_i\left(t, x, D_x V_1, D_x V_2, \ldots, D_x V_N\right) = 0, \quad V_i(T, x) = h_i(x), \quad i = 1, 2, \ldots, N$$

associated with an N-person game a typical assumption is there is a unique feedback control for each player $u_i\left(t, x, D_x V_i\right)$ and it is Lipschitz in all variables and $D_x V_i$ in particular, and independent of the gradient of the other value functions. This is rarely the case and even if it is the case, there is no theory comparable to viscosity solution theory to use to conclude the value functions are the unique solution of the system. For example, consider the two person non-zero sum game. The value functions (V_1, V_2) satisfy the system

$$V_{1,t} + \min_{z_1 \in Z_1} D_x V_1 \cdot f(t, x, z_1, z_2(t, x, D_x V_1, D_x V_2)) = 0$$

$$V_{2,t} + \min_{z_2 \in Z_2} D_x V_2 \cdot f(t, x, z_1(t, x, D_x V_1, D_x V_2), z_2) = 0$$

with $V_i(T, x) = h_i(x), i = 1, 2$. Note that the assumption z_i is independent of $D_x V_i$ is a fairly stringent assumption, as is assuming any sort of regularity of z_i in any variable. Except in special cases a general theory of such a system is nonexistent. See Lenhart [10] and Engler & Lenhart [6] for a situation when certain systems are tractable. Refer also to Bressan & Shen [2] and the references there for recent results on certain first order systems. For systems of second order pde's associated with differential games the story is a bit better (see, e.g., Ishii & Koike [8]).

For two-person zero-sum games the situation is that a system of pde's becomes a single pde. In the two-person zero- sum game the concept of which player goes first reduces the problem for a system to determine the Hamilton-Jacobi equations for the upper value and lower value of the game. Uniqueness results for viscosity solutions of first order Hamilton-Jacobi equations even leads to a result for existence of value in the game. The purpose of this paper is to determine the Hamilton-Jacobi equations for the upper and lower values of a generalized two-person zero-sum game. The main goal of this paper is to define the upper and lower values of such a game and to derive the Isaacs' equations for these values. Our definitions are motivated by considering the penalized version of the games to get rid of the constraints and, as mentioned, even the correct penalization is an issue. Unfortunately, the theory of generalized N-person non-zero sum differential games awaits a breakthrough in the theory of nonlinear first order systems of pde's similar to the breakthrough achieved by Crandall and Lions through viscosity solutions.

Many excellent references exist for the general theory of differential games [1, 3, 4, 5, 9]. Refer also to [1] and [11] for the general theory of viscosity solutions and the connection with differential games.

2. Isaacs' Equations for Generalized Differential Games

The game dynamics are given by (1.1) and we consider this simplest form for a generalized differential game. We will use the following assumption on the dynamics throughout unless specified otherwise. These conditions are stronger than necessary.

$$\left.\begin{array}{l} Y \text{ and } Z \text{ are compact subsets of euclidean spaces} \\ (t,x,y,z)\mapsto f(t,x,y,z) \text{ is continuous,} \\ |f(t,x,y,z)-f(t,x',y,z)|\le C_f|x-x'|, \text{ and } |f(t,x,y,z)|\le C_f(1+|x|) \\ |h(x)-h(x')|\le C_h|x-x'|, \text{ and} \\ g: Y\times Z\to\mathbb{R} \text{ is continuous} \end{array}\right\} \tag{H}$$

for constants $C_f, C_h > 0$. For every $0\le t<\tau\le T$, we set

$$\mathcal{Z}[t,\tau]=\left\{\zeta:[t,\tau]\to Z\subset\mathbb{R}^{q_1} \;\middle|\; \zeta \text{ is Lebesgue measurable}\right\},$$
$$\mathcal{Y}[t,\tau]=\left\{\eta:[t,\tau]\to Y\subset\mathbb{R}^{q_2} \;\middle|\; \eta \text{ is Lebesgue measurable}\right\}.$$

Consider the maps $y\mapsto \max_{z\in Z} g(y,z)$ and $z\mapsto \max_{y\in Y} g(y,z)$ and their $r-$sublevel sets

$$Z_g(r)=\left\{z\in Z \;\middle|\; \max_{y\in Y} g(y,z)\le r\right\}, \qquad Y_g(r)=\left\{y\in Y \;\middle|\; \max_{z\in Z} g(y,z)\le r\right\}. \tag{2.1}$$

We shall define

$$\mathcal{Z}_g(r)[t,\tau]=\left\{\zeta\in\mathcal{Z}[t,\tau] \;\middle|\; \zeta(s)\in Z_g(r), t\le s\le\tau\right\},$$
$$\mathcal{Y}_g(r)[t,\tau]=\left\{\eta\in\mathcal{Y}[t,\tau] \;\middle|\; \eta(s)\in Y_g(r), t\le s\le\tau\right\},$$
$$\mathcal{Z}_\eta[t,\tau]=\{\zeta\in\mathcal{Z}[t,\tau]\mid g(\eta(s),\zeta(s))\le 0, t\le s\le\tau\},$$
$$\mathcal{Y}_\zeta[t,\tau]=\{\eta\in\mathcal{Y}[t,\tau]\mid g(\eta(s),\zeta(s))\le 0, t\le s\le\tau\}.$$

We will frequently write Z_g for $Z_g(0)$ and Y_g for $Y_g(0)$ and similarly for any of the sets depending on r when $r=0$. The strategy set $\Gamma(t)$ (or $\Gamma[t,T]$ if we emphasize the interval) for the maximizer η is the set of all nonanticipating maps $\alpha:\mathcal{Z}[t,T]\to\mathcal{Y}[t,T]$ such that $g(\alpha[\zeta](s),\zeta(s))\le 0, t\le s\le T$, and if $\zeta(\tau)=\widehat{\zeta}(\tau),\ a.e.\, t\le\tau\le s$, for each $t\le s\le T$, then $\alpha[\zeta](\tau)=\alpha[\widehat{\zeta}](\tau),\ a.e.\, t\le\tau\le s$,

$$\Gamma(t)=\{\alpha:\mathcal{Z}[t,T]\to\mathcal{Y}[t,T]\mid \alpha[\zeta]\in\mathcal{Y}_\zeta[t,T], \zeta\in\mathcal{Z}[t,T]\}$$

Similarly, the strategy set for the minimizer ζ is $\Delta(t)$ (or $\Delta[t,T]$), the set of all nonanticipating maps $\beta:\mathcal{Y}[t,T]\to\mathcal{Z}[t,T]$ and $g(\eta(s),\beta[\eta](s))\le 0, t\le s\le T$,

$$\Delta(t)=\{\beta:\mathcal{Y}[t,T]\to\mathcal{Z}[t,T]\mid \beta[\eta]\in\mathcal{Z}_\eta[t,T], \eta\in\mathcal{Y}[t,T]\}$$

The upper and lower values $V^{\pm}:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ are defined by

$$V^+(t,x)=\sup_{\alpha\in\Gamma[t]}\inf_{\zeta\in\mathcal{Z}_g[t,T]}P(\alpha[\zeta],\zeta),\qquad V^-(t,x)=\inf_{\beta\in\Delta[t]}\sup_{\eta\in\mathcal{Y}_g[t,T]}P(\eta,\beta[\eta]).\tag{2.2}$$

Given a control ζ and a strategy α, (or a control η and a strategy β), the trajectory $\xi_{t,x}^{\alpha,\zeta}(\cdot)$ (or $\xi_{t,x}^{\eta,\beta}(\cdot)$) on $[t,T]$ is the solution of (1.1) corresponding to $(\eta=\alpha[\zeta],\zeta)$ (or $(\eta,\zeta=\beta[\eta])$. The upper value V^+ quantifies the most the maximizer can get assuming full knowledge of the minimizer's choice of control and assuming the minimizer will choose his control to minimize the payoff. In the upper value the minimizer plays first. In the generalized upper game, $\zeta\in\mathcal{Z}_g[t,T]$ must satisfy $g(\eta(\tau),\zeta(\tau))\le 0$ for all $t\le\tau\le T$ without knowledge of $\eta\in\mathcal{Y}[t,T]$. This will manifest in the Isaacs equation. Similarly, V^- represents the most the maximizer can lose assuming the maximizer plays first. The worst case is that the minimizer will assume full knowledge of the maximizer's control and then the minimizer chooses a strategy to minimize the maximum payoff. In the lower value the maximizer plays first and $\eta\in\mathcal{Y}_g[t,T]$ must be chosen so that $g(\eta(\tau),\zeta(\tau))\le 0$ for all $t\le\tau\le T$ without knowledge of $\zeta\in\mathcal{Z}[t,T]$.

To see that our definitions are on the right track we must first show that the upper value is at least as large as the lower value.

Lemma 2.1. *Assume that both Z_g and Y_g are nonempty. Then*

$$V^-(t,x)\ \le\ V^+(t,x)\qquad\forall(t,x)\in[0,T]\times\mathbb{R}^n.$$

Proof. By the assumption, both $\mathcal{Z}_g[t,T]$ and $\mathcal{Y}_g[t,T]$ are nonempty. For every $\zeta\in\mathcal{Z}_g[t,T]$, let $\beta_\zeta\in\Delta(t)$ be the constant nonanticipating map such that $\beta_\zeta[\eta]\equiv\zeta$ for all $\eta\in\mathcal{Y}[t,T]$. Since $\zeta\in\mathcal{Z}_g[t,T]$, it holds $\zeta(s)\in Z_g(0)$ and

$$g(\zeta(s),\eta(s))\le\max_{y\in Y}g(\zeta(s),y)\le 0\qquad\forall s\in[t,T],\eta\in\mathcal{Y}[t,T].$$

Thus, $\zeta\in\mathcal{Z}_\eta[t,T]$ for all $\eta\in\mathcal{Y}[t,T]$. Fix $\varepsilon>0$ and choose $\eta_\zeta\in\mathcal{Y}_g[t,T]$ such that

$$P(\eta_\zeta,\beta_\zeta(\eta))=P(\eta_\zeta,\zeta)\ge\sup_{\eta\in\mathcal{Y}_g[t,T]}P(\eta,\zeta)-\varepsilon\ge V^-(t,x)-\varepsilon.$$

Pick an $\eta_0\in\mathcal{Y}_g[t,T]$, let $\alpha_\varepsilon:\mathcal{Z}[t,T]\to\mathcal{Y}[t,T]$ be such that

$$\alpha_\varepsilon[\zeta]=\begin{cases}\eta_\zeta, & \forall\zeta\in\mathcal{Z}_g[t,T],\\ \eta_0, & \forall\zeta\in\mathcal{Z}[t,T]\backslash\mathcal{Z}_g[t,T].\end{cases}$$

Since $\eta \in \mathcal{Y}_\zeta[t,T]$ for all $\zeta \in \mathcal{Z}[t,T]$, $\eta \in \mathcal{Y}_g[t,T]$, we have that $\alpha_\varepsilon[\zeta] \in \mathcal{Y}_\zeta[t,T]$ for all $\zeta \in \mathcal{Z}[t,T]$. Therefore, $\alpha_\varepsilon \in \Gamma(t)$ and

$$V^+(t,x) \ \geq \ \inf_{\zeta \in \mathcal{Z}_g[t,T]} P(\alpha_\varepsilon[\zeta],\zeta) \ = \ \inf_{\zeta \in \mathcal{Z}_g[t,T]} P(\eta_\zeta,\zeta) \ \geq \ V^-(t,x) - \varepsilon,$$

and this complete the proof. □

The formulation of a differential game in (2.2) is due to Elliott and Kalton [5]. Refer to [4] and [1] for a concise introduction to differential games and Hamilton-Jacobi equations. In order to describe the Isaacs' equations for the game we need to introduce the hamiltonians which arise.

Definition 2.2. The upper hamiltonian is

$$H^+(t,x,r,p) = \begin{cases} \min\limits_{z \in Z_g(r)} \max\limits_{y \in Y_z} p \cdot f(t,x,y,z), & \text{if } Z_g(r) \neq \emptyset, \\ +\infty, & \text{if } Z_g(r) = \emptyset. \end{cases}$$

The lower hamiltonian is

$$H^-(t,x,r,p) = \begin{cases} \max\limits_{y \in Y_g(r)} \min\limits_{z \in Z_y} p \cdot f(t,x,y,z), & \text{if } Y_g(r) \neq \emptyset, \\ -\infty, & \text{if } Y_g(r) = \emptyset. \end{cases}$$

Notice that both $Z_g(r)$ and $Y_g(r)$ are compact and increasing with respect to r,and the above upper and lower hamiltonians are well-defined. Moreover, the map $r \mapsto H^+(t,x,r,p)$ is decreasing and lower semicontinuous, and the map $r \mapsto H^-(t,x,r,p)$ is increasing and upper semicontinuous. Thus,

$$(H^+)^{usc}(t,x,r,p) = H^+(t,x,r-0,p) \quad \text{and} \quad (H^-)^{usc}(t,x,r,p) = H^-(t,x,r+0,p) = H^-(t,x,r,p) \tag{2.3}$$

$$(H^+)^{lsc}(t,x,r,p) = H^+(t,x,r+0,p) = H^+(t,x,r,p) \quad \text{and} \quad (H^-)^{lsc}(t,x,r,p) = H^-(t,x,r-0,p) \tag{2.4}$$

where we use the notation $f^{usc}(x) = \limsup_{y \to x} f(y)$ and $f_{lsc}(x) = \liminf_{y \to x} f(y)$ is the upper (resp. lower) semicontinuous envelope of the function f. Also the notation $f(x \pm 0)$, for instance, means $\lim_{\delta \downarrow 0} f(x \pm \delta)$.

Remark 2.3. 1. Observe that for any function $\alpha : Y \times Z \to \mathbb{R}$,

$$\min_{z \in Z_g} \max_{y \in Y_z} \alpha(y,z) \geq \max_{y \in Y_g} \min_{z \in Z_y} \alpha(y,z). \tag{2.5}$$

Indeed, let $(y',z') \in Y_g \times Z_g \subset Y_{z'} \times Z_{y'}$. Then $\max_{y \in Y_{z'}} \alpha(y,z') \geq \alpha(y',z') \geq \min_{z \in Z_{y'}} \alpha(y',z)$. This implies

$$\max_{y \in Y_{z'}} \alpha(y,z') \geq \max_{y \in Y_g} \min_{z \in Z_y} \alpha(y,z),$$

and then

$$\min_{z\in Z_g}\max_{y\in Y_z}\alpha(y,z)\geq\max_{y\in Y_g}\min_{z\in Z_y}\alpha(y,z).$$

2. The justification for these hamiltonians is based on several observations. Consider the upper game in which the minimizer plays first. In the worst case the minimizer must play a control which will guarantee that the constraint is satisfied and the only way to do that is to choose from Z_g. The maximizer will have full knowledge of the minimizer when choosing a control and therefore will choose from Y_z and not Y_g. A second justification of the hamiltonians is based on the penalty method as we will see in the last section of the paper.

Here is the definition of a viscosity solution. See [1] for the basic theory of viscosity solutions.

Definition 2.4. Let u be a locally bounded function. We say that

- u is a viscosity subsolution of $u_t+F(t,x,u,Du)=0$ if for $\varphi\in C^\infty$,

$$(t_0,x_0)\in\arg\max\,(u^{usc}-\varphi)\implies\varphi_t+F^{usc}(t_0,x_0,\varphi,D\varphi)\geq 0.$$

- u is a viscosity supersolution of $u_t+F(t,x,u,Du)=0$ if

$$(t_0,x_0)\in\arg\min\,(u_{lsc}-\varphi)\implies\varphi_t+F_{lsc}(t_0,x_0,\varphi,D\varphi)\leq 0.$$

We may assume $u^{usc}(t_0,x_0)=\varphi(t_0,x_0)$ in the subsolution definition and $u_{lsc}(t_0,x_0)=\varphi(t_0,x_0)$ in the supersolution definition.

Next we derive the Isaacs' equations using these Hamiltonians satisfied by V^+,V^-. Consider the equations

$$V^+_t+H^+(t,x,0,D_xV^+)=0,\quad (t,x)\in[0,T)\times\mathbb{R}^n, \tag{2.6a}$$

$$V^-_t+H^-(t,x,0,D_xV^-)=0,\quad (t,x)\in[0,T)\times\mathbb{R}^n, \tag{2.6b}$$

and terminal condition $V^\pm(T,x)=h(x),\ \ x\in\mathbb{R}^n$. We begin by deriving the Dynamic Programming Principle (DPP) for $V^\pm$ by modifying the argument in [1, Theorem 1.9] or [4, Theorem 3.1]. The proof of this proposition is very similar to the standard proof and is omitted.

Proposition 2.5. *Assume that both Z_g and Y_g are nonempty. Let $r=0$. For every $(t,x)\in[0,T)\times\mathbb{R}^n]$, it holds*

$$\begin{cases} V^+(t,x) = \sup\limits_{\alpha\in\Gamma[t,t+\delta]}\ \inf\limits_{\zeta\in\mathcal{Z}_g[t,t+\delta]} V^+\left(t+\delta,\xi^{\alpha,\zeta}_{t,x}(t+\delta)\right)\\ V^-(t,x) = \inf\limits_{\beta\in\Delta[t,t+\delta]}\ \sup\limits_{\eta\in\mathcal{Y}_g[t,t+\delta]} V^-\left(t+\delta,\xi^{\alpha,\zeta}_{t,x}(t+\delta)\right)\end{cases}\quad \forall\ 0<\delta<T-t. \tag{2.7}$$

From the DPP the next theorem can be proved.

Theorem 2.6. *Assume that both Z_g and Y_g are nonempty. Let (H) hold and $r = 0$. Then V^+ is the unique continuous viscosity solution of* (2.6a) *and V^- is the unique continuous viscosity solution of* (2.6b).

Proof. We shall prove that V^+ the unique continuous viscosity solution of (2.6a). The assertion for V^- is similar.

1. The proof that $V^+(t,x)$ is in fact locally Lipschitz continuous in both variables follows closely the standard proof and is omitted.

2. We claim that V^+ is a subsolution of (2.6a), namely,

$$V_t^+ + \min_{z\in Z_g(0)} \max_{y\in Y_z} D_x V^+ \cdot f(t,x,y,z) \geq 0, \qquad V^+(T,x) = h(x). \tag{2.8}$$

Given $(t,x) \in [0,T[\times\mathbb{R}^n$ and $\varphi \in C^1$, assume that $V^+ - \varphi$ has a max at (t,x). For every $0 < \delta < T - t$, by the DPP

$$\begin{aligned}
0 &= \sup_{\alpha\in\Gamma[t,t+\delta]} \inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} V^+\left(t+\delta, \xi_{t,x}^{\alpha,\zeta}(t+\delta)\right) - V^+(t,x) \\
&\leq \sup_{\alpha\in\Gamma[t,t+\delta]} \inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} \varphi\left(t+\delta, \xi_{t,x}^{\alpha,\zeta}(t+\delta)\right) - \varphi(t,x) \\
&= \sup_{\alpha\in\Gamma[t,t+\delta]} \inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} \int_t^{t+\delta} \left[\varphi_t(t,x) + D_x\varphi(t,x)\cdot f(t,x,\alpha[\zeta](s),\zeta(s))\right] ds + o(\delta) \\
&= \varphi_t(t,x)\cdot\delta + \sup_{\alpha\in\Gamma[t,t+\delta]} \inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} \int_t^{t+\delta} D_x\varphi(t,x)\cdot f(t,x,\alpha[\zeta](s),\zeta(s))\, ds + o(\delta).
\end{aligned}$$

For every $\alpha \in \Gamma[t,t+\delta]$ and $\zeta \in \mathcal{Z}_g(0)[t,t+\delta]$, since $\alpha[\zeta](s) \in Y_{\zeta(s)}$ for a.e $s \in [t,t+\delta]$, one has

$$\int_t^{t+\delta} D_x\varphi(t,x)\cdot f(t,x,\alpha[\zeta](s),\zeta(s))ds \leq \int_t^{t+\delta} \max_{y\in Y_{\zeta(s)}} D_x\varphi(t,x)\cdot f(t,x,y,\zeta(s))ds,$$

and this implies that

$$0 \ \leq\ \varphi_t(t,x)\cdot\delta + \inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} \int_t^{t+\delta} \max_{y\in Y_{\zeta(s)}} D_x\varphi(t,x)\cdot f(t,x,y,\zeta(s))\, ds + o(\delta). \tag{2.9}$$

In particular, for every $z \in Z_g(-\delta)$, choosing $\zeta(s) = z$ for all $s \in [t,t+\delta]$, we have

$$0 \ \leq\ \varphi_t(t,x)\cdot\delta + \delta\cdot\max_{y\in Y_z} D_x\varphi(t,x)\cdot f(t,x,y,z) + o(\delta).$$

By letting $\delta \to 0+$, we get

$$0 \ \leq\ \varphi_t(t,x) + \min_{z\in Z_g(0-0)} \max_{y\in Y_z} D_x\varphi(t,x)\cdot f(t,x,y,z).$$

This says V^+ is a subsolution of (2.6a).

3. We now show that V^+ is a supersolution of (2.6a). Given $(t,x) \in [0,T[\times\mathbb{R}^n$ and $\varphi \in C^1$, assume that $V^+ - \varphi$ has a min at (t,x). Again by the DPP

$$\begin{aligned}
0 &= \sup_{\alpha\in\Gamma[t,t+\delta]} \inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} V^+\left(t+\delta, \xi_{t,x}^{\alpha,\zeta}(t+\delta)\right) - V^+(t,x) \\
&\geq \sup_{\alpha\in\Gamma[t,t+\delta]} \inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} \varphi\left(t+\delta, \xi_{t,x}^{\alpha,\zeta}(t+\delta)\right) - \varphi(t,x) \\
&\geq \varphi_t(t,x)\cdot\delta + \sup_{\alpha\in\Gamma[t,t+\delta]} \inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} \int_t^{t+\delta} D_x\varphi(t,x)\cdot f(t,x,\alpha[\zeta](s),\zeta(s))\,ds + o(\delta) \\
&\geq \varphi_t(t,x)\cdot\delta + \sup_{\alpha\in\Gamma[t,t+\delta]} \inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} \int_t^{t+\delta} D_x\varphi(t,x)\cdot f(t,x,\alpha[\zeta](s),\zeta(s))\,ds + o(\delta)
\end{aligned}$$

Consider a mutually disjoint covering $Z_1, Z_1, \ldots, Z_N$ of $Z_g(0)$ such that $\sup_{z_1,z_2\in Z_i}|z_1 - z_2| \leq \varepsilon$ for all $i \in \{1,2,\ldots,N\}$. Given any $\zeta \in \mathcal{Z}_g(0)[t,t+\delta]$, we denote by

$$\mathcal{I}^\zeta := \left\{i \in \{1,2,\ldots,N\} \;\middle|\; E_i^\zeta := \{s \in [t,t+\delta] : \zeta(s) \in Z_i\} \neq \emptyset\right\}.$$

Picking $z_i \in E_i^\zeta$ for every $i \in \mathcal{I}^\zeta$, we approximate ζ by the piecewise constant function $\tilde{\zeta}(s) = \sum_{i\in\mathcal{I}^\zeta} z_i \cdot \chi_{E_i^\zeta}(s)$. It is clear that

$$\tilde{\zeta} \in \mathcal{Z}_g(0)[t,t+\delta] \qquad \text{and} \qquad \big|z_i - \zeta(s)\big| \leq \varepsilon \quad \forall s \in [t,t+\delta].$$

Let $\eta_\zeta \in \mathcal{Y}[t,t+\delta]$ be the piecewise constant function

$$\eta_\zeta(s) \;=\; y_i \;\in\; \arg\max_{y\in Y_{z_i}} D_x\varphi(t,x)\cdot f(t,x,y,z_i), \quad \forall s \in E_i^\zeta, i \in \mathcal{I}^\zeta.$$

Define $\alpha_\varepsilon : \mathcal{Z}[t,t+\delta] \to \mathcal{Y}_\zeta[t,t+\delta]$ by $\alpha_\varepsilon[\zeta] = \eta_\zeta$. We get for $s \in E_i^\zeta, i \in \mathcal{I}^\zeta$,

$$\begin{aligned}
D_x\varphi(t,x)\cdot f(t,x,\alpha_\varepsilon[\zeta](s),\tilde{\zeta}(s)) &= D_x\varphi(t,x)\cdot f(t,x,y_i,\tilde{\zeta}(s)) \\
&= \max_{y\in Y_{z_i}} D_x\varphi(t,x)\cdot f(t,x,y,z_i) + o(\varepsilon) \\
&\geq \min_{z\in Z_g(0)} \max_{y\in Y_z} D_x\varphi(t,x)\cdot f(t,x,y,z) + o(\varepsilon).
\end{aligned}$$

In particular,

$$\begin{aligned}
&\inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} \int_t^{t+\delta} D_x\varphi(t,x)\cdot f(t,x,\alpha_\varepsilon[\zeta](s),\zeta(s))\,ds \\
&\qquad \geq \inf_{\zeta\in\mathcal{Z}_g(0)[t,t+\delta]} \int_t^{t+\delta} D_x\varphi(t,x)\cdot f(t,x,\alpha_\varepsilon[\zeta](s),\tilde{\zeta}(s))\,ds + o(\varepsilon)\delta \\
&\qquad\qquad \geq \delta\cdot \min_{z\in Z_g(0)} \max_{y\in Y_z} D_xV^+\cdot f(t,x,y,z) + o(\varepsilon)\delta,
\end{aligned}$$

and this yields

$$0 \geq \varphi_t(t,x)\cdot\delta + \delta\cdot \min_{z\in Z_g(0)} \max_{y\in Y_z} D_xV^+ \cdot f(t,x,y,z) + o(\varepsilon)\delta + o(\delta).$$

Thus, V^+ is a supersolution of (2.6a) as well.

4. Finally, it is direct to show that

$$|H^{\pm}(t,x,0,p) - H^{\pm}(t,x,0,p')| \leq C_f\cdot(1+|x|)\cdot|p-p'|,\ t\in[0,T],\ \ x,p,p'\in\mathbb{R}^n$$
$$|H^{\pm}(t,x,0,p) - H^{\pm}(t,x',0,p)| \leq C_f\cdot|p|\cdot|x-x'|,\ t\in[0,T],\ \ x,x',p\in\mathbb{R}^n.$$

This is enough to conclude that viscosity solutions of (2.6a) and (2.6b) are unique. □

Remark (2.3) also shows that since $H^+(t,x,0,p_x) \geq H^-(t,x,0,p_x)$ it will follow by standard comparison theorems that $V^+(t,x) \geq V^-(t,x)$. Recall that value exists in a game if and only if $V^+ = V^-$. The uniqueness of viscosity solutions for the Hamilton-Jacobi equations allows us to conclude that if the hamiltonians are the same, then the game must have a value.

Corollary 2.7. *If $H^+(t,x,0,p) = H^-(t,x,0,p), t\in[0,T], x,p\in\mathbb{R}^n$, then the game has value $V(t,x) = V^{\pm}(t,x)$.*

The next theorem gives an equivalent formulation of the Hamilton-Jacobi equations.

Theorem 2.8. *Let (H) hold. V^+ is a viscosity solution of* (2.6a) *if and only if it is a viscosity solution of*

$$\min_{z\in Z}\max_{y\in Y}\max\{V_t^+ + D_xV^+\cdot f(t,x,y,z),\ g(y,z)\} = 0. \tag{2.10}$$

Similarly, V^- is a viscosity solution of (2.6b) *if and only if it is a viscosity solution of*

$$\max_{y\in Y}\min_{z\in Z}\min\{V_t^- + D_xV^-\cdot f(t,x,y,z),\ -g(y,z)\} = 0. \tag{2.11}$$

The advantage of the formulations in (2.10) and (2.11) is that these equations submit to standard theorems in viscosity solution theory regarding existence, uniqueness, and numerical solution.

Proof. We will only show that our assertion is true for V^+ since the proof for V^- is similar.

Suppose V^+ is a viscosity subsolution of (2.6a). Then, from (2.3), one has that for $\varphi\in C^\infty$,

$$(t_0,x_0)\in\arg\max(V^+-\varphi) \implies \varphi_t(t_0,x_0) + H^+(t_0,x_0,0-0,D\varphi(t_0,x_0)) \geq 0.$$

By the monotone decreasing property of $r\mapsto H^+(t,x,r,p)$, for each small enough $\delta>0$, it holds $\forall\ z\in Z_g(-\delta)$,

$$\varphi_t(t_0,x_0) + \max_{y\in Y} D\varphi(t_0,x_0)\cdot f(t_0,x_0,y,z) \geq \varphi_t(t_0,x_0) + \max_{y\in Y_z} D\varphi(t_0,x_0)\cdot f(t_0,x_0,y,z) \geq 0. \tag{2.12}$$

Notice that if $z \in Z \setminus Z_g(-\delta)$, then $\max_{y\in Y} g(y,z) > -\delta$, and we have

$$\min_{z\in Z}\max_{y\in Y}\max\{\varphi_t + D\varphi\cdot f(t_0,x_0,y,z),\ g(y,z)\} \geq -\delta.$$

Sending $\delta \to 0$, this says, V^+ is a subsolution of (2.10).

Now suppose V^+ is a supersolution of (2.6a) and $(t_0,x_0) \in \arg\min(V^+ - \varphi)$ for $\varphi \in C^\infty$. Then, from (2.4), one has

$$\varphi_t(t_0,x_0) + H^+(t_0,x_0,0+0,D\varphi(t_0,x_0)) \ = \ \varphi_t(t_0,x_0) + H^+(t_0,x_0,0,D\varphi(t_0,x_0)) \leq 0.$$

Thus, there exists $z_0 \in Z_g(0)$ such that $Y_{z_0} = Y$ and

$$\varphi_t(t_0,x_0) + \max_{y\in Y} D\varphi(t_0,x_0)\cdot f(t_0,x_0,y,z_0) \leq 0, \qquad \max_{y\in Y} g(y,z_0) \ \leq \ 0.$$

Consequently,

$$\begin{aligned}\min_{z\in Z}\max_{y\in Y}&\max\{\varphi_t(t_0,x_0) + D\varphi\cdot f(t_0,x_0,y,z),\ g(y,z)\}\\ &\leq \max_{y\in Y}\max\{\varphi_t(t_0,x_0) + D\varphi(t_0,x_0)\cdot f(t_0,x_0,y,z_0), g(y,z_0)\} \ \leq \ 0,\end{aligned}$$

which means V^+ is a supersolution of (2.6a).

Now suppose V^+ is a subsolution of (2.10) and let $(t_0,x_0) \in \arg\max(V^+ - \varphi)$. If V^+ is not a subsolution of (2.6a), then there exists $\delta > 0$ such that

$$\varphi_t + \min_{z\in Z_g(-\delta)}\max_{y\in Y_z} D\varphi(t_0,x_0)\cdot f(t_0,x_0,y,z) \leq -\delta < 0$$

There exists $\overline{z} \in Z_g(-\delta)$ such that $\varphi_t + D\varphi(t_0,x_0)\cdot f(t_0,x_0,y,\overline{z}) \leq -\delta$ for every $y \in Y = Y_{\overline{z}}$. Since $\max_{y\in Y} g(y,\overline{z}) \leq -\delta$, we conclude

$$\min_{z\in Z}\max_{y\in Y}\max\{\varphi_t + D\varphi(t_0,x_0)\cdot f(t_0,x_0,y,\overline{z}), g(y,\overline{z})\} \leq -\delta,$$

which contradicts the fact that V^+ is a subsolution of (2.10).

Finally suppose V^+ is a supersolution of (2.10). Let $\varphi \in C^1$ and $(t_0,x_0) \in \arg\min(V^+ - \varphi)$ and then

$$\min_{z\in Z}\max_{y\in Y}\max\{\varphi_t + D\varphi\cdot f(t_0,x_0,y,z),\ g(y,z)\} \leq 0.$$

There is a $\overline{z} \in Z$ such that $\varphi_t + \max_{y\in Y} D\varphi\cdot f(t_0,x_0,y,\overline{z}) \leq 0$, and $\max_{y\in Y} g(y,\overline{z}) \leq 0$. Consequently, $\overline{z} \in Z_g(0), Y_{\overline{z}} = Y$ and so

$$\varphi_t + \min_{z\in Z_g(0)}\max_{y\in Y_z} D\varphi\cdot f(t_0,x_0,y,z) \leq 0,$$

which means V^+ is a supersolution of (2.6a). $\square$

Remark 2.9. It was pointed out by the referee that the equivalence also follows from the observation that

$$\{(p_t, p_x) \in \mathbb{R}^{n+1} \mid p_t + H^+(t, x, 0-0, p_x) \geq 0\}$$
$$= \{(p_t, p_x) \in \mathbb{R}^{n+1} \mid \min_{z \in Z} \max_{y \in Y} (p_t + p_x \cdot f(t, x, y, z)) \vee g(y, z) \geq 0\}$$

for the subsolution part and

$$\{(p_t, p_x) \in \mathbb{R}^{n+1} \mid p_t + H^+(t, x, 0+0, p_x) \leq 0\}$$
$$= \{(p_t, p_x) \in \mathbb{R}^{n+1} \mid \min_{z \in Z} \max_{y \in Y} (p_t + p_x \cdot f(t, x, y, z)) \vee g(y, z) \leq 0\}$$

for the supersolution part. A similar equivalence holds for V^-. The proofs of these set equalities, although along the same lines as in the theorem, are independent of any theory of viscosity solutions.

Existence of value means that the order of play wouldn't matter in the game. But when there are shared constraints on the controls it is obvious that the order of play would be essentially important. For instance the first player could use up an entire shared resource leaving the second player with nothing. Even linear dynamics and linear constraints do not guarantee value will exist. We exhibit several examples. One would expect, and Example 3 below shows the critical nature of the constraint on the value or lack thereof. These examples also raise an important question about the role of convexity (or concave-convexity) in generalized games and how one would go about relaxing a game without value to obtain a value, similar to the classical theory.

Example 2.10. *1. Consider the game with dynamics*

$$\dot{\xi}(s) = (\eta(s) - \zeta(s))^2, \ \xi(t) = x \in \mathbb{R}, Y = Z = [0, 1], \quad g(y, z) = 1 - y - z,$$

and payoff $P(y, z) = \xi(T)$. We have

$$\begin{aligned} H^+(t, x, 0, p) &= \min_{z=1} \max_{y+z \geq 1} p(y - z)^2 = \max\{p, 0\}, \\ H^-(t, x, 0, p) &= \max_{y=1} \min_{y+z \geq 1} p(y - z)^2 = \min\{p, 0\}. \end{aligned}$$

These Hamiltonians lead to the equations

$$\begin{cases} V_t^+ + H^+(t, x, 0, V_x^+) = 0, \ \ V^+(T, x) = x \implies V^+(t, x) = x + (T - t), \\ V_t^- + H^-(t, x, 0, V_x^-) = 0, \ \ V^-(T, x) = x \implies V^-(t, x) = x. \end{cases}$$

Since $V^+(t, x) > V^-(t, x)$ for all $t < T$, the game does not have a value.

2. If we have the linear dynamics $\dot{\xi} = \eta + \zeta$ with the same payoff as above and with the constraint $g(y,z) = y + z - 1 \le 0, Y = Z = [0,1]$, then

$$\begin{aligned} H^+(t,x,0,p) &= \min_{z=0} \max_{y+z\le 1} p(y+z) = \max\{p,0\}, \\ H^-(t,x,0,p) &= \max_{y=0} \min_{y+z\le 1} p(y+z) = \min\{p,0\}. \end{aligned}$$

Thus, we have that

$$V^+(t,x) = x + (T-t) > x = V^-(t,x) \qquad \forall x \in \mathbb{R}, t < T,$$

and this game does not have value even though the dynamics and the constraint are linear.

3. This example is a game which does have a value. The dynamics is $\dot{\xi} = (\eta - a)^2 + (\zeta - b)^2$ where a and b are constants with $a \ge 1/2$. The control sets are $Y = Z = [0,1]$ and the constraint is $g(y,z) = y \cdot z - 1/2 \le 0$. Then,

$$Z_g = [0,1/2] \quad \text{and} \quad Y_z = \begin{cases} [0,1], & 0 \le z \le 1/2 \\ [0,1/(2z)], & 1/2 < z \le 1 \end{cases}.$$

We obtain

$$\begin{aligned} H^+(t,x,0,p) &= \min_{0\le z\le 1/2} \max_{0\le y\le 1} p\big[(y-a)^2 + (z-b)^2\big] = \max_{0\le y\le 1} p(y-a)^2 + \min_{0\le z\le 1/2} p(z-b)^2 \\ &= \max\{pa^2, 0\} + \min\left\{p \max\left\{b^2, (1/2-b)^2\right\}, 0\right\}, \end{aligned}$$

and by symmetry

$$\begin{aligned} H^-(t,x,0,p) &= \max_{0\le y\le 1/2} \min_{0\le z\le 1} p\big[(y-a)^2 + (z-b)^2\big] = \max_{0\le y\le 1/2} p(y-a)^2 + \min_{0\le z\le 1} p(z-b)^2 \\ &= \max\left\{pa^2, 0\right\} + \min\left\{p \max\{b^2, (1-b)^2\}, 0\right\}. \end{aligned}$$

In particular,

$$H^+(t,x,0,p) = H^-(t,x,0,p) = pa^2 \qquad \forall p \ge 0,$$

and the game has a value $V = V^+ = V^- = x + (T-t)a^2$.

On the other hand, if everything else stays the same but if we assume that $a < 1/2$, the situation changes dramatically. In this case, we have

$$H^+(t,x,0,p) = p(1-a)^2 > p \max\{a^2, (1/2-a)^2\} = H^-(t,x,0,p) \qquad \forall p \ge 0,$$

and this game does not have value.

3. Penalization

In this section we will show that the generalized differential game with constraints is the limit of a standard game in which we penalize when the constraint is violated. Throughout, we take the constraint to be $g(y,z) \leq 0$ and denote by

$$g^+(y,z) = \max\{g(y,z), 0\} \qquad \forall y \in Y, z \in Z.$$

Theorem 3.1. *Assume that both Z_g and Y_g are nonempty. Let W^ε be the unique viscosity solution of*

$$W_t^\varepsilon + \min_{z\in Z}\max_{y\in Y}\left[D_x W^\varepsilon \cdot f(t,x,y,z) + \frac{1}{\varepsilon}g^+(y,z)\right] = 0, \quad W^\varepsilon(T,x) = h(x), \tag{3.1}$$

and W_ε the unique solution of

$$W_{\varepsilon,t} + \max_{y\in Y}\min_{z\in Z}\left[D_x W_\varepsilon \cdot f(t,x,y,z) - \frac{1}{\varepsilon}g^+(y,z)\right] = 0, \quad W_\varepsilon(T,x) = h(x). \tag{3.2}$$

Then $\lim_{\varepsilon\to 0+} W^\varepsilon = V^+$, $\lim_{\varepsilon\to 0+} W_\varepsilon = V^-$.

Remark 3.2. 1. The unique viscosity solution of (3.1) is the upper value function

$$W^\varepsilon(t,x) = \sup_{\alpha\in\Gamma[t]}\inf_{\zeta\in\mathcal{Z}[t,T]} h(\xi(T)) + \frac{1}{\varepsilon}\int_t^T g^+(\alpha[\zeta](s),\zeta(s))\,ds$$

where $\Gamma(t) = \{\alpha : \mathcal{Z}[t,T] \to \mathcal{Y}[t,T] \mid \alpha \text{ is non-anticipating}\}$. A similar remark applies to W_ε.

2. The penalty term $+\frac{1}{\varepsilon}g^+$ is appropriate for the upper game in (3.1) because the minimizer plays first and so will pay a severe penalty for violating the constraint. Similarly, $-\frac{1}{\varepsilon}g^+$ is appropriate for the lower game in (3.2) because the maximizer plays first in the lower game.

Proof. Recall that $Z_g = \{z \in Z \mid \max_{y\in Y} g(y,z) \leq 0\}$ and $Y_g = \{y \in Y \mid \max_{z\in Z} g(y,z) \leq 0\}$. We use the following lemma.

Lemma 3.3. *For a given $\alpha \in C(Y\times Z, \mathbb{R})$, it holds*

$$\begin{cases} \lim_{\varepsilon\to 0+}\min_{z\in Z}\max_{y\in Y}\left[\alpha(y,z) + \frac{1}{\varepsilon}g^+(y,z)\right] = \min_{z\in Z_g}\max_{y\in Y_z}\alpha(y,z), \\ \lim_{\varepsilon\to 0+}\max_{y\in Y}\min_{z\in Z}\left[\alpha(y,z) - \frac{1}{\varepsilon}g^+(y,z)\right] = \max_{y\in Y_g}\min_{z\in Z_y}\alpha(y,z). \end{cases} \tag{3.3}$$

Indeed, for any $z \in Z_g$, one has that $Y = Y_z$ and $g^+(y,z) = 0$ for all $y \in Y$. This yields

$$\lim_{\varepsilon\to 0+}\min_{z\in Z}\max_{y\in Y}\left[\alpha(y,z) + \frac{1}{\varepsilon}g^+(y,z)\right] \leq \lim_{\varepsilon\to 0+}\min_{z\in Z_g}\max_{y\in Y}\left[\alpha(y,z) + \frac{1}{\varepsilon}g^+(y,z)\right] = \min_{z\in Z_g}\max_{y\in Y_z}\alpha(y,z). \tag{3.4}$$

Thus, for every $\varepsilon > 0$, let $z_\varepsilon \in Z$ be such that

$$\min_{z\in Z}\max_{y\in Y}\left[\alpha(y,z) + \frac{1}{\varepsilon}g^+(y,z)\right] = \max_{y\in Y}\left[\alpha(y,z_\varepsilon) + \frac{1}{\varepsilon}g^+(y,z_\varepsilon)\right],$$

we have that $\max_{y\in Y} g^+(y,z_\varepsilon) \le o(\varepsilon)$. By the continuity of the map $z \mapsto \max_{y\in Y} g^+(y,z)$, there exists a sequence $\varepsilon_n \to 0+$ such that $\lim_{n\to\infty} z_{\varepsilon_n} = \widehat{z}$ and

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \min_{z\in Z} \max_{y\in Y} \left[\alpha(y,z) + \frac{1}{\varepsilon} g^+(y,z)\right] &= \lim_{n\to\infty} \min_{z\in Z} \max_{y\in Y} \left[\alpha(y,z) + \frac{1}{\varepsilon_n} g^+(y,z)\right] \\
&= \lim_{n\to\infty} \max_{y\in Y} \left[\alpha(y,z_{\varepsilon_n}) + \frac{1}{\varepsilon_n} g^+(y,z_{\varepsilon_n})\right] \\
&\ge \lim_{n\to\infty} \max_{y\in Y} \alpha(y,z_{\varepsilon_n}) = \max_{y\in Y} \alpha(y,\widehat{z}) = \max_{y\in Y_{\widehat{z}}} \alpha(y,\widehat{z}).
\end{aligned}$$

Observe that since $\max_{y\in Y} g^+(y,z_{\varepsilon_n}) \le o(\varepsilon_n)$, we have $\widehat{z} \in Z_g$. This implies the first equality in (3.3). Similarly, one can prove the second equality in (3.3).

As a consequence of Lemma 3.3, the hamiltonians satisfy

$$\begin{cases}
\lim_{\varepsilon\to 0} \min_{z\in Z} \max_{y\in Y} \left[p\cdot f(t,x,y,z) + \frac{1}{\varepsilon} g^+(y,z)\right] = H^+(t,x,0,p) \\
\lim_{\varepsilon\to 0} \max_{y\in Y} \min_{z\in Z} \left[p\cdot f(t,x,y,z) - \frac{1}{\varepsilon} g^+(y,z)\right] = H^-(t,x,0,p)
\end{cases} \tag{3.5}$$

and then in a way similar to [1, Theorem 1.7 and Corr. 1.8] we have $W^\varepsilon \to V^+$ and $W_\varepsilon \to V^-$. Here is a short sketch of the proof for V^+. For every $(t,x) \in]0,T] \times \mathbb{R}^n$, let us introduce the weak limits

$$\overline{W}^+(t,x) = \limsup_{(\varepsilon,s,y)\to(0+,t,x)} W^\varepsilon(s,y) \qquad \text{and} \qquad \underline{W}^+(t,x) = \liminf_{(\varepsilon,s,y)\to(0+,t,x)} W^\varepsilon(s,y).$$

It is clear that $\overline{W}^+$ and $\underline{W}^+$ are, respectively, upper and lower semicontinuous and satisfies

$$\overline{W}^+(t,x) \ \ge \ \underline{W}^+(t,x) \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n.$$

2. We first show that $\overline{W}^+$ is a subsolution of (2.6a). Let $\varphi \in C^\infty$ and (t_0,x_0) is a locally strict maximum for $\overline{W}^+ - \varphi$. Then there exists a sequence $\varepsilon_n \to 0+$ such that $W^{\varepsilon_n} - \varphi$ achieve a max at $(t_{\varepsilon_n}, x_{\varepsilon_n}) \to (t_0,x_0) \in \arg\max \overline{W}^+ - \varphi$. For every $n \ge 1$, we have

$$\varphi_t\left(t_{\varepsilon_n}, x_{\varepsilon_n}\right) + \min_{z\in Z} \max_{y\in Y} \left[D_x\varphi\left(t_{\varepsilon_n}, x_{\varepsilon_n}\right) \cdot f\left(t_{\varepsilon_n}, x_{\varepsilon_n}, y, z\right) + \frac{1}{\varepsilon_n} g^+(y,z)\right] \ge 0,$$

and this yields

$$\varphi_t\left(t_0,x_0\right) + \min_{z\in Z} \max_{y\in Y} \left[D_x\varphi\left(t_0,x_0\right) \cdot f\left(t_0,x_0,y,z\right) + \frac{1}{\varepsilon_n} g^+(y,z)\right] \ge o\left(|t_{\varepsilon_n} - t_0| + |x_{\varepsilon_n} - x_0|\right).$$

Sending $n \to \infty$ and recalling the first equality in (3.5) we get

$$\varphi_t(t_0,x_0) + H^+(t_0,x_0,0-0,D_x\varphi(t_0,x_0)) \ge 0.$$

Hence $\overline{W}^+$ is a subsolution of (2.6a).

3. Now suppose $\underline{W}^+ - \varphi$ achieves a min at (t_0, x_0) and $W^{\varepsilon_n} - \varphi$ achieves a min at $(t_{\varepsilon_n}, x_{\varepsilon_n}) \to (t_0, x_0)$ as $\varepsilon_n \to 0+$. We have

$$\varphi_t\left(t_{\varepsilon_n}, x_{\varepsilon_n}\right) + \min_{z\in Z}\max_{y\in Y}\left[D_x\varphi\left(t_{\varepsilon_n}, x_{\varepsilon_n}\right)\cdot f\left(t_{\varepsilon_n}, x_{\varepsilon_n}, y, z\right) + \frac{1}{\varepsilon_n} g^+(y,z)\right] \leq o\left(|t_{\varepsilon_n} - t_0| + |x_{\varepsilon_n} - x_0|\right),$$

and this implies

$$\varphi_t\left(t_0, x_0\right) + \min_{z\in Z}\max_{y\in Y}\left[D_x\varphi\left(t_0, x_0\right)\cdot f\left(t_0, x_0, y, z\right) + \frac{1}{\varepsilon_n} g^+(y,z)\right] \leq 0.$$

Sending $n \to \infty$ and recalling the first equality in (3.5) we get

$$\varphi_t\left(t_0, x_0\right) + H^+(t_0, x_0, 0, D\varphi(t_0, x_0)) \leq \lim_{n\to\infty} o\left(|t_{\varepsilon_n} - t_0| + |x_{\varepsilon_n} - x_0|\right) = 0,$$

which means $\underline{W}^+$ is a supersolution of (2.6a). □

4. From Theorem (2.8), $\overline{W}^+$ and $\underline{W}^+$ are a subsolution and supersolution, respectively, of (2.10) with $\underline{W}^+(T,x) = \overline{W}^+(T,x) = h(x)$. Since V^+ is the unique viscosity solution of (2.10) with $V^+(T,x) = h(x)$, one finally has that $\lim_{\varepsilon\to 0+} W^\varepsilon = V^+$.

The proof of the assertion (2.6b) for V^- is similar and is omitted. □

Remark 3.4. It seems natural to choose the penalization in the theorem but because we have a min max and not just min, where $1/\varepsilon$ would be the standard penalization, how does the max affect this? Maybe we should have used $-1/\varepsilon$? Consider what happens if we change the sign of $1/\varepsilon$ in the penalization. Let W^ε be the unique viscosity solution of

$$W^\varepsilon_t + \min_{z\in Z}\max_{y\in Y}\left[D_x W^\varepsilon\cdot f(t,x,y,z) - \frac{1}{\varepsilon} g^+(y,z)\right] = 0, \quad W^\varepsilon(T,x) = h(x), \tag{3.6}$$

and W_ε the unique solution of

$$W_{\varepsilon,t} + \max_{y\in Y}\min_{z\in Z}\left[D_x W_\varepsilon\cdot f(t,x,y,z) + \frac{1}{\varepsilon} g^+(y,z)\right] = 0, \quad W_\varepsilon(T,x) = h(x). \tag{3.7}$$

Then we claim that $\lim\limits_{\varepsilon\to 0+} W^\varepsilon = U^+$, $\lim\limits_{\varepsilon\to 0+} W_\varepsilon = U^-$ where U^+ is the unique viscosity solution of

$$U^+_t + \min_{z\in Z}\max_{y\in Y_z} D_x U^+\cdot f(t,x,y,z) = 0, \quad U^+(T,x) = h(x), \tag{3.8}$$

and U^- the unique solution of

$$U^-_t + \max_{y\in Y}\min_{z\in Z_y} D_x U^-\cdot f(t,x,y,z) = 0, \quad U^-(T,x) = h(x). \tag{3.9}$$

To verify the claim, consider the following analog of Lemma 3.3.

Lemma 3.5. *For a given $\alpha \in C(Y \times Z, \mathbb{R})$,*

$$\begin{cases} \lim\limits_{\varepsilon\to 0+} \min\limits_{z\in Z} \max\limits_{y\in Y} \left[\alpha(y,z) - \dfrac{1}{\varepsilon}\cdot g^+(y,z)\right] = \min\limits_{z\in Z}\max\limits_{y\in Y_z} \alpha(y,z), \\ \lim\limits_{\varepsilon\to 0+} \max\limits_{y\in Y} \min\limits_{z\in Z} \left[\alpha(y,z) + \dfrac{1}{\varepsilon}\cdot g^+(y,z)\right] = \max\limits_{y\in Y}\min\limits_{z\in Z_y} \alpha(y,z). \end{cases} \tag{3.10}$$

Proof. Indeed, for every $z \in Z$, one has

$$\max_{y\in Y}\left[\alpha(y,z) - \frac{1}{\varepsilon}\cdot g^+(y,z)\right] \;\geq\; \max_{y\in Y_z}\left[\alpha(y,z) - \frac{1}{\varepsilon}\cdot g^+(y,z)\right] \;=\; \max_{y\in Y_z}\alpha(y,z),$$

and this yields

$$\lim_{\varepsilon\to 0+}\min_{z\in Z}\max_{y\in Y}\left[\alpha(y,z) - \frac{1}{\varepsilon}\cdot g^+(y,z)\right] \;\geq\; \min_{z\in Z}\max_{y\in Y_z}\alpha(y,z). \tag{3.11}$$

On the other hand, for every $s > 0$, set $Y_z(s) = \{y \in Y \mid g(y,z) \leq s\}$, we have

$$\begin{aligned} \max_{y\in Y}\left[\alpha(y,z) - \frac{1}{\varepsilon}\cdot g^+(y,z)\right] \;&\leq\; \max\left\{\max_{y\in Y_z(s)}\alpha(y,z),\ \max_{y\in Y_z(s)}\left[\alpha(y,z) - \frac{1}{\varepsilon}\cdot g^+(y,z)\right]\right\} \\ &\leq\; \max\left\{\max_{y\in Y_z(s)}\alpha(y,z),\ \max_{(y,z)\in Y\times Z}\alpha(y,z) - \frac{s}{\varepsilon}\right\}, \end{aligned}$$

and this implies that

$$\limsup_{\varepsilon\to 0+}\min_{z\in Z}\max_{y\in Y}\left[\alpha(y,z) - \frac{1}{\varepsilon}\cdot g^+(y,z)\right] \;\leq\; \min_{z\in Z}\max_{y\in Y_z(s)}\alpha(y,z).$$

Thus,

$$\begin{aligned} \limsup_{\varepsilon\to 0+}\min_{z\in Z}\max_{y\in Y}\left[\alpha(y,z) - \frac{1}{\varepsilon}\cdot g^+(y,z)\right] \;&\leq\; \lim_{s\to 0+}\min_{z\in Z}\max_{y\in Y_z(s)}\alpha(y,z) \\ &=\; \min_{z\in Z}\max_{y\in Y_z}\alpha(y,z) \end{aligned}$$

and (3.11) yields the first inequality in (3.10). Similarly, one can get the second inequality in (3.10). ☐

Using Lemma 3.5, the hamiltonians satisfy

$$\begin{cases} \lim\limits_{\varepsilon\to 0}\min\limits_{z\in Z}\max\limits_{y\in Y}\left[p\cdot f(t,x,y,z) - \dfrac{1}{\varepsilon}g^+(y,z)\right] = \min\limits_{z\in Z}\max\limits_{y\in Y_z} p\cdot f(t,x,y,z) \\ \lim\limits_{\varepsilon\to 0}\max\limits_{y\in Y}\min\limits_{z\in Z}\left[p\cdot f(t,x,y,z) + \dfrac{1}{\varepsilon}g^+(y,z)\right] = \max\limits_{y\in Y}\min\limits_{z\in Z_y} p\cdot f(t,x,y,z) \end{cases} \tag{3.12}$$

and then just as in Theorem 3.4 we have $W^\varepsilon \to U^+$ and $W_\varepsilon \to U^-$.

However, because of the fact that we are decreasing the upper hamiltonian in W_ε^+ and increasing the hamiltonian in W_ε^- with this penalization, we may not get $U^+ \geq U^-$. For example, if we have the dynamics $\dot{\xi} = (\eta - \zeta)^2, \xi(T) = x \in \mathbb{R}, Y = Z = [0,1], g(y,z) = 1 - y - z \leq 0$, and payoff $P(\eta,\zeta) = \xi(T)$, we obtain, after a simple computation, the equations for U^+ and U^-, respectively,

$$U_t^+ + \min(U_x^+, 0) + \frac{1}{9}\max(U_x^+, 0) = 0, \;\; U^+(T,x) = x \implies U^+(t,x) = x + \frac{1}{9}(T-t),$$
$$U_t^- + \max(U_x^-, 0) + \frac{1}{9}\min(U_x^-, 0) = 0, \;\; U^-(T,x) = x \implies U^-(t,x) = x + (T-t).$$

Since $U^+(t,x) < U^-(t,x)$ for all $0 \leq t < T$, this cannot be the correct penalization.

Remark 3.6. 1. If there is a running cost so the payoff is of the form

$$P(y,z) = h(\xi(T)) + \int_t^T k(s, \xi(s), \eta(s), \zeta(s))\, ds,$$

the Isaacs' equations have an inhomogeneous term. For instance, the upper value would satisfy

$$V_t^+ + \min_{z \in Z_g} \max_{y \in Y_z} \big[D_x V^+ \cdot f(t,x,y,z) + k(t,x,y,z)\big] = 0, \;\; (t,x) \in [0,T) \times \mathbb{R}^n, \; V^+(T,x) = h(x).$$

2. It is clear how to extend the results of this paper if we have multiple constraints on the controls such as $g_1(y,z) \leq 0, \ldots, g_n(y,z) \leq 0$. In addition one may consider state and control constraints of the form $g(t,x,y,z) \leq 0$ but the situation becomes considerably more technical because one needs to impose conditions on what happens when $g(t,x,y,z) = 0$. We leave this open.

References

[1] M. Bardi and I. Capuzzo-Dolcetta, **Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations,** With appendices by Maurizio Falcone and Pierpaolo Soravia, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 1997.

[2] A. Bressan and W. Shen, *Semi-cooperative strategies for differential games,* Int. J. Game Theory, **32**, 2003, 561–593.

[3] P. Cardaliaguet, *Introduction to Differential Games*, 2010, https://www.ceremade.dauphine.fr/ cardalia/.

[4] L.C. Evans and P. Souganidis, *Differential Games and Representation Formulas for Solutions of Hamilton-Jacobi-Isaacs Equations,* Indiana Univ. Math. J., **33**, (1984), 773–797.

[5] R. Elliott and N. Kalton, *The existence of value in differential games*, Memoirs AMS, **126**, 1972.

[6] H. Engler, S.M. Lenhart, *Viscosity solutions for weakly coupled systems of Hamilton–Jacobi equations*, Proc. London Math. Soc., **63**, 1991, 212-240.

[7] F. Facchinei and C. Kanzow, *Generalized Nash Equilibrium Problems*, Annals of Operations Research, **175**, 2010, 177–211.

[8] H. Ishii, S. Koike, *Viscosity solutions for monotone systems of second-order parabolic PDE's* Comm. Partial Differential Equations, 16 (1991), pp. 1095-1128

[9] N.N. Krasovskii and A.I. Subbotin, **Game-Theoretical Control Problems**, Springer, New York, 1988.

[10] S.M. Lenhart, *Viscosity solutions for weakly coupled systems of first order PDEs* J. Math. Anal. Appl., **131**, 1988, 180-193.

[11] A.I.Subbotin, **Generalized Solution of First-Order PDEs**, Birkhauser,Boston, 1995.

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