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A NEW FEEDBACK FORM OF OPEN-LOOP STACKELBERG STRATEGY IN A GENERAL LINEAR-QUADRATIC DIFFERENTIAL GAME

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Abstract. In this paper, we consider a general form of linear-quadratic Stackelberg deterministic differential game model, which consists of one leader and one follower. Each of their utility functions includes all possible squared terms, cross terms and single terms of states and controls of the two players, and constant terms. The time-consistent state feedback form of Stackelberg equilibrium strategy is obtained. Its explicit expression is in terms of the solutions of three decoupled symmetric Riccati differential equations. These decoupled symmetric Riccati differential equations are independent of the state and can be solved backward in time one by one. The proposed model and theory are applied to some classical Stackelberg games.

1. Introduction.

The Stackelberg game model is originally proposed by Stackelberg in 1934 [19], which involves players with asymmetric roles, one leading (called the leader) and the other following (called the follower). In the Stackelberg game, the leader announces the policy first, and the follower reacts it. In fact, the leader knows the objective function of the follower and aims to minimize his own objective function, taking into account the follower's optimal response made based on the leader's optimal strategy being announced.

Compared with the classical Nash equilibrium non-cooperative game, the Stackelberg game, captures more real characteristics in many practical problems. To date, the Stackelberg game model has been widely used in various fields, especially in the study of operational management, marketing channels [10, 16], and water resource management [17]. In addition, Stackelberg games are important in the field of optimization, because they can be regarded as bi-level optimization problems [3, 7].

For some Stackelberg games that are uncertain or governed by stochastic process with appropriate distribution function, they can be transformed into deterministic problems [6, 14, 15]. The linear-quadratic Stackelberg stochastic differential game model is studied in [25], where the corresponding Riccati differential equations for the follower is a backward stochastic differential equation. To obtain the

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follower-based feedback strategy, it is necessary to solve an two-point boundary value problem governed by the coupled system of forward stochastic differential equations and backward stochastic differential equations. The state feedback form of the Stackelberg equilibrium strategy in terms of the solutions of these two stochastic differential equations is given. However, the expression of the utility function of each of the two players does not involve the cross terms of the state and control of the two players. Furthermore, they do not contain the control of the other player. In addition, no practical computational method is developed to solve the resulting two-point boundary value problem. In [4], the corresponding maximum principles are derived for open-loop and closed-loop Stackelberg stochastic differential games. In particular, for the linear quadratic Stackelberg game model, due to the two different information structures, their respective maximum principles are also different. In addition, in this linear quadratic Stackelberg game model, the expression of the utility function of each of the two the players does not involve the cross terms of the states and controls of both players. Also, they do not contain the control of the other player. As for the case considered in [25], no practical computational method is developed to solve the resulting two-point boundary value problem.

For the linear-quadratic Stackelberg deterministic differential game model, the sufficient and/or necessary conditions for the existence and uniqueness of equilibrium strategy and are obtained by using various methods in the existing literatures, for example, see [1, 2, 8, 9, 12, 18, 20, 24]. On the other hand, in order to obtain the solution of each of the problems mentioned above, it is necessary to solve the corresponding two-point boundary value problem governed by three coupled asymmetric Riccati differential equations. However, no practical effective computational method has been developed. In [23], by introducing a new costate, the Stackelberg strategy, which is expressed in terms of three decoupled symmetric Riccati differential equations, is obtained for a two-player game in discrete-time dynamic setting. Later, this idea is applied to the study of the two-person linear-quadratic Stackelberg game of time delay discrete time model and the continuous time model in [22]. However, their utility functions do not involve cross terms. In addition, for each case, the expression of the Stackelberg strategy involves the transition matrix and its inverse at each time point in [22].

In [11], the focus is on the development of differential game theory and numerical methods in economic and management applications. It is pointed out that the linear-quadratic model involving time-dependent parameters and the interaction between the controls and between the controls and states of the players, can be a good approximation to a more general non-linear model. For example, if the market demand is affected by the retail price, then the utility function of each of the two players will include the product of the wholesale price and the retail price. In general, the wholesale price is determined by the supplier as the leader, and the retail price is determined by the retailer as the follower. In addition, the differential game problem involving only one state variable, is can be solved by using the Hamilton-Jacobi-Bellman (HJB) approach.

The main contributions of this paper are threefold:

(i) A general linear-quadratic Stackelberg game model is presented, in which the utility function of each of the two players contains all the quadratic terms, including cross terms and individual terms of the players' controls and the states. All parameters can be time-dependent.

(ii) Due to reformulating the leader's optimal control as a new linear-quadratic optimal control problem, the expression of the equilibrium strategy contains only four variables which can be solved backward in time one by one. In other words, for the Stackelberg game model, there is no need to solve the two-point boundary value problem anymore.

(iii) The existence and uniqueness of the game equilibrium strategy are analyzed. And the Stackelberg equilibrium strategy in the form of time-consistent state feedback is presented.

In addition, to demonstrate the applicability of the proposed model and theory, some classical examples are considered.

The remainder of the paper is organized as follows. In the next section, we present the description of our model. Section 3 is devoted to the study of the follower's optimal control problem with the leader's strategy being given. The corresponding symmetric Riccati differential equation is obtained. The leader's optimal control problem is discussed in Section 4. The state feedback form of Stackelberg equilibrium strategy is given, which is expressed in terms of the solutions of three decoupled symmetric Riccati differential equations and one ordinary differential equations. The three decoupled symmetric Riccati differential equations are independent of the state variable x, and they can be solved backward in time from $t = T$ to $t = 0$ one by one. Thus, the time-consistent state feedback form of Stackelberg equilibrium strategy is obtained. In Section 5, some classical Stackelberg games are presented. Section 6 concludes the paper.

2. **Problem formulation.** We consider a linear-quadratic Stackelberg game involving two players labeled as the follower and the leader. The underlying dynamical system is described by the following system of differential equations.

$$\begin{cases} \dfrac{dx(t)}{dt} = f(t, x(t), u^1(t), u^2(t)),\ t \in (0, T], \\ x(0) = x^o, \end{cases} \tag{1}$$

where

$$x(t) = \left[x_1(t), \cdots, x_n(t)\right]^\top \in \mathbb{R}^n,$$

is the state vector, and for $i = 1, 2$

$$u^i(t) = \left[u^i_1(t), \cdots, u^i_{m_i}(t)\right]^\top \in \mathbb{R}^{m_i},$$

are the follower's (respectively, leader's) control vector at time $t \in [0, T]$, $x^o \in \mathbb{R}^n$ is a given vector, and $n, m_i \in \mathbb{N}^+$, $i = 1, 2$, while

$$f(t, x, u^1, u^2) = A(t)x + B_1(t)u^1(t) + B_2(t)u^2(t) + c(t)$$

with $A(t) \in \mathbb{R}^{n\times n}$, $B_i(t) \in \mathbb{R}^{n\times m_i}$ for $i = 1, 2$, and $c(t) \in \mathbb{R}^n$.

To proceed further, $x(t)$ is abbreviated as x when no confusion can arise. Similar abbreviations are applied to all other notations throughout.

For the follower and the leader, their respective objective functions are given by

$$J_F(u^1, u^2) = e^{-\rho T}\Phi_1(x(T)) + \int_0^T e^{-\rho t}\mathcal{L}_1(t, x(t), u^1(t), u^2(t))dt, \tag{2}$$

and

$$J_L(u^1, u^2) = e^{-\rho T}\Phi_2(x(T)) + \int_0^T e^{-\rho t}\mathcal{L}_2(t, x(t), u^1(t), u^2(t))dt, \tag{3}$$

where ρ is a given depreciation constant, for $i, j = 1, 2,\ i \neq j$,

$$\begin{aligned}\mathcal{L}_i(t, x, u^1, u^2) =& \frac{1}{2}x^\top Q_i x + \frac{1}{2}(u^i)^\top R_{ii} u^i + x^\top S_i u^i + x^\top \omega^i + (u^i)^\top \mu^i \\ &+ \frac{1}{2}(u^j)^\top R_{ij} u^j + (u^i)^\top \widetilde{R}_i u^j + x^\top \widetilde{S}_i u^j + (u^j)^\top \widetilde{\mu}^i + \beta_i, \\ \Phi_i(x) =& \frac{1}{2}x^\top G_i x + x^\top \alpha^i,\end{aligned}$$

and

$$\begin{aligned}&Q_i = Q_i^\top \in \mathbb{R}^{n\times n},\ R_{ii} = R_{ii}^\top \in \mathbb{R}^{m_i\times m_i},\ S_i \in \mathbb{R}^{n\times m_i},\ \omega^i \in \mathbb{R}^n,\ \mu^i \in \mathbb{R}^{m_i},\\ &R_{ij} = R_{ij}^\top \in \mathbb{R}^{m_j\times m_j},\ \widetilde{R}_i \in \mathbb{R}^{m_i\times m_j},\ \widetilde{S}_i \in \mathbb{R}^{n\times m_j},\ \widetilde{\mu}^i \in \mathbb{R}^{m_j},\ \beta_i \in \mathbb{R},\\ &G_i = G_i^\top \in \mathbb{R}^{n\times n},\ \alpha^i \in \mathbb{R}^n.\end{aligned}$$

Furthermore, let E, O, and 0 denote, respectively, the identity matrix, zero matrix and zero vector with appropriate dimension.

In Stackelberg game, the leader aims to minimize his own objective function, taking into account the follower's optimal response made based on the leader's optimal strategy being announced. The optimal strategy of the leader and the optimal reaction of the followers are called Stackelberg equilibrium or Stackelberg solution. The mathematical expression, which can be found in [2, 16], is stated as a definition given below.

Definition 2.1. A pair of strategies $(u^{1,*}(\cdot), u^{2,*}(\cdot))$ is called a Stackelberg equilibrium if $u^{2,*}(\cdot)$ is the minimum solution of the leader's optimal control problem given below

$$\inf_{u^2(\cdot)} J_L(\bar{u}(u^2(\cdot)), u^2(\cdot))$$

subject to

$$\bar{u}(u^2(\cdot)) = \arg\inf_{u^1(\cdot)} J_F(u^1(\cdot), u^2(\cdot))$$

such that $u^{1,*}(\cdot) = \bar{u}(u^{2,*}(\cdot))$.

Let this problem be referred to as Problem (P_S), for which the Stackelberg equilibrium strategy in feedback form is given in Theorem 4.3 and its computational procedure is detailed in Algorithm (P_S) in Section 4.

To obtain these results, we need to obtain the follower's optimal control with reference to the given leader's strategy in Section 3. Then, the leader's optimal control problem is reformulated as a new linear quadratic optimal problem in Section 4.

3. Optimal control for the follower.

Suppose that the leader's strategy

$$u^2(\cdot) = \widehat{u}^2(\cdot) = \left\{\widehat{u}^2(t) : t \in [0, T]\right\}$$

is given. Then, the follower's optimal control problem is: Subject to the dynamical system (1) with $u^2(\cdot)$ taken as $\widehat{u}^2(\cdot)$, find a $u^1(\cdot)$ such that the follower's objective function

$$J_F(u^1(\cdot), \widehat{u}^2(\cdot)) \tag{4}$$

is minimized. Let this problem be referred to as Problem (P_F).

Theorem 3.1. *Consider Problem* (P_F) *and assume that* R_{11} *is a symmetric positive definite matrix. Then the follower's optimal strategy is given by*

$$u^{1,*} = \bar{u}(\widehat{u}^2(\cdot)) = R_{11}^{-1}\Big(-(S_1 + P_1 B_1)^\top x - B_1^\top \varphi^1 - \widetilde{R}_1 \widehat{u}^2 - \mu^1 \Big), \tag{5}$$

where $P_1 = P_1^\top \in \mathbb{R}^{n\times n}$ *and* $\varphi^1 \in \mathbb{R}^n$ *are such that the following systems of differential equations are satisfied:*

$$\begin{cases} \dfrac{dP_1}{dt} = -Q_1 - A^\top P_1 - P_1 A + \rho P_1 \\ \qquad\quad +(S_1 + P_1 B_1) R_{11}^{-1} (S_1 + P_1^\top B_1)^\top, \\ P_1(T) = G_1, \end{cases} \tag{6}$$

and

$$\begin{cases} \dfrac{d\varphi^1}{dt} = -\Big(A - B_1 R_{11}^{-1}(S_1 + P_1 B_1)^\top - \rho E\Big)^\top \varphi^1 \\ \qquad\quad -\Big(\widetilde{S}_1 + P_1 B_2 - (S_1 + P_1 B_1) R_{11}^{-1} \widetilde{R}_1\Big)\widehat{u}^2 \\ \qquad\quad -(\omega^1 + P_1 c) + (S_1 + P_1 B_1) R_{11}^{-1} \mu^1, \\ \varphi^1(T) = \alpha^1. \end{cases} \tag{7}$$

Proof. The proof of this theorem is similar to that of Theorem 4.1, and so only the proof of Theorem 4.1 will be given. □

Remark 1. The differential equations (6) are the symmetric Riccati differential equations, the existence and uniqueness of these equations are well-known.

4. **Optimal control for the leader.** Recall that the follower's optimal response $\bar{u}$ is given by (5). Then, substituting $\bar{u}$ for u^1 in the dynamical system (1) and J_L, the leader's optimal control problem, which is referred to as Problem (P_L), can be described as follows: Given the dynamical system

$$\frac{d}{dt}\begin{bmatrix} x \\ \varphi^1 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ O & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} x \\ \varphi^1 \end{bmatrix} + \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} u^2 + \begin{bmatrix} \mathcal{C}^1 \\ \mathcal{C}^2 \end{bmatrix} \tag{8}$$

with $x(0) = x^o$ and $\varphi^1(T) = \alpha^1$, find a $u^2(\cdot)$ such that the leader's objective function

$$e^{-\rho T}\Phi(x(T), \varphi^1(T)) + \int_0^T e^{-\rho t}\mathcal{L}(t, x(t), \varphi^1(t), u^2(t))\, ds$$

is minimized, where

$$\begin{aligned} \mathcal{L} =& \frac{1}{2}\begin{bmatrix} x \\ \varphi^1 \end{bmatrix}^\top \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^\top & \mathcal{Q}_{22} \end{bmatrix} \begin{bmatrix} x \\ \varphi^1 \end{bmatrix} + \frac{1}{2}(u^2)^\top \mathcal{R} u^2 \\ &+ \begin{bmatrix} x \\ \varphi^1 \end{bmatrix}^\top \begin{bmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{bmatrix} u^2 + \begin{bmatrix} x \\ \varphi^1 \end{bmatrix}^\top \begin{bmatrix} \mathcal{W}^1 \\ \mathcal{W}^2 \end{bmatrix} + (u^2)^\top \mathcal{K} + \mathcal{D}, \\ \Phi =& \frac{1}{2}\begin{bmatrix} x \\ \varphi^1 \end{bmatrix}^\top \begin{bmatrix} G_2 & O \\ O & O \end{bmatrix} \begin{bmatrix} x \\ \varphi^1 \end{bmatrix} + \begin{bmatrix} x \\ \varphi^1 \end{bmatrix}^\top \begin{bmatrix} \alpha^2 \\ 0 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{A}_{11} = A - B_1 R_{11}^{-1}(S_1 + P_1^\top B_1)^\top,\ \mathcal{A}_{12} = -B_1 R_{11}^{-1} B_1^\top, \\ &\mathcal{A}_{22} = -A^\top + (S_1 + P_1 B_1) R_{11}^{-1} B_1^\top + \rho E, \\ &\mathcal{B}_1 = B_2 - B_1 R_{11}^{-1}\widetilde{R}_1,\ \mathcal{B}_2 = -\widetilde{S}_1 + S_1 R_{11}^{-1}\widetilde{R}_1 - P_1 \mathcal{B}_1, \\ &\mathcal{C}^1 = c - B_1 R_{11}^{-1}\mu^1,\ \mathcal{C}^2 = -\omega^1 + S_1 R_{11}^{-1}\mu^1 - P_1 \mathcal{C}^1, \end{aligned}$$

$$
\begin{aligned}
&\mathcal{Q}_{11} = Q_2 + \big((S_1 + P_1 B_1) R_{11}^{-1} R_{21} - 2\widetilde{S}_2\big) R_{11}^{-1} (S_1 + P_1^\top B_1)^\top,\\
&\mathcal{Q}_{12} = \big((S_1 + P_1 B_1) R_{11}^{-1} R_{21} - \widetilde{S}_2\big) R_{11}^{-1} B_1^\top,\\
&\mathcal{Q}_{22} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^\top,\\
&\mathcal{R} = R_{22} + (\widetilde{R}_1^\top R_{11}^{-1} R_{21} - 2\widetilde{R}_2) R_{11}^{-1} \widetilde{R}_1,\\
&\mathcal{S}_1 = S_2 - \widetilde{S}_2 R_{11}^{-1} \widetilde{R}_1 + (S_1 + P_1 B_1) R_{11}^{-1} (R_{21} R_{11}^{-1} \widetilde{R}_1 - \widetilde{R}_2^\top),\\
&\mathcal{S}_2 = B_1 R_{11}^{-1} (R_{21} R_{11}^{-1} \widetilde{R}_1 - \widetilde{R}_2^\top),\\
&\mathcal{W}^1 = \omega^2 - \widetilde{S}_2 R_{11}^{-1} \mu^1 - (S_1 + P_1 B_1) R_{11}^{-1} (R_{21} R_{11}^{-1} \mu^1 + \widetilde{\mu}^2),\\
&\mathcal{W}^2 = -B_1 R_{11}^{-1} (R_{21} R_{11}^{-1} \mu^1 + \widetilde{\mu}^2),\\
&\mathcal{K} = \mu^2 - \widetilde{R}_2 R_{11}^{-1} \mu^1 - \widetilde{R}_1^\top R_{11}^{-1} (R_{21} R_{11}^{-1} \mu^1 + \widetilde{\mu}^2),\\
&\mathcal{D} = \beta_2 + \frac{1}{2} (\mu^1)^\top R_{11}^{-1} (R_{21} R_{11}^{-1} \mu^1 - 2\widetilde{\mu}^2).
\end{aligned}
$$

Theorem 4.1. *Consider Problem (P_L) and assume that $\overline{\mathcal{R}}$ is a symmetric positive definite matrix. Then, the leader's optimal strategy $u^{2,*}$ is given by*

$$
u^{2,*} = \overline{\mathcal{R}}^{-1}\Big(-\overline{\mathcal{B}}_1^\top \begin{bmatrix}\varphi^1\\ \varphi^2\end{bmatrix} + \overline{\mathcal{B}}_2^\top \begin{bmatrix}\lambda^2\\ x\end{bmatrix} - \mathcal{K}\Big), \tag{9}
$$

where

$$
\begin{bmatrix}\varphi^1\\ \varphi^2\end{bmatrix}, \begin{bmatrix}\lambda^2\\ x\end{bmatrix} \in \mathbb{R}^{2n}
$$

are such that the following systems of differential equations are satisfied:

$$
\frac{d}{dt}\begin{bmatrix}\varphi^1\\ \varphi^2\end{bmatrix} = \overline{\mathcal{A}}_1 \begin{bmatrix}\varphi^1\\ \varphi^2\end{bmatrix} + \overline{\mathcal{B}}_2 u^{2,*} + \overline{\mathcal{C}}^2, \tag{10}
$$

$$
\frac{d}{dt}\begin{bmatrix}\lambda^2\\ x\end{bmatrix} = \overline{\mathcal{A}}_2 \begin{bmatrix}\varphi^1\\ \varphi^2\end{bmatrix} - \overline{\mathcal{A}}_1^\top \begin{bmatrix}\lambda^2\\ x\end{bmatrix} + \overline{\mathcal{B}}_1 u^{2,*} + \overline{\mathcal{C}}^1 + \rho \begin{bmatrix}\lambda^2\\ x\end{bmatrix} \tag{11}
$$

with

$$
\begin{bmatrix}\varphi^1(T)\\ \varphi^2(T)\end{bmatrix} = \begin{bmatrix}\alpha^1\\ \alpha^2\end{bmatrix}, \ \begin{bmatrix}\lambda^2(0)\\ x(0)\end{bmatrix} = \begin{bmatrix}0\\ x^o\end{bmatrix}, \tag{12}
$$

and

$$
\begin{aligned}
&\overline{\mathcal{A}}_1 = \begin{bmatrix}\mathcal{A}_{22} & O\\ -P_2\mathcal{A}_{12} - \mathcal{Q}_{12} & -\mathcal{A}_{11}^\top + \rho E\end{bmatrix}, \ \overline{\mathcal{A}}_2 = \overline{\mathcal{A}}_2^\top = \begin{bmatrix}\mathcal{Q}_{22} & \mathcal{A}_{12}^\top\\ \mathcal{A}_{12} & O\end{bmatrix},\\
&\overline{\mathcal{B}}_1 = \begin{bmatrix}\mathcal{S}_2\\ \mathcal{B}_1\end{bmatrix}, \ \overline{\mathcal{B}}_2 = \begin{bmatrix}\mathcal{B}_2\\ -P_2\mathcal{B}_1 - \mathcal{S}_1\end{bmatrix}, \ \overline{\mathcal{C}}^1 = \begin{bmatrix}\mathcal{W}^2\\ \mathcal{C}^1\end{bmatrix}, \ \overline{\mathcal{C}}^2 = \begin{bmatrix}\mathcal{C}^2\\ -P_2\mathcal{C}^1 - \mathcal{W}^1\end{bmatrix},\\
&\overline{\mathcal{R}} = \frac{\mathcal{R} + \mathcal{R}^\top}{2}.
\end{aligned}
$$

Moreover, $P_2 = P_2^\top \in \mathbb{R}^{n\times n}$ satisfies

$$
\begin{cases}
\dfrac{dP_2}{dt} = -\dfrac{\mathcal{Q}_{11} + \mathcal{Q}_{11}^\top}{2} - \mathcal{A}_{11}^\top P_2 - P_2\mathcal{A}_{11} + \rho P_2,\\
P_2(T) = G_2.
\end{cases} \tag{13}
$$

Proof. Define

$$
H(t, x, \varphi^1, \lambda^1, \lambda^2, u^2) = \begin{bmatrix}\lambda^1\\ \lambda^2\end{bmatrix}^\top \Big(\begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ O & \mathcal{A}_{22}\end{bmatrix} \begin{bmatrix}x\\ \varphi^1\end{bmatrix} + \begin{bmatrix}\mathcal{B}_1\\ \mathcal{B}_2\end{bmatrix} u^2 + \begin{bmatrix}\mathcal{C}^1\\ \mathcal{C}^2\end{bmatrix} \Big) - \mathcal{L},
$$

where $\lambda^1, \lambda^2 \in \mathbb{R}^n$.

To apply the necessary conditions for optimality:

$$\frac{\partial H(t, x(t), \varphi^1(t), \lambda^1(t), \lambda^2(t), u^2(t))}{\partial u^2} = 0,$$

we obtain

$$\mathcal{B}_1^\top \lambda^1 + \mathcal{B}_2^\top \lambda^2 - \frac{\mathcal{R} + \mathcal{R}^\top}{2} u^2 - \mathcal{S}_1^\top x - \mathcal{S}_2^\top \varphi^1 - \mathcal{K} = 0 \tag{14}$$

on $[0, T]$, where λ^1 and λ^2 are such that the following systems of differential equations

$$\frac{d\lambda^1}{dt} = -\frac{\partial H}{\partial x} + \rho\lambda^1, \;\; \frac{d\lambda^2}{dt} = -\frac{\partial H}{\partial \varphi^1} + \rho\lambda^2 \tag{15}$$

are satisfied with $\lambda^1(T) = -G_2 x(T) - \alpha^2$ and $\lambda^2(0) = 0$.

Using the backward sweep method [5, 21], we set

$$\lambda^1 = -P_2 x - \varphi^2. \tag{16}$$

Then, we have $P_2(T) = G_2$ and $\varphi^2(T) = \alpha^2$. Now, according to

$$\frac{d\lambda^1}{dt} + \frac{dP_2}{dt} x + P_2 \frac{dx}{dt} + \frac{d\varphi^2}{dt} = 0, \tag{17}$$

we can get

$$\begin{aligned} &\Big(\mathcal{A}_{11}^\top P_2 + \frac{\mathcal{Q}_{11} + \mathcal{Q}_{11}^\top}{2} - \rho P_2 + \frac{dP_2}{dt} + P_2 \mathcal{A}_{11}\Big) x \\ &\quad + \mathcal{A}_{11}^\top \varphi^2 + \mathcal{Q}_{12}\varphi^1 + \mathcal{S}_1 u^2 + \mathcal{W}^1 - \rho\varphi^2 \\ &\quad + P_2\Big(\mathcal{A}_{12}\varphi^1 + \mathcal{B}_1 u^2 + \mathcal{C}^1\Big) + \frac{d\varphi^2}{dt} = 0. \end{aligned} \tag{18}$$

Thus, (13) holds. Using (13) and (16), we rearrange equations (8), (15), and (18) to get

$$\frac{d}{dt}\begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} = \overline{\mathcal{A}}_1 \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} + \overline{\mathcal{B}}_2 u^2 + \overline{\mathcal{C}}^2, \tag{19}$$

$$\frac{d}{dt}\begin{bmatrix} \lambda^2 \\ x \end{bmatrix} = (-\overline{\mathcal{A}}_1^\top + \rho E)\begin{bmatrix} \lambda^2 \\ x \end{bmatrix} + \overline{\mathcal{A}}_2 \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} + \overline{\mathcal{B}}_1 u^2 + \overline{\mathcal{C}}^1. \tag{20}$$

From (14) and (16), it follows that

$$\frac{\mathcal{R} + \mathcal{R}^\top}{2} u^2 = \begin{bmatrix} \mathcal{B}_2 \\ -P_2\mathcal{B}_1 - \mathcal{S}_1 \end{bmatrix}^\top \begin{bmatrix} \lambda^2 \\ x \end{bmatrix} + \begin{bmatrix} -\mathcal{S}_2 \\ -\mathcal{B}_1 \end{bmatrix}^\top \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} - \mathcal{K}, \tag{21}$$

i.e.,

$$u^2 = \overline{\mathcal{R}}^{-1}\Big(-\overline{\mathcal{B}}_1^\top \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} + \overline{\mathcal{B}}_2^\top \begin{bmatrix} \lambda^2 \\ x \end{bmatrix} - \mathcal{K}\Big). \tag{22}$$

Substituting the expression of u^2 given by (22) into (19) and (20), the validity of (10) and (11) is established. This completes the proof. □

Remark 2. In Theorem 4.1, unlike the results obtained in [23], there is no second-order term in Riccati differential equation of P_2 given by (13) in our model. Problem (P_L) is transformed into system (9)-(11) with boundary conditions (12). In fact, the differential equation (13) is Lyapunov differential equation, of which the unique solution is shown to exist in [2] (Corollary 1.1.6.).

Next, the leader's optimal control problem can be reformulated as a new linear-quadratic optimal control problem, which is referred to as Problem $(\overline{P}_L)$, as given below: Given system (19) with initial condition

$$\begin{bmatrix} \varphi^1(0) \\ \varphi^2(0) \end{bmatrix} = \overline{\varphi}^o, \tag{23}$$

find a u^2 such that the following objective function

$$e^{-\rho T}\overline{\Phi}(\varphi^1(T), \varphi^2(T)) + \int_0^T e^{-\rho t}\overline{\mathcal{L}}(t, \varphi^1(t), \varphi^2(t), u^2(t))\, ds$$

is minimized, where

$$\begin{aligned} \overline{\mathcal{L}} =& \frac{1}{2}\begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix}^\top \overline{\mathcal{A}}_2 \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} + \frac{1}{2}(u^2)^\top \overline{\mathcal{R}} u^2 + \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix}^\top \overline{\mathcal{B}}_1 u^2 \\ &+ \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix}^\top \overline{\mathcal{C}}^1 + (u^2)^\top \mathcal{K}, \\ \overline{\Phi} =& \frac{1}{2}\begin{bmatrix} \varphi^1(T) \\ \varphi^2(T) \end{bmatrix}^\top \overline{\mathcal{G}} \begin{bmatrix} \varphi^1(T) \\ \varphi^2(T) \end{bmatrix} + \begin{bmatrix} \varphi^1(T) \\ \varphi^2(T) \end{bmatrix}^\top \overline{\alpha}, \\ \overline{\mathcal{G}} =& \overline{\mathcal{G}}^\top \in \mathbb{R}^{2n\times 2n},\ \overline{\alpha} \in \mathbb{R}^{2n},\ \overline{\varphi}^o \in \mathbb{R}^{2n}. \end{aligned}$$

In other words,

$$\begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} \text{ and } \begin{bmatrix} \lambda^2 \\ x \end{bmatrix}$$

are considered as the state vector and the co-state vector, respectively.

Next, we will show how to construct $\overline{\mathcal{G}}$, $\overline{\alpha}$, and $\overline{\varphi}^o$ such that (12) holds. Using the backward sweep method, we set

$$\begin{bmatrix} \lambda^2 \\ x \end{bmatrix} = -\overline{\mathcal{P}} \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} - \overline{\varphi}, \tag{24}$$

so $\overline{\mathcal{P}} \in \mathbb{R}^{2n\times 2n}$ and $\overline{\varphi} \in \mathbb{R}^{2n}$ can be solved with the initial conditions

$$\overline{\mathcal{P}}(0) = O \text{ and } \overline{\varphi}(0) = -\begin{bmatrix} 0 \\ x^o \end{bmatrix}. \tag{25}$$

Thus,

$$\begin{bmatrix} \lambda^2(0) \\ x(0) \end{bmatrix} = \begin{bmatrix} 0 \\ x^o \end{bmatrix}. \tag{26}$$

Next, we note that $\left[(\varphi^1)^\top, (\varphi^2)^\top\right]^\top$ can be obtained through solving (10) and (24) with the boundary conditions

$$\begin{bmatrix} \varphi^1(T) \\ \varphi^2(T) \end{bmatrix} = \begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix}. \tag{27}$$

Thus, by setting

$$\overline{\mathcal{G}} = \overline{\mathcal{P}}(T),\ \overline{\alpha} = \overline{\varphi}(T),\ \overline{\varphi}^o = \begin{bmatrix} \varphi^1(0) \\ \varphi^2(0) \end{bmatrix},$$

Problem $(\overline{P}_L)$ is equivalent to Problem (P_L).

Theorem 4.2. *For a symmetric matrix $\Psi \in \mathbb{R}^{2n\times 2n}$ and a vector $\psi \in \mathbb{R}^{2n}$ such that*

$$\begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} = \Psi \begin{bmatrix} \lambda^2 \\ x \end{bmatrix} + \psi. \tag{28}$$

Then, Ψ and ψ satisfy

$$\begin{aligned} \frac{d\Psi}{dt} =& \overline{\mathcal{B}}_2 \overline{\mathcal{R}}^{-1} \overline{\mathcal{B}}_2^\top - \rho\Psi + \left(\overline{\mathcal{A}}_1 - \overline{\mathcal{B}}_2 \overline{\mathcal{R}}^{-1} \overline{\mathcal{B}}_1^\top\right)\Psi \\ &+ \Psi\left(\overline{\mathcal{A}}_1^\top - \overline{\mathcal{B}}_1 \overline{\mathcal{R}}^{-1} \overline{\mathcal{B}}_2^\top\right) - \Psi\left(\overline{\mathcal{A}}_2 - \overline{\mathcal{B}}_1 \overline{\mathcal{R}}^{-1} \overline{\mathcal{B}}_1^\top\right)\Psi \end{aligned} \tag{29}$$

with $\Psi(T) = O$, and

$$\begin{aligned} \frac{d\psi}{dt} =& -\Psi\left(\overline{\mathcal{A}}_2 - \overline{\mathcal{B}}_1 \overline{\mathcal{R}}^{-1} \overline{\mathcal{B}}_1^\top\right)\psi + \left(\overline{\mathcal{A}}_1 - \overline{\mathcal{B}}_2 \overline{\mathcal{R}}^{-1} \overline{\mathcal{B}}_1^\top\right)\psi \\ &- \Psi\left(\overline{\mathcal{C}}^1 - \overline{\mathcal{B}}_1 \overline{\mathcal{R}}^{-1} \mathcal{K}\right) + \left(\overline{\mathcal{C}}^2 - \overline{\mathcal{B}}_2 \overline{\mathcal{R}}^{-1} \mathcal{K}\right) \end{aligned} \tag{30}$$

with

$$\psi(T) = \begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix}, \tag{31}$$

respectively.

Proof. Since system (11)-(10) with boundary conditions (12) has a unique solution, it follows from (12) that

$$\begin{bmatrix} \varphi^1(T) \\ \varphi^2(T) \end{bmatrix} = O \begin{bmatrix} \lambda^2(T) \\ x(T) \end{bmatrix} + \begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix},$$

where O is a zero matrix. Recall (28) as follows:

$$\begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} = \Psi \begin{bmatrix} \lambda^2 \\ x \end{bmatrix} + \psi. \tag{32}$$

Then

$$\Phi(T) = O, \ \psi(T) = \begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix},$$

and

$$\frac{d}{dt}\begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} = \frac{d\Psi}{dt}\begin{bmatrix} \lambda^2 \\ x \end{bmatrix} + \Psi \frac{d}{dt}\begin{bmatrix} \lambda^2 \\ x \end{bmatrix} + \frac{d\psi}{dt}, \tag{33}$$

which implies (29) and (30). This completes the proof. □

Note that $\lambda^2(0) = 0$. Then, by (5) and (9), the optimal controls in feedback form are given, respectively, by

$$\begin{aligned} u^{1,*}(x^o, u^2) =& R_{11}^{-1}\Big(\begin{bmatrix} O \\ -S_1 - P_1 B_1 \end{bmatrix}^\top + \begin{bmatrix} -B_1 \\ O \end{bmatrix}^\top \Psi \Big) \begin{bmatrix} 0 \\ x^o \end{bmatrix} \\ &- R_{11}^{-1} \widetilde{R}_1 u^2 + R_{11}^{-1}\Big(\begin{bmatrix} -B_1 \\ O \end{bmatrix}^\top \psi - \mu^1 \Big) \end{aligned} \tag{34}$$

and

$$u^{2,*}(x^o) = \overline{\mathcal{R}}^{-1}\left(\overline{\mathcal{B}}_2^\top - \overline{\mathcal{B}}_1^\top \Psi\right) \begin{bmatrix} 0 \\ x^o \end{bmatrix} + \overline{\mathcal{R}}^{-1}\left(-\overline{\mathcal{B}}_1^\top \psi - \mathcal{K}\right). \tag{35}$$

We may now summarize the computational procedure to compute the optimal control strategies $u^{1,*}$ and $u^{2,*}$ in feedback form as detailed by Algorithm (P_S).

Algorithm (P_S) for Problem (P_S)

1. Compute P_1 through solving the symmetric Riccati differential equation (6) with $P_1(T) = G_1$;
2. Compute P_2 through solving the symmetric Riccati differential equation (13) with $P_2(T) = G_2$;
3. Compute Ψ through solving the symmetric Riccati differential equation (29) with $\Psi(T) = O$;
4. Compute ψ through solving the ODEs (30) with $\psi(T) = \begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix}$;
5. Compute $u^{1,*}(x^o, u^2)$ and $u^{2,*}(x^o)$ using (34) and (35).

Remark 3. Note that in the above algorithm, the dynamics governing the four sets of variables P_1, P_2, Ψ and ψ are independent of x, and hence they can be solved in sequence from $t = T$ to $t = 0$ successively. Furthermore, we note that $\lambda^2(t) = 0$ for any $t \in [0, T]$. Thus, the expression of the equilibrium strategy contains only $P_1(t)$, $P_2(t)$, $\Psi(t)$, $\psi(t)$, and $x(t)$. Therefore, the optimal control strategies (34) and (35) are time-consistent and in feedback form.

Summarizing the above results, we have the following theorem.

Theorem 4.3. *Consider Problem (P_S). Suppose that R_{11}, $\overline{\mathcal{R}}$ are both symmetric positive definite matrixes, and that the three decoupled matrix differential equations (6), (13), (29), and the ordinary differential equations (30) can be solved uniquely. Then, the time-consistent state feedback Stackelberg equilibrium strategy $(u^{1,*}(x^o, u^2), u^{2,*}(x^o))$ is given by (34) and (35).*

Remark 4. Consider the case as in [1, 2, 9, 18], where only the following terms

$$A(t),\ B_i(t),\ Q_i(t),\ R_{ii}(t),\ R_{ij}(t),\ G_i,\ i, j = 1, 2,\ i \neq j,$$

are considered and $\rho = 0$. Then our results reduce to

$$u^{1,*}(t) = -R_{11}^{-1}(t) B_1^\top(t) \Big(P_1(t) + \Psi_{12}(t) \Big) x(t), \tag{36}$$

$$u^{2,*}(t) = -R_{22}^{-1}(t) B_2^\top(t) \Big(P_2(t) + \Psi_{22}(t) \Big) x(t), \tag{37}$$

where

$$\Psi(t) = \begin{bmatrix} \Psi_{11}(t) & \Psi_{12}(t) \\ \Psi_{12}^\top(t) & \Psi_{22}(t) \end{bmatrix},\ \Psi_{ij}(t) \in \mathbb{R}^{n \times n},\ i, j = 1, 2,$$

and

$$\begin{aligned}
\frac{dP_1}{dt} &= -Q_1 - A^\top P_1 - P_1 A + P_1 U_1 P_1, \\
\frac{dP_2}{dt} &= -\Big(Q_2 + P_1 U_{21} P_1 \Big) - \Big(A - U_1 P_1 \Big)^\top P_2 - P_2 \Big(A - U_1 P_1 \Big), \\
\frac{d\Psi}{dt} &= \begin{bmatrix} -A^\top + P_1 U_1 & P_1 U_2 \\ P_2 U_1 - P_1 U_{21} & -A^\top + P_1 U_1 + P_2 U_2 \end{bmatrix} \Psi \\
&\quad + \Psi \begin{bmatrix} -A^\top + P_1 U_1 & P_1 U_2 \\ P_2 U_1 - P_1 U_{21} & -A^\top + P_1 U_1 + P_2 U_2 \end{bmatrix}^\top \\
&\quad - \Psi \begin{bmatrix} U_{21} & -U_1 \\ -U_1 & -U_2 \end{bmatrix} \Psi + \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} U_2 \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^\top,
\end{aligned}$$

$$\frac{dx}{dt} = \Big(A - U_1(P_1 + \Psi_{12}) - U_2(P_2 + \Psi_{22})\Big)x$$

with

$$P_1(T) = G_1,\ P_2(T) = G_2,\ \Psi(T) = O,$$
$$U_1 = B_1 R_{11}^{-1} B_1^\top,\ U_2 = B_2 R_{22}^{-1} B_2^\top,\ U_{21} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^\top.$$

Obviously, P_1, P_2, Ψ and x can be solved from $t = T$ to $t = 0$ successively.

5. **Classical examples.** In this section, the results and the computational algorithm developed, respectively, in Sections 3 and 4 are applied to solve some classical Stackelberg games. Matlab code can be downloaded from GitHub[1].

Example 1 ([18])**.** Consider the nonzero-sum velocity-controlled pursuit-evasion game. The performance criteria are

$$J_F(u^1, u^2) = \frac{1}{2}x(T)^2 + \frac{1}{2}\int_0^T \frac{1}{c_p}(u^1(t))^2\,dt,$$
$$J_L(u^1, u^2) = -\frac{1}{2}x(T)^2 + \frac{1}{2}\int_0^T \frac{1}{c_e}(u^2(t))^2\,dt$$

with

$$\begin{cases} \dfrac{dx(t)}{dt} = u^1(t) - u^2(t),\ t \in (0, T], \\ x(0) = x^o, \end{cases}$$

where u^1 and u^2 are, respectively, the velocities of the pursuer as the follower and the evader as leader, x is their relative position, and $T = 1$, $c_p > 0$, $c_e > 0$, $c_p c_e = 1$.

By Algorithm (P_S), we obtain, for each $t \in [0, 1]$,

$$P_1(t) = \frac{1/c_p}{1 + 1/c_p - t},\ P_2(t) = -P_1(t)^2,\ \psi(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and $\Psi(t)$ satisfies

$$\begin{aligned} \frac{d\Psi}{dt} =& \Psi \begin{bmatrix} c_p P_1 & c_p P_2 \\ c_e P_1 & c_p P_1 + c_e P_2 \end{bmatrix} + \begin{bmatrix} c_p P_1 & c_e P_1 \\ c_p P_2 & c_p P_1 + c_e P_2 \end{bmatrix} \Psi \\ & - \Psi \begin{bmatrix} 0 & -c_p \\ -c_p & -c_e \end{bmatrix} \Psi + c_e \begin{bmatrix} P_1 P_1 & P_1 P_2 \\ P_1 P_2 & P_2 P_2 \end{bmatrix} \end{aligned} \tag{38}$$

with

$$\Psi(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The optimal controls in feedback form are given by

$$u^1(t) = -c_p\Big(P_1(t) + \Psi_{12}(t)\Big)x(t),\ u^2(t) = c_e\Big(P_2(t) + \Psi_{22}(t)\Big)x(t).$$

In Figure 1, we can verify numerically that

$$u^1(0) = \frac{-c_p}{(1 + c_p) - c_e/(1 + c_p)}x^o,\ u^2(0) = \frac{-c_e/(1 + c_p)}{(1 + c_p) - c_e/(1 + c_p)}x^o,$$

as in [18]. Note that a two-point boundary-value problem is required to be solved in [18], where three coupled matrix equations are involved. This is clearly not easy. On the other hand, only three decoupled symmetric Riccati differential equations are required to be solved one by one for the approach presented in this paper.

[1] https://github.com/LiYuTJUFE/Stackelberg-Game.

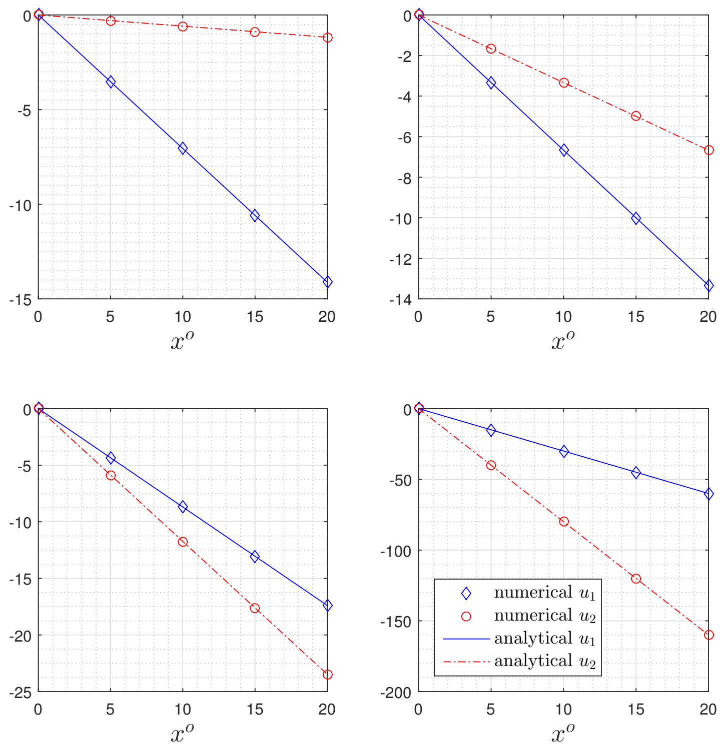

FIGURE 1. $c_e = 0.5, 1.0, 1.5, 2.0$ from left to right and from up to down.

In the above example, there is no cross terms involved in each player's objective function, and the parameters are time invariant. A more general cases are considered in Examples 2 and 3.

Example 2 ([13])**.** Consider a supply chain composed of a supplier and a retailer. The supply chain faces time-dependent endogenous demand that depends on the retail price.

The retailer as the follower must decide on the retail price $p(t)$ and the order quantity $u(t)$. The supplier as the leader, on the other hand, decides the wholesale price $w(t)$ only. This Stackelberg game is played over a season of length T, which includes a short promotional period $[t_s, t_f)$. The performance criteria are

$$J_F(u,p,w) = -\int_0^T \Big(p(t)(a(t) - b(t)p(t)) - c_r u(t) - w(t)u(t) - h(x(t),u(t))\Big)\,dt,$$

and

$$J_L(u,p,w) = -\int_0^T \Big(w(t)u(t) - c_s u(t)\Big)\,dt,$$

while the governing dynamical system is:

$$\begin{cases} \dfrac{dx(t)}{dt} = u(t) - \Big(a(t) - b(t)p(t)\Big), \ t \in (0, T], \\ x(0) = 0, \end{cases}$$

where $x(t)$ denotes the inventory level at time t.

$$a(t) = \begin{cases} a_1, & \text{if } t < t_s \text{ and } t \geq t_f \\ a_2, & \text{if } t_s \leq t < t_f, \end{cases}, \ a_1 \leq a_2,$$

$$b(t) = \begin{cases} b_1, & \text{if } t < t_s \text{ and } t \geq t_f \\ b_2, & \text{if } t_s \leq t < t_f, \end{cases}, \ b_1 \leq b_2,$$

$$h(x, u) = \frac{1}{2}\gamma_1 x^2 + \frac{1}{2}\gamma_2 u^2,$$

and

$$\begin{aligned} &12/5\, a_1 = a_2 = 6000, \ 3\, b_1 = b_2 = 30, \ \gamma_1 = \gamma_2 = 0.01, \\ &2\, c_r = c_s = 60, \ t_s = 100, \ t_f = 300, \ T = 1000. \end{aligned}$$

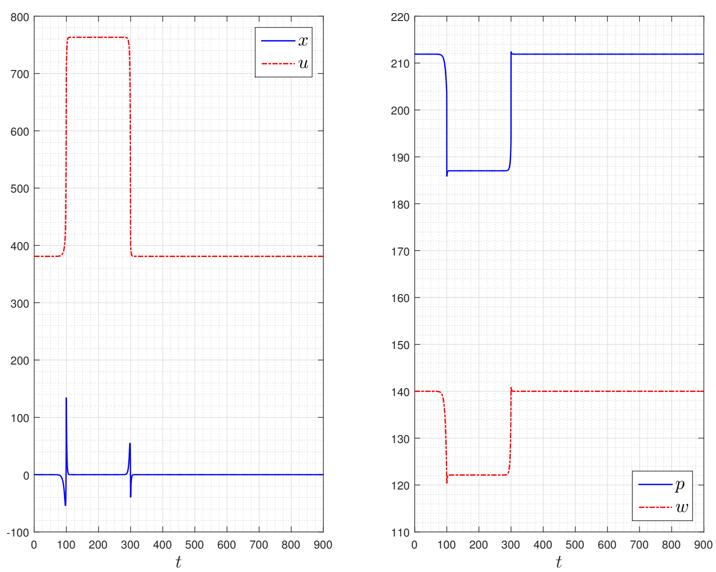

FIGURE 2. The optimal polices under promotion.

For this example, it was assumed in [13] that the supplier is restricted to set a constant wholesale price w_1 in the regular periods and $w_2 \leq w_1$ in the promotion period $[t_s, t_f)$. However, this assumption is not imposed using our approach. Interestingly, the optimal solutions obtained, which are depicted in Figure 2, turn out to be in the form as specified in this assumption. From Figure 3, we observe that the retailer starts to lower the price even before the promotion starts, at the time when it is out of stock. This outcome is consistent with the sketch shown in Figure 4.6 in [13]. In addition, there will be a surplus in the inventory level towards the

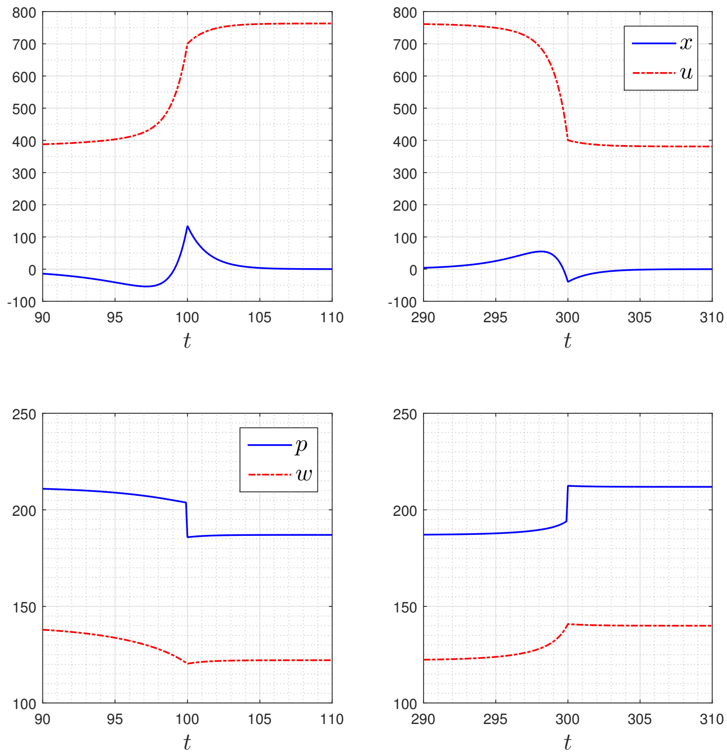

FIGURE 3. Zoom in at t_s and t_f.

end of the promotion period. The above conclusions are also shared in the study reported in [13] (Chapter 4.3).

In the next example, the model considered in [13] is extended to include three state variables. The solution obtained clearly shows that our algorithm can handle cases involving multi state variables.

Example 3. We consider the situation where there are different demand rates

$$d_i = a_i(t) - b_i(t)p(t),\ i = 1, 2, 3,$$

for three locations denoted by 1, 2, 3. The retailer will determine the total order quantity u_1 to be delivered the location 1, and allocate v_{12}, v_{13} from location 1 to locations 2 and 3, respectively. Here, the price sensitivity $b_i(t)$, $i = 1, 2, 3$, is the same, but $a_i(t)$, $i = 1, 2, 3$, are different. Specifically,

$$a_1(t) = a_2(t) < a_3(t),\ b_1(t) = b_2(t) = b_3(t).$$

In addition, the retailer as the follower also determines on the retail price $p(t)$ and the supplier as the leader determines on the wholesale price $w(t)$ only. The performance criteria are:

$$
\begin{aligned}
J_F(u_1, v_{12}, v_{13}, p, w) = -\int_0^T \Big(& \sum_{i=1}^{3} p(t)(a_i(t) - b_i(t)p(t)) \\
& - c_r u_1(t) - c_{r12} v_{12}(t) - c_{r13} v_{13}(t) \\
& - w(t)u_1(t) - h(x_1(t), x_2(t), x_3(t), u_1(t), v_{12}(t), v_{13}(t)) \Big)\, dt,
\end{aligned}
$$

and

$$
J_L(u_1, v_{12}, v_{13}, p, w) = -\int_0^T \Big(w(t)u_1(t) - c_s u_1(t) \Big)\, dt.
$$

The governing dynamical system is:

$$
\begin{cases}
\dfrac{dx_1(t)}{dt} = u_1(t) - \Big(v_{12}(t) + v_{13}(t) \Big) - \Big(a_1(t) - b_1(t)p(t) \Big), \\
\dfrac{dx_2(t)}{dt} = v_{12}(t) - \Big(a_2(t) - b_2(t)p(t) \Big), \\
\dfrac{dx_3(t)}{dt} = v_{13}(t) - \Big(a_3(t) - b_3(t)p(t) \Big), \\
x_1(0) = x_2(0) = x_3(0) = 0,
\end{cases}
$$

where $x_i(t)$, $i = 1, 2, 3$, denote the respective inventory at location i at time t, and for $i = 1, 2, 3$,

$$
a_i(t) = \begin{cases} a_{i1}, & \text{if } t < t_s \text{ and } t \geq t_f \\ a_{i2}, & \text{if } t_s \leq t < t_f \end{cases}, \; a_{i1} \leq a_{i2},
$$

$$
b_i(t) = \begin{cases} b_{i1}, & \text{if } t < t_s \text{ and } t \geq t_f \\ b_{i2}, & \text{if } t_s \leq t < t_f \end{cases}, \; b_{i1} \leq b_{i1},
$$

$$
h(x_1, x_2, x_3, u_1, v_{12}, v_{13}) = \frac{1}{2} \Big(\sum_{i=1}^{3} \alpha_i x_i^2 + \gamma_1 u_1^2 + \eta_{12} v_{12}^2 + \eta_{13} v_{13}^2 \Big),
$$

and

$$
\begin{aligned}
& T = 1000, \; t_s = 100, \; t_f = 300, \\
& \alpha_1 = \alpha_2 = \alpha_3 = \gamma_1 = \eta_{12} = \eta_{13} = 0.01, \\
& 8/3\, a_{11} = 8/3\, a_{21} = 2\, a_{31} = a_{12} = a_{22} = a_{32} = 2000, \\
& 3\, b_{11} = 3\, b_{21} = 3\, b_{31} = b_{12} = b_{22} = b_{32} = 10, \\
& 2\, c_{r1} = 2\, c_{r12} = 2\, c_{r13} = c_{s1} = 60.
\end{aligned}
$$

Due to the different the demand rates of the three locations are different, the three locations react differently during the promotion period. The outcomes are depicted in Figures 4 and 5. In location 3, due to the increase in the price sensitivity and $b_{31} < b_{32}$, although $a_{31} < a_{32}$, the demand rate does not increase but decrease during the promotion period. For location 1, the demand rate increases substantially.

6. **Conclusions.** In this paper, we considered a general form of linear-quadratic Stackelberg deterministic differential game model. The explicit expression of the Stackelberg equilibrium strategy in feedback form is obtained. A practical effective computational procedure is developed. The results obtained are applied to some classical Stackelberg game problems. Some useful managerial insights are drawn.

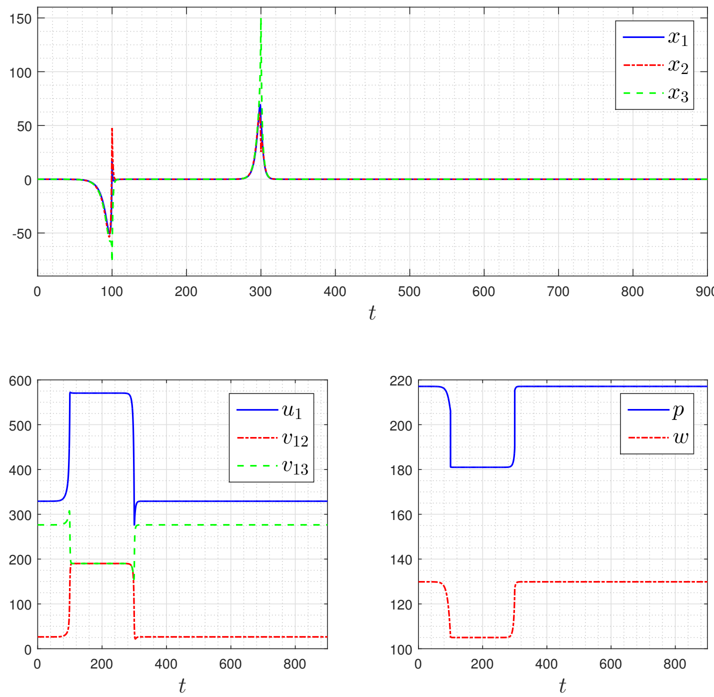

FIGURE 4. The optimal polices under promotion.

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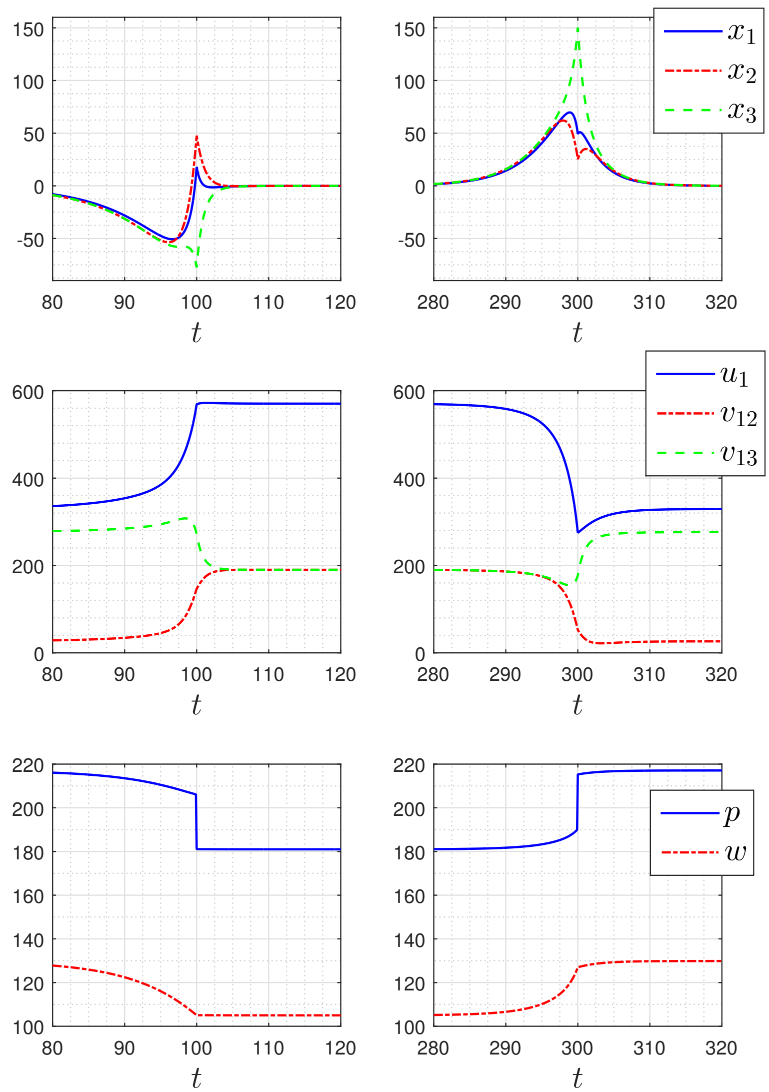

FIGURE 5. Zoom in at t_s and t_f.

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